Introduction to Black Hole Thermodynamics

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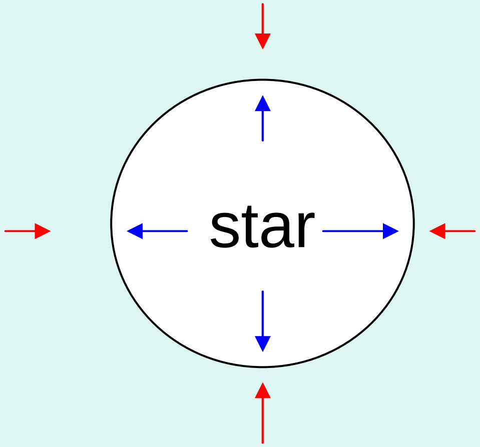
Plan of the talk

- [1] Overview of BH thermodynamics
  - ･causal structure of horizon
  - ･Hawking radiation
  - ･stringy picture of BH entropy

- [2] Hawking radiation via quantum anomalies
  - ･universality of Hawking radiation

- [3] Conclusion
  - ･ towards quantum nature of space-time

[1] Overview of BH thermodynamics



Pressure caused by nuclear fusion in the star stabilizes it against gravitational collapse.

All nuclear fuel used up

Massive stars end their lives by supernova explosion and the remnants become **black holes**.

No hair theorem

Stationary black holes are characterized by 3 quantities. **(M, Q, J)**

mass, charge, and angular momentum

Schwarzschild black holes

$$ds^2 = f(r)dt^2 - \frac{dr^2}{f(r)} - r^2 d\Omega^2$$

$$f(r) = 1 - \frac{r_H}{r}$$

horizon radius: $$r_H = \frac{2GM}{c^2}$$

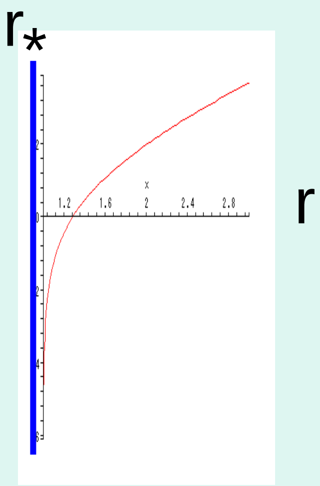
Curvature: $$R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} = \frac{12r_H^2}{r^6}$$

Curvature is singular at r=0 but nothing is singular at the horizon.

Causal structure of horizon

Tortoise coordinate:

$$r_* = r + r_H \ln(\frac{r}{r_H} - 1)$$



Null coordinates:

$$u = t - r_*, \qquad v = t + r_*$$

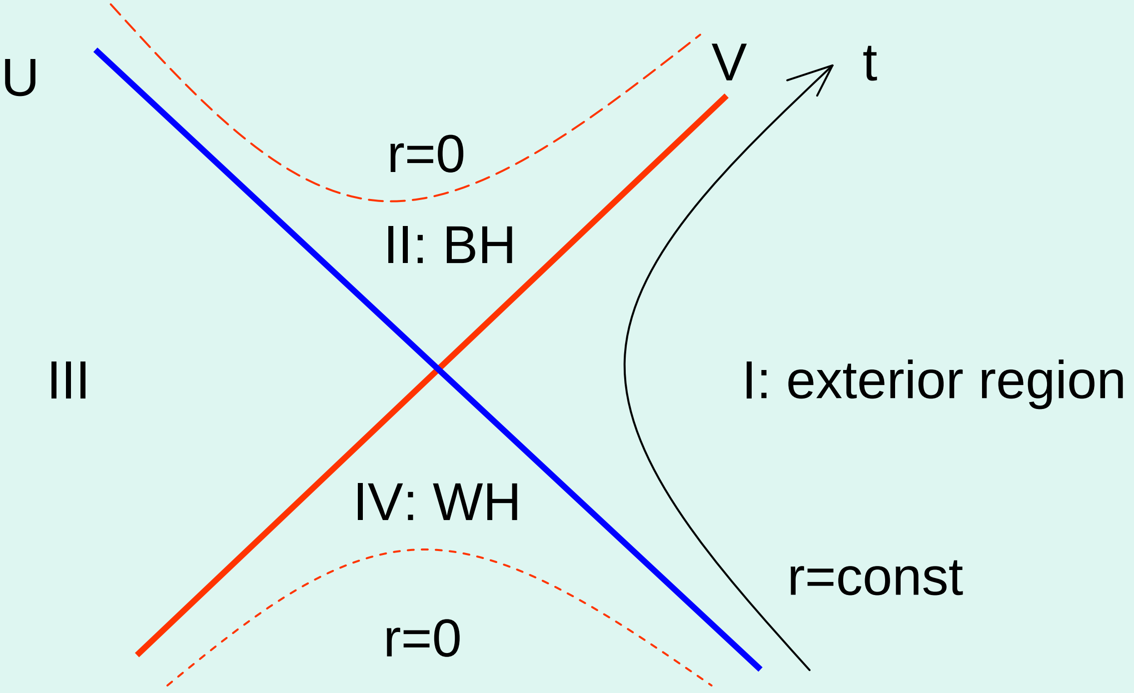
$$ds^2 = f(r)(dt^2 - dr_*^2) - r^2 d\Omega^2 = f(r) du dv - r^2 d\omega^2$$

