Orbifold as Calabi–Yau manifold

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Heterotic String and M-Theory
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Heterotic string

A top-down approach.

Objective: Obtain the Standard Model from the first principle.

Heterotic string: closed string with uniform distribution of charge

$$j^i(z)j^j(0) \sim \frac{k\delta^{ij}}{2z^2} + i\frac{f^{ijk}}{z}j^k(0)$$

Consistency condition $Z(\tau) = Z(\tau+1) = Z(-1/\tau)$ τ complex structure of worldseet torus

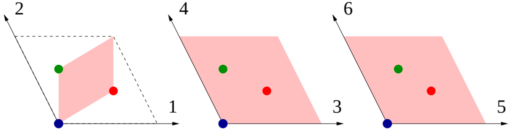
$Z(\tau) =$ 

$E_8 \times E_8$ or $SO(32)$, 10D, 16 SUSY ... Too much symmetry!
Break it by associating with symmetries of internal manifold.

Toroidal orbifold [Dixon, Harvey, Vafa, Witten] …

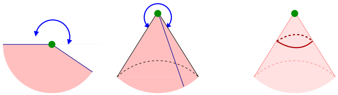
Toroidal orbifold: Very simple but very rich structure.

ex. $T^6/\mathbf{Z}_3, \phi = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ $z_i \sim e^{2\pi i \phi_i}, z_i \sim z_i + 1$



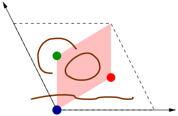
- Worldsheet Eq. of motion exactly solvable: $\alpha' M^2 =$ oscillator + windings
- 1/4 SUSY $\mathbf{4} = \mathbf{3} + \mathbf{1}$, chiral fermions
- gauge symmetry breaking associated with rotation, translation

Fixed points. locally $\mathbf{C}^3/\mathbf{Z}_3 : z_1 z_2 z_3 = w^3$.



Singular: Mathematics is not well-defined. ex. Euler no cannot be obtained from homology.

Physicswise:



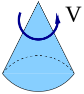
1. String is well-behaving—twisted string.
2. Singularity = signal of massless fields

This talk: Resolve the singularity and compare two theories.

Comparison strategy

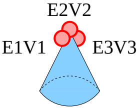
We will compare the orbifold compactification and its blown-up as follows.

1. Choose the orbifold ϕ. $z_i \sim e^{2\pi i \phi_i}, z_i \sim z_i + 1$
   Symmetry breaking is specified by the shift vector $(V_1, V_2, \dots, V_{16})$.



$\phi \longrightarrow V$

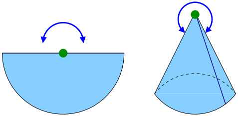
2. Calculate the spectrum using CFT.
3. Find the corresponding blown-up orbifold
   : A smooth Calabi–Yau manifold, embracing the orbifold
   Symmetry breaking specified by line bundles $(L^{V_1}, L^{V_2}, \dots, L^{V_{16}})$,
   supported by a number of smooth cycles $E_1, E_2, \dots E_l$.



4. Calculate the spectrum using index theorem.

Compare the spectra to identify the moduli, see the changes in symmetry breaking, etc.

Resolution of $\mathbf{C}^2/\mathbf{Z}_2$ $(Z_1, Z_2) \sim (-Z_1, -Z_2)$ [Lust Reffert Scheidegger Stieberger] [GrootNibbelink Trapletti Walter]



singular at $(Z_1, Z_2) = (0, 0)$

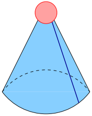
Introduce another coordinate x and mod out $(\mathbf{C}^3 - F)/\mathbf{C}^*$, $\quad F = \{z_1 = z_2 = 0\}$,

$$\mathbf{C}^* : (z_1, z_2, x) \longrightarrow (\lambda z_1, \lambda z_2, \lambda^{-2} x).$$

