Multiloop Calculations: towards R at Order $\mathcal{O}({\alpha_s}^4)$

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(talk based on results obtained by Karlsruhe group:
P. Baikov, K. Ch, J. H. Kühn /massless propagators/
+ Ch. Sturm /massive tadples/)

Phys. Rev. Lett. 88 (2002) 012001; Phys. Rev. D67 (2003) 074026
Phys. Letters. B559 (2003) 245
Nucl. Phys. Proc. Suppl. 135 (2004) 243-246
Eur. Phys. J. C 40, 261 (2005) Phys. Rev. Lett. 95, 012003 (2005)
+ few manuscripts in preparation

RADCOR 05

- Intro: α_s from LEP and GIGA-Z
- R(s): Theoretical Aspects
- New **Complete** Results at Order $\mathcal{O}({\alpha_s}^4)$
- summary and outlook

α_s from LEP from a review /S. Bethke (2004)/

$R_\ell = \Gamma_h/\Gamma_\ell = 20.767 \pm 0.025$ (1.2 ‰ !)

($\sim 10^6$ leptonic events)

$\alpha_s = 0.1226 \pm 0.0038 \;{}^{+0.0028}_{-0.0}(M_H = {}^{900}_{100}\text{GeV})$

$\sigma_\ell \sim \frac{\Gamma_\ell^2}{\Gamma_{tot}^2} = 2.003 \pm 0.0027\text{pb}$

(luminosity)

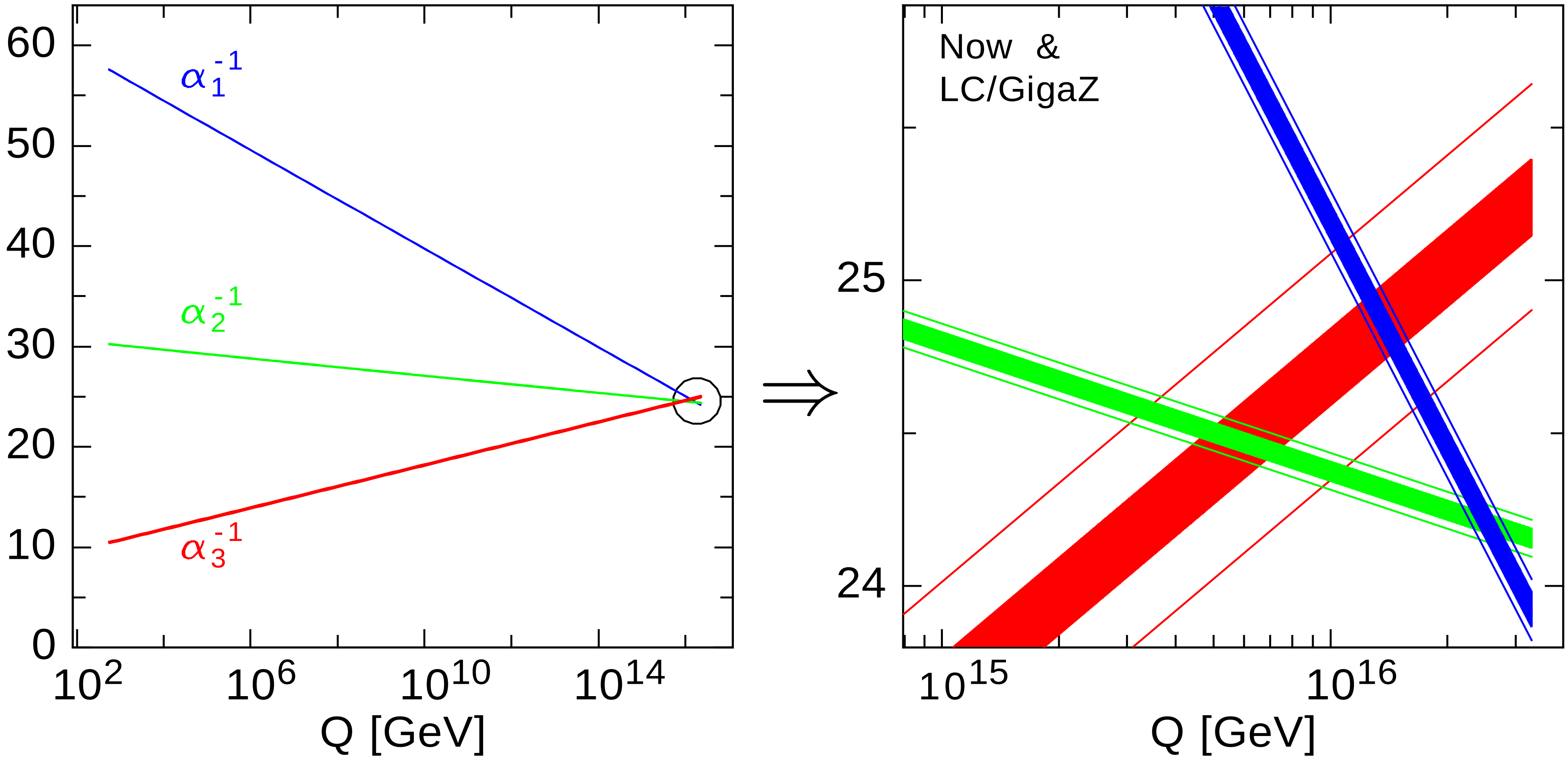
$\alpha_s = 0.1183 \pm 0.0030 \;{}^{+0.0022}_{-0.0}(M_H = {}^{900}_{100}GeV)$