Kruskal coordinates: $U = \frac{-1}{\kappa}\exp(-\kappa u), \qquad V = \frac{1}{\kappa}\exp(\kappa v)$

surface gravity: $\kappa = \frac{1}{2r_H} = \frac{GM}{r_H^2}$

$$ds^2 = \frac{r_H}{r}e^{-r/r_H}dU\,dV - r^2 d\Omega^2$$

regular at horizon


U

V

t

r=0

II: BH

III

I: exterior region

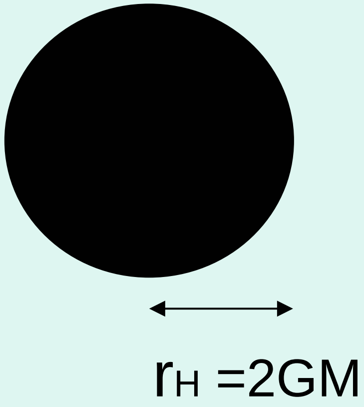
IV: WH

r=const

r=0

U=0, V=0 at horizon

U=0 future horizon
V=0 past horizon



Horizon is a null hypersurface.

No one can come out of the horizon.

↓

BH mass always increases classically.

↓

Horizon area never decreases
like entropy in thermodynamics.

$$d A \geq 0$$

Analogy with Thermodynamics

Equilibrium Thermodynamics	Black Hole
0th law T=const.	0th law κ =const.
1st law $dE = T\,dS$	1st law $dM = \kappa /(8\pi G)\, dA$
2nd law $dS \geqq 0$	2nd law $dA \geqq 0$

Classical correspondence

Hawking radiation from black hole

In 1974 Hawking found that black hole radiates.
This really gave sense to the analogy with thermodynamics.

Hawking temperature:

$$T_H = \frac{\hbar\kappa}{2\pi}$$

Entropy of BH:

$$S_{BH} = \frac{A}{4\hbar G} = \frac{A}{4l_{PL}^2}$$

They are quantum effects!

For BH with 10 solar mass

$T_H \sim 6 \times 10^{-9}$ K very low temperature

$S_{BH} \sim 10^{79}$ k_B huge entropy

cf. Entropy of sun ~ 10^{58}

In the classical limit,

$$T_H \longrightarrow 0$$

$$S_{BH} \longrightarrow \infty$$

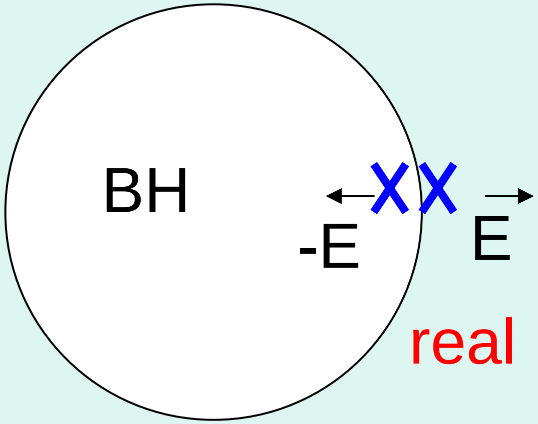
Hawking radiation = universal quantum effect
for matters in Black holes.