In terms of invariant local coordinates

$$Z_1 = z_1 x^{\frac{1}{2}}, \quad Z_2 = z_2 x^{\frac{1}{2}}.$$

1. $P_1 = \{x \neq 0\}$: the original orbifold,
   $\lambda^{-2} x = 1 \Longrightarrow \lambda = \pm\sqrt{x}$, the $\mathbf{Z}_2$ ambiguity.
   Origin $(Z_1, Z_2) = 0$ was singular: $\{z_1 = z_2 = 0\} = F$ removed



2. Replaced by $P_2 = \{x = 0\} \Longrightarrow$ smooth $\mathbf{CP}^1$
   well-describing non-singular origin $(Z_1, Z_2) = 0$.

cf. total space of $\mathbf{CP}^1$ with fiber $\mathcal{O}(-2)$. [Eguchi–Hanson, Gibbons–Hawking]

Subspaces “divisors”

Two types of subspaces, $(z_1, z_2, x) \to (\lambda z_1, \lambda z_2, \lambda^{-2} x)$

subspace	equation	topology	line bundle [de Rham]
ordinary D_1	$\{z_1 = 0\}$	$\mathbf{C}$	$[D_1]$
ordinary D_2	$\{z_2 = 0\}$	$\mathbf{C}$	$[D_2]$
exceptional E	$\{x = 0\}$	$\mathbf{CP}^1$	$[E]$

Cech: Transition function equivalently defines the line bundles on D_i: $\phi_{jk} = z_j/z_k$, on E: $\phi_{jk} = (z_j/z_k)^{-2}$.
Linear equivalence relation

$$-2D_1 \sim -2D_2 \sim E$$

The first chern class

$$\begin{aligned} c_1(T) &\equiv \sum \text{all the line bundles} \\ &= D_1 + D_2 + E \\ &\sim -\frac{1}{2}E - \frac{1}{2}E + E = 0 \end{aligned}$$

Calabi–Yau!

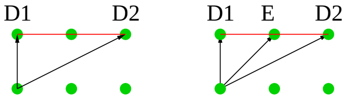
Toric geometry

$$Z_1 = z_1 x^{\frac{1}{2}}, Z_2 = z_2 x^{\frac{1}{2}}, (z_1, z_2, x) \to (\lambda z_1, \lambda z_2, \lambda^{-2} x)$$

Plot powers of scaling

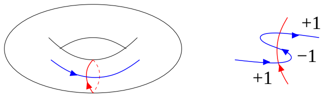
$$z_1 : (1,0), \quad z_2 : (0,1), \quad x : (\tfrac{1}{2}, \tfrac{1}{2})$$

$$D_1, \quad D_2, \quad E$$



det $e_i = 2$ (order).

- CY: points on the same line (plane)
- Smoothness: each cone spanning the entire integral lattice
- Inside point: compact divisor



- Minimal triangle: intersection number 1
  $D_1 E = 1, D_2 E = 1,$
  $E^2 = -2, D_1 D_2 = -\frac{1}{2}.$

Splitting principle

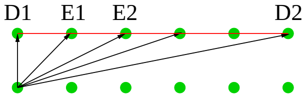
$$c(T) \equiv c_1(T) + c_2(T) + \cdots = \prod (1 + D_i) \prod (1 + E_a)$$

$$c_2(T) = D_1 E + D_2 E + D_1 D_2 = \tfrac{3}{2} \text{Euler \# of } \mathbf{C}^2/\mathbf{Z}_2$$

We can patch on 16 fixed points of $T^4/\mathbf{Z}_2$
$16 \times \frac{3}{2} = 24 = c_2(\text{K3})$

Various resolutions

Resolution of $\mathbf{C}_2/\mathbf{Z}_l$



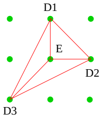
- Intersection numbers of $E_iE_j = \begin{pmatrix} -2 & 1 & & & \\ 1 & -2 & 1 & & \\ & 1 & -2 & 1 & \\ & & \ddots & & \\ & & & 1 & -2 \end{pmatrix} = (-1)\cdot$Cartan matrix
- McKay correspondence between A_{l-1} groups and $\mathbf{C}^2/\mathbf{Z}_l$ singularities.