SM-fit: $\alpha_s = 0.1188 \pm 0.0027$

based on $\sim 10^7$ Z-events/experiment

GIGA-Z: 10^9 events $\Longrightarrow \delta\alpha_s = 0.0009$ /M. Winter/

from Zerwas



$\alpha_s = 0.1183 \pm 0.0027$ vs ± 0.0009

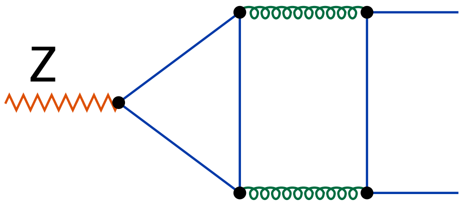
$\delta\alpha_s/\alpha_s = 2.3\,\%$ $\delta\alpha_s/\alpha_s = 0.8\,\%$

extraction of α_s from $\Gamma(Z \to \text{hadrons})$ based on

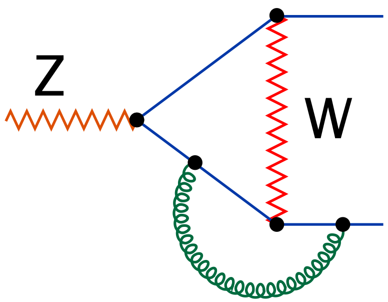
$$\Gamma_{\text{had}} = \Gamma_0 \left(R(s) \equiv 1 + \frac{\alpha_s}{\pi} + 1.409 \frac{\alpha_s^2}{\pi^2} - 12.767 \frac{\alpha_s^3}{\pi^3} + ? \frac{\alpha_s^4}{\pi^4} \right)$$

$+$ corrections:

$\Big(\sim m_b^2/M_z^2$ /known already to $\alpha_s{}^4$!; see below / $+$

singlet terms 

$+ \ldots$ /not decoupled top mass logs!, **known to $\alpha_s{}^3$ only!/** $+$

mixed QCD $\otimes$ electroweak 

$+ \ldots \Big)$

dominant theory error:
uncalculated higher orders! α_s^4 (massless diagrams!)
to have idea about their size: we use
optimization schemes (PMS,FAC): $-97\left(\frac{\alpha_s}{\pi}\right)^4$ (Kataev)

estimates for uncertainty:

- conservative: last calculated term (α_s^3)
  $\Longrightarrow \delta\alpha_s = 0.002; \quad \delta\alpha_s/\alpha_s = 1.7\%$
- "standard" (optimistic): estimated α_s^4 term
  $\Longrightarrow \delta\alpha_s = 0.0006; \quad \delta\alpha_s/\alpha_s = 0.5\%$
- scale variation: $\mu = \frac{1}{3}\sqrt{s} - 3\sqrt{s}$

$\delta\alpha_s = \begin{matrix} +0.002 \\ +0.00016 \end{matrix}$ $\delta\alpha_s/\alpha_s = 1.7 - 0.1\%$ (asymmetric!)

theory error

smaller than **present** experimental error (but not much!)

highly relevant for GIGA-Z

$\Longrightarrow$ calculation of α_s^4-term required for GIGA-Z

and even more so for $R_\tau = \Gamma(\tau \to \nu \text{ had})/\Gamma(\tau \to e\nu\nu)$

Technical Aspects

$$R(s) \approx \Im\,\Pi(s - i\delta)$$

$$3Q^2\Pi(q^2 = -Q^2) = \int e^{iqx}\langle 0|T[\ j^v_\mu(x) j^v_\mu(0)\ ]|0\rangle dx$$