BH entropy = universal quantum gravity effect
(geometrical quantity)

Physical picture of Hawking radiation

virtual pair creation of particles

-E XX E



real pair creation

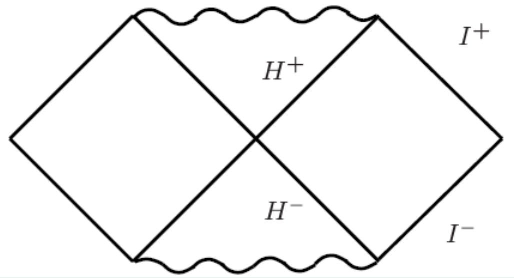
Hawking radiation

thermal spectrum with T

$$k_B T = \frac{\hbar c^3}{8\pi G M}$$

Derivation of Hawking radiation by Unruh for eternal BH

$$\begin{aligned}\Phi &= \sum_{\omega>0} \left(a^B_\omega \varphi^B_\omega + \tilde{a}^B_\omega \tilde{\varphi}^B_\omega + h.c.\right) \\ &= \sum_{\omega>0} \left(a^U_\omega \varphi^U_\omega + \tilde{a}^U_\omega \tilde{\varphi}^U_\omega + h.c.\right)\end{aligned}$$



$\varphi^B_\omega \longrightarrow e^{-i\omega v}$ near I^-

$\tilde{\varphi}^B_\omega \longrightarrow e^{-i\omega u}$ near H^-

$\varphi^U_\omega \longrightarrow e^{-i\omega v}$ near I^-

$\tilde{\varphi}^U_\omega \longrightarrow e^{-i\omega U}$ near H^-

Boulware vacuum : $a^B_\omega|0,B\rangle = \tilde{a}^B_\omega|0,B\rangle = 0,$

Unruh vacuum : $a^U_\omega|0,U\rangle = \tilde{a}^U_\omega|0,U\rangle = 0$

$$N(w) = \langle 0,U|(\tilde{a_\omega}^B)^\dagger \tilde{a_\omega}^B|0,U\rangle = \frac{1}{e^{(\omega-\phi)/T_H} \pm 1}$$

$\langle 0,B|\ T^r_t\ |0,B\rangle \longrightarrow 0 \quad (r \longrightarrow \infty)$

$\langle 0,B|\ T_{UU}\ |0,B\rangle\Big|_{H+} = \infty$

$\langle 0,U|\ T^r_t\ |0,U\rangle \longrightarrow \frac{\pi}{12}T_H^2 \quad (r \longrightarrow \infty)$

$\langle 0,U|\ T_{UU}\ |0,U\rangle\Big|_{H+} < \infty$

Hawking radiation reduces BH mass.

↓

Area decreases and 2nd law is violated.

Generalized 2nd law

$S_{tot} = S_{BH} + S_{rad}$ $\quad\quad d\, S_{tot} \geqq 0$

Microscopic (statistical) meaning of BH entropy?

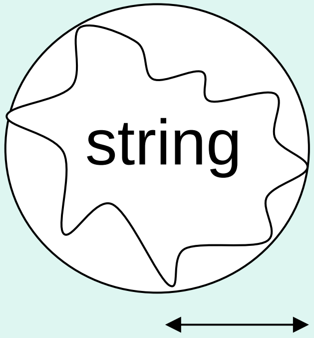
(1) Thermal Hawking radiation contradicts with the unitary evolution of quantum states. → information paradox

(2) Microscopic understanding of BH entropy?
needs quantum nature of space-time?