Res.$\mathbf{C}_3/\mathbf{Z}_3$

$(z_1, z_2, z_3, x) \sim (\lambda z_1, \lambda z_2, \lambda z_3, \lambda^{-3} x)$

$D_1D_2E = D_2D_3E = D_3D_1E = 1$

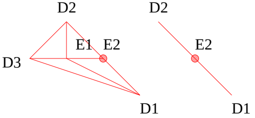
$D_1 \sim D_2 \sim D_3 \sim -\frac{1}{3}E$



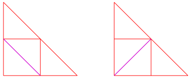
Res.$\mathbf{C}^3/\mathbf{Z}_4$

The exceptional divisor E_2 is noncompact

$\phi = (\frac{1}{4}\ \frac{1}{4}\ \frac{1}{2}), 2\phi = (\frac{1}{2}\ \frac{1}{2}\ 1)$



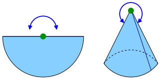
Res.$\mathbf{C}^3/(\mathbf{Z}_2 \times \mathbf{Z}_2)$: Two ‘flops’



Spectrum and matching

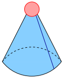
We can understand what happened after blowing-up, by investigating the change of the spectrum. [Candelas et al] [Groot Nibbelink et al]

- $\mathbf{C}^3/\mathbf{Z}_3$ orbifold model: obtained by CFT



background	$3v = V = (1\,1\,0\,0\,0\,0\,0\,0)(-2\,0\,0\,0\,0\,0\,0\,0)$
group	$E_7 \times SO(14) \times U(1) \times U(1)$
matter	$\frac{1}{9}[(\mathbf{56};\mathbf{1})_{2,2} + (\mathbf{1};\mathbf{1})_{-4,-4} + (\mathbf{1};\mathbf{64})_{-1,2} + (\mathbf{1};\mathbf{14})_{2,-4}]$ $+(\mathbf{1};\mathbf{14})_{2,0} + (\mathbf{1};\mathbf{1})_{-4,0} + 3(\mathbf{1};\mathbf{1})_{0,4}$
superpotential	$W = (\mathbf{1},\mathbf{1})_{-4,0}(\mathbf{1};\mathbf{14})_{2,0}(\mathbf{1};\mathbf{14})_{2,0} + \ldots$

- Resolution I: obtained by index theorem



background	$\mathcal{V} = (L^2, L^2, 0, 0, 0, 0, 0, 0)(L^2, 0, 0, 0, 0, 0, 0, 0)$
group	$E_7 \times SO(14) \times U(1)^2$
matter	$\frac{1}{9}[(\mathbf{56};\mathbf{1})_{2,2} + (\mathbf{1};\mathbf{1})_{-4,-4} + (\mathbf{1};\mathbf{64})_{-1,2} + (\mathbf{1};\mathbf{14})_{2,-4}] + 3(\mathbf{1};\mathbf{1})_{4,4}$

no renormalizable superpotential

  - modulus $\langle(\mathbf{1};\mathbf{1})_{-4,0}\rangle = v$ radius of the resolution space.
  - The Goldstone $T(x)$ becomes axion $(\mathbf{1};\mathbf{1})_{0,4} = e^{iT}(\mathbf{1};\mathbf{1})_{-4,4}$, $(\mathbf{1};\mathbf{14})_{2,0} = e^{-iT/2}(\mathbf{1};\mathbf{14})_{0,0}$
  - anomalous $U(1)$s
  - $(\mathbf{1};\mathbf{14})$ became massive and decoupled.