- one needs only **pole** part (e.g. UV counterterm) of Π^B to get $R(s)$ and this is **MUCH** simpler to compute

$$\Pi^B(Q^2, \alpha_s^B) = \sum \frac{a_{ij}}{\epsilon^i}(\frac{\mu^2}{Q^2})^{\epsilon j} \ \ (D = 4 - 2\epsilon)$$

$$\frac{1}{\epsilon^n}(\frac{\mu^2}{Q^2})^{\epsilon n} \xrightarrow{\Uparrow} \mathrm{Ln}^n(\mu^2/Q^2) + \cdots \xrightarrow{\Im} \mathrm{Ln}^{n-1}(\mu^2/s) + \ldots$$

$$n = 0(\text{constant}) \xrightarrow{\Im} 0$$

The same is true for other correlators (scalar, tensor..) and also for massive corrections like

$\mathbf{m_q^2/s, m_q^4/s^2}$, etc. / J. Kühn, K.Ch (91,94)/

For Π at 5 loop

$$\frac{\partial}{\partial \log(Q^2)}\Pi = \gamma^{ph}(a_s) - \left(\beta(a_s)\frac{\partial}{\partial a_s}\right)\Pi$$

anom.dim. of the photon field; comes from 5-loop integrals: most complicated part of calculations

4-loop integrals only contribute due to the factor of $\beta(a_s)$

- to find Log-dependent part of Π at 5-loops one needs 5-loop anomalous dimension γ^{ph} and 4-loop Π

- (5) loop anom.dim. reducible to 4-loop massless propagators /K. Ch., Smirnov (1984)/;involved combinatorics resolved and automatized /K.Ch. 1997/

COMMON STRATEGY

1. reduce (with the use of the traditional IBP method) to master integrals

2. evaluate masters (better analytically)

COMMON PROBLEMS

1. IBP identities are *extremely* complicated at higher loops/legs

2. master integrals are difficult to perform analytically (numerical integration is possible but not simple: an art by itself)

5 ways to reduce a Feynman integral to Masters

- Empiric /sit and think/ way (/Mincer,Matad/); new: 3-loop splitting functions! /Moch,Vermaseren, Vogt (2004)/

- Arithmetic way: direct solution of /thousands or even millions!/ IBP eqs. /Laporta, Remiddi (96); Gehrmann, Schröder, Anastasiou, Czakon, Melnikov, Czarnecky, Bonciani, Mastrolia, Sturm ...

- Gröbner Basis Technique /Tarasov (98), Gerdt, very new: Smirnov & Smirnov (wait for his talk!)/

- New Representation for CF's /Baikov (96), Steinhauser, Smirnov ... /

$$\Downarrow$$

- $1/D$ expansion of CF's /Baikov (98-04) /

Feynman parameters: New parameters:

$$\int d\ \alpha\ \mathrm{e}^{i\alpha(m^2-p^2)} \approx \frac{1}{m^2-p^2} \approx \int\ \frac{d\ x}{x}\ \delta(x-(m^2-p^2))$$

Now for a given topology one can make loop integrations once and forever with the result:

Baikov's Representation:

$$F(\underline{n}) \ \sim \ \int \cdots \int \frac{\mathbf{d}x_1 \ldots \mathbf{d}x_N}{x_1^{n_1} \ldots x_N^{n_N}} \left[P(\underline{x})\right]^{(D-h-1)/2} ,$$

where $P(\underline{x})$ is a polynomial on $x_1, \ldots, x_N$ (and masses and external momenta)

New representation obviously meets the same set IBP'id as the original integral but it has much more flexibility! (Due to choice of the integration contours)

MAIN IDEA: **to use (1) as a "template" for the very CF's!**

Reduction to Masters: $1/D$ expansion[1]

- coefficient functions in front of *master integrals* depend on D in simple way:

$$C^{\alpha}(D) = \frac{P^n(D)}{Q^m(D)} \underset{D\to\infty}{=\!=\!=} \sum_k C_k^{\alpha} \ (1/D)^k$$

- The terms in the $\mathbf{1/D}$ expansion expressible (with the use of the Baikov's representation) through simple Gaussian integrals

- sufficiently many terms in $1/D$ and $C_k^{\alpha} \longrightarrow C^{\alpha}(D)$

- computing time and required resources: could be huge (the price for full automatization); to cope with it we use parallel FORM /Vermaseren, Retey, Fliegner, Tentyukov, ...(2000 – . . .)