Basic idea to understand BH entropy in strings

Strings: both of matters and space-time (graviton) are excitations of strings

(4d) Newton constant $G \sim (g_s\, l_s)^2$



$r_H = 2GM$

At strong coupling, string with mass M becomes BH when its Schwarzschild radius equals the string length.

$(2GM \sim l_s)$

$S = k_B \log N(M) = k_B\, l_s\, M/\hbar$

$\sim k_B\, (GM)^2/(\hbar G) = S_{BH}$

$N(M) = \exp(l_s\, M/\hbar)$

Extrapolation to strong coupling is not reliable.

Instead of fundamental strings, we can use specific D-brane configurations. (cf. Wadia’s lecture)

(D1+D5+momentum along D1)

In this way, BH entropy can be understood microscopically in string theory.

Furthermore Hawking radiation can be also understood as a unitary process of closed string emission from D-branes.

Is everything understood in strings?

No!

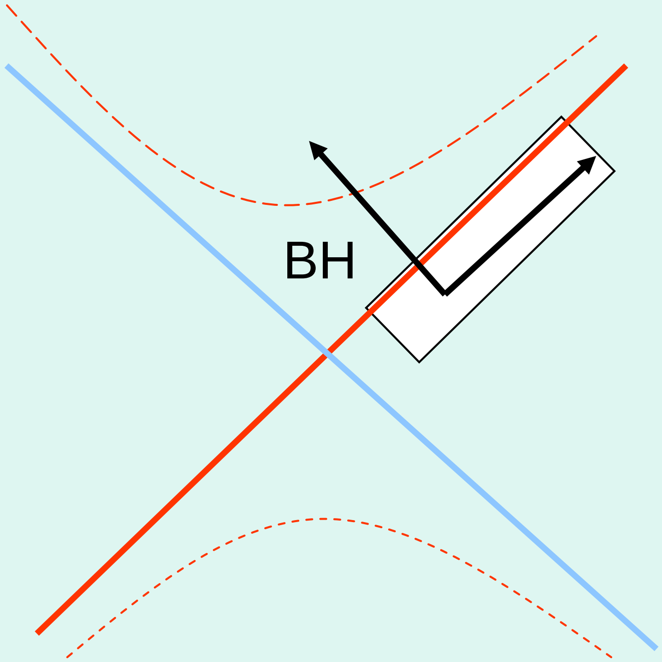
- Once D-branes are in the horizon, they are invisible from outside the BH.
  Why are these d.o.f seen as entropy to an outside observer?
- Information paradox is not yet well understood.

BH thermodynamics will be more universal phenomena irrespective of the details of quantum gravity formulation?

[2] Hawking radiation and quantum anomalies

Robinson Wilczek (05)
Iso Umetsu Wilczek (06)

Quantum fields
in black holes.



(1)Near horizon, each partial wave of d-dim quantum field behaves as d=2 massless free field.

Outgoing modes = right moving
Ingoing modes = left moving

Effectively 2-dim **conformal** fields

(2) Ingoing modes are decoupled
once they are inside the horizon.

→ These modes are ***classically irrelevant*** for the physics in exterior region.

So we first neglect ingoing modes near the horizon.

→ The effective theory becomes **chiral** in the two-dimensional sense.

gauge and gravitational anomalies
= breakdown of gauge and general coordinate invariance

(3) But the underlying theory is NOT anomalous.

Anomalies must be cancelled by **quantum effects** of the classically irrelevant ingoing modes.
(∼Wess-Zumino term)

⇛ flux of Hawking radiation

Charged black hole (Ressner-Nordstrom solution).

Iso Umetsu Wilczek (06)

Metric and gauge potential

$$ds^2 = f(r)dt^2 - \frac{1}{f(r)}dr^2 - r^2 d\Omega^2, \qquad A = -\frac{Q}{r}dt,$$

$$f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} = \frac{(r-r_+)(r-r_-)}{r^2} \qquad r_\pm = M \pm \sqrt{M^2 - Q^2}$$

r_+: outer horizon

r_-: inner horizon

Charged scalar field

$$S = \int d^4x\sqrt{-g}\left[g^{\mu\nu}\left(\partial_\mu + ieA_\mu\right)\phi^*\left(\partial_\nu - ieA_\nu\right)\phi - m^2\phi^*\phi + V(\phi)\right].$$