Other directions

$\mathbf{C}^3/\mathbf{Z}_3$ orbifold [Groot Nibbelink et al. 0802]

background	$3v = V = (1\,1\,0\,0\,0\,0\,0\,0)(-2\,0\,0\,0\,0\,0\,0\,0)$
group	$E_7 \times SO(14) \times U(1)^2$
matter	$\frac{1}{9}[(\mathbf{56};\mathbf{1})_{2,2} + (\mathbf{1};\mathbf{1})_{-4,-4} + (\mathbf{1};\mathbf{64})_{-1,2} + (\mathbf{1};\mathbf{14})_{2,-4}]$ $+(\mathbf{1};\mathbf{14})_{2,0} + (\mathbf{1};\mathbf{1})_{-4,0} + 3(\mathbf{1};\mathbf{1})_{0,4}$
superpotential	$W = (\mathbf{1},\mathbf{1})_{-4,0}(\mathbf{1};\mathbf{14})_{2,0}(\mathbf{1};\mathbf{14})_{2,0} + \ldots$

Resolution I

background	$(L^2, L^2, 0, 0, 0, 0, 0, 0)(L^2, 0, 0, 0, 0, 0, 0, 0)$
group	$E_7 \times SO(14) \times U(1) \times U(1)$
matter	$\frac{1}{9}[(\mathbf{56};\mathbf{1})_{2,2} + (\mathbf{1};\mathbf{1})_{-4,-4} + (\mathbf{1};\mathbf{64})_{-1,2} + (\mathbf{1};\mathbf{14})_{2,-4}] + 3(\mathbf{1};\mathbf{1})_{4,4}$
modulus	$\langle(\mathbf{1};\mathbf{1})_{-4,0}\rangle = v$
axion	$(\mathbf{1};\mathbf{1})_{0,4} = e^{iT}(\mathbf{1};\mathbf{1})_{-4,4},\ (\mathbf{1};\mathbf{14})_{2,0} = e^{-iT/2}(\mathbf{1};\mathbf{14})_{0,0}$

Resolution II

background	$(L, L, 0, 0, 0, 0, 0, 0)(L^3, L^1, 0, 0, 0, 0, 0, 0)$
group	$E_7 \times U(1) \times SO(12) \times U(1) \times U(1)$
matter	$\frac{1}{9}[(\mathbf{56};\mathbf{1})_{-1,2,-2} + (\mathbf{1};\mathbf{32})_{-1,2,2} + (\mathbf{1};\overline{\mathbf{32}})_{2,2,0} + (\mathbf{1};\mathbf{1})_{-4,-4,0}]$ $+\frac{1}{9}[(\mathbf{1};\mathbf{12})_{-1,-4,2} + (\mathbf{1};\mathbf{1})_{2,-4,\pm4}] + (\mathbf{1};\mathbf{12})_{3,0,-2} + 3(\mathbf{1};\mathbf{1})_{4,4,0}$
modulus	$\langle(\mathbf{1};\mathbf{14})_{2,0}\rangle = v$
axion	$(\mathbf{1};\mathbf{1})_{2,0,-4} = e^{-iT/2}(\mathbf{1};\mathbf{1})_{0,0,4}, \ldots$

No superpotential for $3(\mathbf{1};\mathbf{1})_{0,4}$: all the resolutions are found.

VEV of twisted matter fields = moduli of resolution: back-reactions

Anstropic resolution [KSC]

For one resolution, there are many models not admitting the orbifold limits.