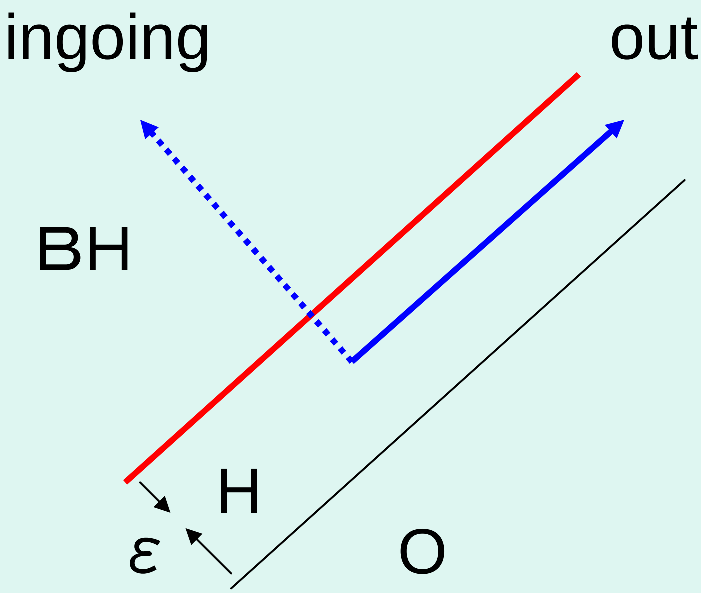
Partial wave decomposition: $\phi = \sum \phi_{lm}(t,r)Y_{lm}(\Omega)$

$$S = \sum_{l,m}\int dtdr_*\ r^2(r_*)\Big[|\left(\partial_t - ieA_t\right)\phi_{lm}|^2 + |\partial_{r_*}\phi_{lm}|^2 + f(r(r_*))\left(-m^2|\phi_{lm}|^2 + \frac{l(l+1)}{r^2}|\phi_{lm}|^2 + V(\phi_{lm})\right)\Big].$$

Near horizon, potential terms can be suppressed.

↓

Each partial wave behaves as **d=2 conformal field**.



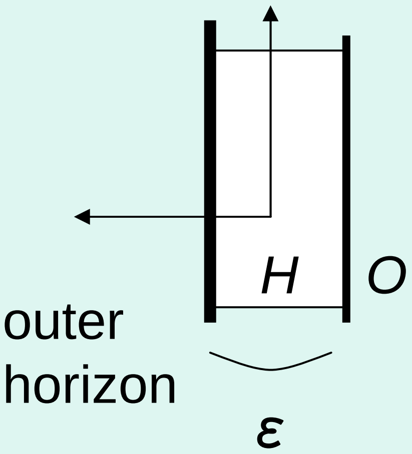
For calculational convenience, we divide the exterior region into H and O.

H: $[r_+, r_+ + \varepsilon]$

O: $[r_+ + \varepsilon, \infty]$

First neglect the classically irrelevant ingoing modes in region H.

Gauge current and gauge anomaly



The theory becomes chiral in H.

↓

Gauge current has anomaly in region H.

$$\nabla_\mu J^\mu = +\frac{e^2}{4\pi\sqrt{-g}}\epsilon^{\mu\nu}\partial_\mu A_\nu$$

consistent current

We can define a covariant current by

$$\tilde{J}^\mu = J^\mu + \frac{e^2}{4\pi\sqrt{-g}}A_\lambda \epsilon^{\lambda\mu}$$

which satisfies

$$\nabla_\mu \tilde{J}^\mu = +\frac{e^2}{4\pi\sqrt{-g}}\epsilon_{\mu\nu}F^{\mu\nu}$$

In region *O*, $\partial_r J^r_{(o)} = 0$

In near horizon region H, $\partial_r J^r_{(H)} = \frac{e^2}{4\pi}\partial_r A_t$ consistent current

$$\boxed{\begin{aligned} J^r_{(o)} &= c_o, \\ J^r_{(H)} &= c_H + \frac{e^2}{4\pi}\left(A_t(r) - A_t(r_+)\right) \end{aligned}}$$

c_o = current at infinity

c_H = value of consistent current at horizon

c_o c_H are integration constants.

Current is written as a sum of two regions.

$$J^\mu = J^\mu_{(o)}\Theta_+(r) + J^\mu_{(H)}H(r)$$

where $\Theta_+(r) = \Theta(r - r_+ - \epsilon)$ $H(r) = 1 - \Theta_+(r)$

Variation of the effective action under gauge tr.

$$-\delta W = \int d^2x \sqrt{-g_{(2)}}\lambda \nabla_\mu J^\mu_{(2)}$$

Using anomaly eq.

$$-\delta W = \int d^2x \lambda \left[\delta(r - r_+ - \epsilon) \left(J^r_o - J^r_H + \frac{e^2}{4\pi} A_t \right) + \partial_r \left(\frac{e^2}{4\pi} A_t H \right) \right]$$

↑

Impose $\delta W + \delta W' = 0$

W' = contribution from ingoing modes (WZ term)

⇓

$$c_o = c_H - \frac{e^2}{4\pi} A_t(r_+)$$

cancelled by WZ term

- Determination of c_H

We assume that the covariant current should vanish at horizon. ⟷ Unruh vac.

$$\tilde{J}^r = J^r + \frac{e^2}{4\pi}A_t(r)H(r) \longrightarrow c_o = -\frac{e^2}{2\pi}A_t(r_+) = \frac{e^2 Q}{2\pi r_+}$$

Reproduces the correct Hawking flux

$$J^r = e\int_0^\infty \frac{d\omega}{2\pi}\left[\frac{1}{e^{\beta(\omega-c)}+1} - \frac{1}{e^{\beta(\omega+c)}+1}\right]$$

$$(\beta = 1/T_H) \qquad c = \frac{eQ}{r_+}$$

Total current including ingoing modes near the horizon

$$J^{\mu}_{total} = J^{\mu} + K^{\mu}$$

should be conserved!

ingoing mode -------

$$K^{\mu} = -\frac{e^2}{4\pi} A_t(r) H(r)$$

outgoing mode ------

$$\boxed{\begin{aligned} J^{r}_{(o)} &= c_o, \\ J^{r}_{(H)} &= c_H + \frac{e^2}{4\pi} \left(A_t(r) - A_t(r_+)\right) \end{aligned}}$$

EM tensor and Gravitational anomaly

Effective d=2 theory contains background of graviton, gauge potential and dilaton.

Under diffeo. they transform

$$\delta g^{\mu\nu} = -(\nabla^\mu \xi^\nu + \nabla^\nu \xi^\mu)$$

$$\delta A_\mu = \xi^\nu \partial_\nu A_\mu + \partial_\mu \xi^\nu A_\nu$$

$$\delta\sigma = \xi^\mu \partial_\mu \sigma$$

Ward id. for the partition function $Z = \int \mathcal{D}\phi \exp(iS)$

$$-i \int d^n x \left[\delta g^{\mu\nu}(x) \underset{\overset{\uparrow}{T_{\mu\nu}}}{\frac{\delta}{\delta g^{\mu\nu}(x)}} + \delta A_\mu(x) \underset{\overset{\uparrow}{J^\mu}}{\frac{\delta}{\delta A_\mu(x)}} + \delta\sigma(x) \frac{\delta}{\delta\sigma(x)} \right] Z[g_{\mu\nu}, A_\mu, \sigma] = \text{anomaly}$$

Gravitational anomaly

$$\nabla_\mu T^\mu_\nu = \frac{1}{96\pi\sqrt{-g}} \epsilon^{\beta\delta}\partial_\delta\partial_\alpha\Gamma^\alpha_{\nu\beta} = \mathcal{A}_\nu$$ consistent current

$$\nabla_\mu \tilde{T}^\mu_\nu = \frac{1}{96\pi\sqrt{-g}} \epsilon_{\mu\nu}\partial^\mu R = \tilde{\mathcal{A}}_\nu$$ covariant current

In the presence of gauge and gravitational anomaly, Ward id. becomes

$$\nabla_\mu T^\mu{}_\nu = F_{\mu\nu}J^\mu + A_\nu\nabla_\mu J^\mu - \underset{\substack{\uparrow \\ \text{non-universal}}}{\partial_\nu(e^\sigma)\mathcal{L}} + \mathcal{A}_\nu$$

Solve $\nu = t$ component of Ward.id.