model	V_1	V_2	gauge group spectrum	mod. inv.
a	$(2^6\ 0^{10})$	(0^{16})	$U(6)\times SO(20)$ $\frac{2}{9}(\mathbf{6},\mathbf{20})_2+\frac{14}{9}(\mathbf{15},\mathbf{1})_4$	no
b	$(1^8\ 0^8)$	$(2^8,0^8)$	$U(8)\times SO(16)$ $\frac{1}{9}(\mathbf{8},\mathbf{16})_{1+2}+\frac{4}{9}(\mathbf{28},\mathbf{1})_{2+4}$	yes
b′	$(1^8\ 0^8)$	$(-1^8,0^8)$	$U(8)\times SO(16)$ $\frac{1}{9}(\mathbf{8},\mathbf{16})_{1-1}+\frac{4}{9}(\mathbf{28},\mathbf{1})_{2-2}$	yes
c	$(1^2\ 2^6\ 0^8)$	$(2^2,1^6,0^8)$	$U(2)\times U(6)\times SO(16)$ $\frac{1}{9}(\mathbf{2},\mathbf{6},\mathbf{1})_{1,-1}+\frac{1}{9}(\mathbf{2},\mathbf{1},\mathbf{16})_{1,0}+\frac{1}{9}(\mathbf{1},\mathbf{6},\mathbf{16})_{0,1}+\frac{7}{9}(\mathbf{2},\mathbf{6},\mathbf{1})_{1,1}$ $+\frac{10}{9}(\mathbf{1},\mathbf{1},\mathbf{1})_{2,0}+\frac{10}{9}(\mathbf{1},\mathbf{15},\mathbf{1})_{0,2}$	no
d	$(1^8\ 0^{10})$	$(0^6,-1^2,2^3,0^5)$	$U(6)\times U(2)\times U(3)\times SO(10)$ $\frac{1}{9}(\mathbf{2},\mathbf{6},\mathbf{1})_{1,-1}+\frac{1}{9}(\mathbf{2},\mathbf{1},\mathbf{16})_{1,0}+\frac{1}{9}(\mathbf{1},\mathbf{6},\mathbf{16})_{0,1}+\frac{7}{9}(\mathbf{2},\mathbf{6},\mathbf{1})_{1,1}$ $+\frac{10}{9}(\mathbf{1},\mathbf{1},\mathbf{1})_{2,0}+\frac{10}{9}(\mathbf{1},\mathbf{15},\mathbf{1})_{0,2}$	yes

Table: Some typical A_3 resolution of $\mathbf{C}^2/\mathbf{Z}_3$ with anistropic flux support. V_i denotes the background flux supported on exceptional divisors E_i, $i=1,2$ respectively. Only the model no. 2 admits blowdown limit.

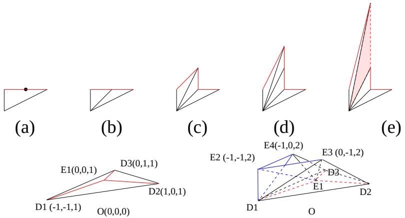
- Orbifold $\mathbf{C}^{2n}/\mathbf{Z}_l$ has one fixed points;
  Resolution has more than one divisors which can support the flux.

  ‘Wilson line’ inside the singularity.
- Bianchi identity similar to modular invariance, but contents are different.

Non-CY resolution [KSC]

Lesson: all the matter fields play the role of moduli.

- Resolution is not unique: infinitely many resolutions. cf. CY resolution unique.
- SUSY breaking direction?



- Different Euler number: topology change.
  :In fact, there were no well-defined topology for the orbifold.
  :String knows.
- Toric geometry valid
  :Divisor vectors are not on the same plane
- Anisotropic compactification
- Singularity cascade

Non-CY: spectrum

Index theorem still valid for fermions

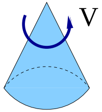
model	V	V'	gauge group matter spectrum	mod. inv.
a	$(0^{14}\ 1^2)$	$(0^{14}\ \text{-}1^2)$	$SO(28) \times U(2)$ $\frac{5}{8}(\mathbf{28},\mathbf{2})_{1-1} + \frac{23}{8}(\mathbf{1},\mathbf{1})_{2-2}$	yes
b	$(0^{13}\ 1^2\ 2)$	$(0^{13}\ 1^2\ 2)$	$SO(26) \times U(2) \times U(1)$ $\frac{1}{8}(\mathbf{26},\mathbf{2})_{1,0} + \frac{7}{8}(\mathbf{26},\mathbf{1})_{0,2} + \frac{1}{4}(\mathbf{1},\mathbf{2})_{1,2}$ $+\frac{5}{4}(\mathbf{1},\mathbf{2})_{1,-2} + \frac{3}{8}(\mathbf{1},\mathbf{1})_{2,0}$	yes
c	$(0^{12}\ 1^4)$	(0^{16})	$SO(24) \times U(4)$ $\frac{2}{8}(\mathbf{24},\mathbf{4})_1 + \frac{11}{8}(\mathbf{1},\mathbf{6})_2$	no
d	(0^{16})	$(0^4\ 1^{12})$	$SO(8) \times U(12)$ $0(\mathbf{8},\mathbf{12})_1 + \frac{3}{8}(\mathbf{1},\mathbf{66})_2$	no
e	$(0^{12}\ 1^4)$	$(0^{11}\ 1^2\ 0^3)$	$SO(22) \times U(1)^2 \times U(3)$ ...	no
f	$(1^2\ 0^{14})$	$(0^{10}\ 1^6)$	$U(2) \times SO(16) \times U(6)$ ...	yes