(1) In region *O*

$$\partial_r T^r_{t(o)} = F_{rt} J^r_{(o)}$$

(2) In region *H*

(near horizon)

$$\partial_r T^r_{t(H)} = \underbrace{F_{rt} J^r_{(H)} + A_t \nabla_\mu J^\mu_{(H)}}_{F_{rt} \tilde{J}^r_{(H)}} + \partial_r N^r_t$$

Using

$$J^r_{(o)} = c_o \qquad \tilde{J}^r_{(H)} = c_o + \frac{e^2}{2\pi} A_t(r)$$

$$T^r_{t(o)} = a_o + c_o A_t(r)$$

$$T^r_{t(H)} = a_H + \int_{r_+}^{r} dr \partial_r \left(c_o A_t + \frac{e^2}{4\pi} A_t^2 + N^r_t \right)$$

Variation of effective action under diffeo.

$$\int d^2x \sqrt{-g_{(2)}}\ \xi^t \nabla_\mu T^\mu_t$$

$$= \int d^2x\ \xi^t \left[c_o \partial_r A_t(r) + \partial_r \left(\frac{e^2}{4\pi} A_t^2 + N^r_t \right) \quad + \quad \left(T^r_{t\ (o)} - T^r_{t\ (H)} + \frac{e^2}{4\pi} A_t^2 + N^r_t \right) \delta(r - r_+ - \epsilon) \right]$$

(1) (2) (3)

(1) classical effect of background electric field

(2) cancelled by induced WZ term of ingoing modes

(3) Coefficient must vanish.

$$a_o = a_H + \frac{e^2}{4\pi} A_t^2(r_+) - N^r_t(r_+)$$

Determination of a_H

We assume that the covariant current to vanish at horizon.

↓

since

$$\tilde{T}^r_t = T^r_t + \frac{1}{192\pi}(ff'' - 2(f')^2)$$

we can determine $a_H = \kappa^2/24\pi = 2N^r_t(r_+)$

and therefore flux at infinity is given by

$$a_o = \frac{e^2Q^2}{4\pi r_+^2} + N^r_t(r_+) = \frac{e^2Q^2}{4\pi r_+^2} + \frac{\pi}{12\beta^2}$$

Reproduces the flux of Hawking radiation

The derivation of Hawking radiation made use of only the very fundamental property of horizon.

We have used only the following two
- horizon is null hypersurface
- ingoing modes at horizon can communicate with the exterior region only through anomaly

↓

Universality of Hawking radiation

[3] Conclusions

The deepest mysteries of BH are

Black hole entropy & information paradox

$$S_{BH} = \frac{A}{4\hbar G} = \frac{A}{4 l_{PL}^2}$$

- geometrical
- quantum

→

Do we really need details of Quantum gravity ?

SBH can be calculated by various geometrical ways once we assume the temperature of the BH.

Various geometrical ways to obtain S_{BH}

- Euclidean method
  calculate partition function for BH
  by using Einstein action with a boundary term

- conical singularity method
  dependence of partition function on the deficit
  angle (related to temperature)

- Wald formula
  BH entropy as Noether charge
  surface integral of Noether current on horizon
  associated with general coordinate tr.

But they cannot answer its microscopic origin.

Some proposals

- asymptotic symmetry
  number of general coordinate tr. that keep the asymptotic form of the metric invariant
  (successful in d=3 case)

- near horizon conformal symmetry (Carlip)

- ingoing graviton modes on the horizon
  may be relevant to the entropy

As blackbody radiation played an important role
in discovering the quantum mechanics,

black hole physics will play a similar role
to understand the quantum geometry.

Still there are many mysteries.