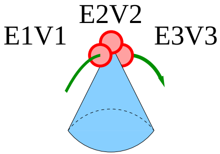
Table: Some non-Calabi–Yau resolutions of $\mathbf{C}^2/\mathbf{Z}_2$ with anistropic flux. V, V' denotes the background flux supported on exceptional divisors E, E' respectively.

1. Dynamic fluctuation of SUSY theory in SUSY breaking directions
2. Instability $\alpha' M^2 < 0$
   : Geometry tend to come back to SUSY.
3. Spontaneous symmetry breaking?
   : Non-CY Ground state is stable minimum.

Summary: difference between orbifold and CY

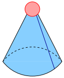
Calabi–Yau	orbifold
background flux supported by blown-up cycle $\frac{\mathcal{F}}{2\pi} = \sum E_i \mathcal{V}_i.$	nontrivial cycle encircling the singularity $\phi \to V$
line bundles	shift vector
$\mathcal{V} = (L^{V_1}, L^{V_2}, \ldots, L^{V_{16}})$	$(V_1, V_2, \ldots V_{16})$
unbroken group	
commutant to the structure group of $\mathcal{V}$	commutant to V modulo integer
consistency condition	
Bianchi identity of H $V^2 = 2\mathrm{ch}_2(\mathcal{M})$	modular invariance $V^2 = \phi^2$ modulo order
low energy spectrum	
$\mathrm{index}\,\not\nabla = \int \mathrm{ch}(V)\mathrm{Td}(T)$	CFT calculation $\alpha' M^2 = \ldots$





Phenomenology

We can obtain sufficiently realistic model using orbifold, as a zeroth order approximation.

- VEVs always give rise to resolution of the space.
  :Back-reaction of geometry
  $\langle \phi \rangle =$ 
- Vacuum configuration necessary
  :We give VEVs to singlet scalar fields to break symmetry and fit the phenomenological data.

Blowing-up inevitable.

- Some gauge group is broken by resolution and CY manifold.
  ex. $SU(3) \times E_6 \times E_8$ at the orbifold point $\longrightarrow E_6 \times E_8$
  We cannot make use of $SU(3)$ for SM gauge group.
  Orbifold as approximation.
- Interaction becomes different [Stromger Witten] [Strominger]:
  Calculated done at the orbifold point in the moduli space. [Hamidi Vafa] [Dixon et al] ... [KSC Kobayasahi]
  Perturbation valid for sufficiently small VEVs.

Conclusion

We can obtain semi-realistic model using toroidal orbifold.

- Some special points in the moduli space of internal manifold.
- We can explore nearby Calabi–Yau spaces by blowing-up.
- Toric geometry is useful for calculating topological quantitites.
- Some matter fields played the role of moduli fields.:
  Change of spectrum.

Future Direction

- Obtaining a realistic model
- Blowing up known models
- Fluxes $H = dB + \dots$
  :Warped geometry, different vacua [Strominger]
- Understanding CY more.
- What manifold/flux will give rise to our world?
  Who gave us CY manifold?
  :Dynamical explanation of compactification