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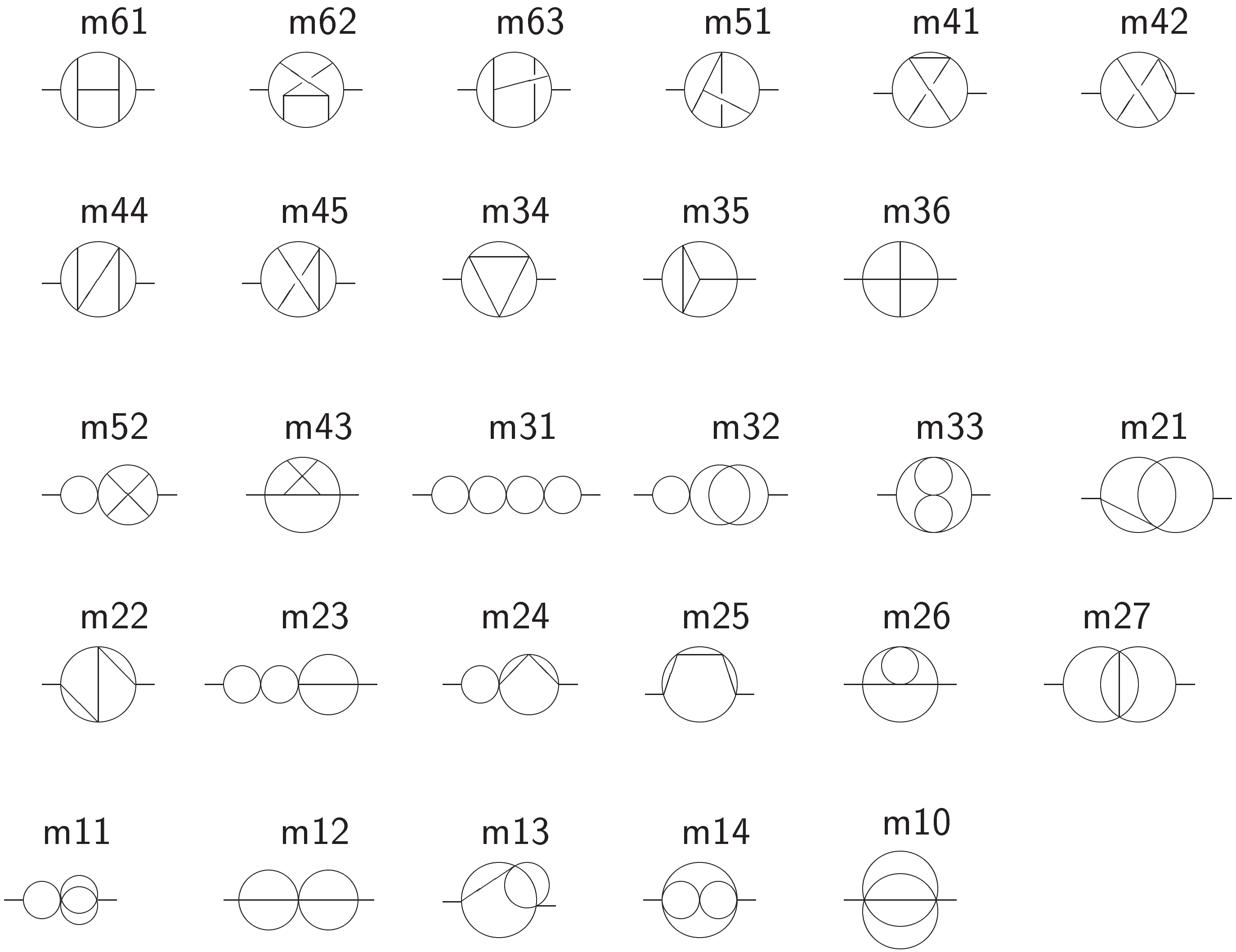
[1] Baikov, Phys. Lett. B385 (1996) 403; B474 (2000) 385; Nucl.Phys.Proc.Suppl.116:378-381,2003

NEW All Master Integrals solved analytically (2004)

(method: “glue and cut” (Chetyrkin, Tkachov /1981/)) + reduction to masters



Summary of Technical Aspects

4-loop propagators (including real parts) and the absorpive parts of 5-loop ones can *in principle* be *analytically* computed in any massless theory*

in QCD it means the correlators to $\mathcal{O}(\alpha_s^4)$ /5-loops!/

* in practice: severe constraints come from the finitnes of computer resourses, see below

List of Recent Results*

1: the first complete five-loop result in QCD: $\mathcal{O}({\alpha_s}^4 m_q^2/s)$ contribution to $R(s)$

2: $\mathcal{O}({\alpha_s}^3 m_s^2/s)$ contribution (including real part) to the charged current correlator $\longrightarrow$ full $\mathcal{O}({\alpha_s}^3 m_s^2/M_\tau^2)$ contribution to the semileptonic decay rate of τ-lepton

3: full $\mathcal{O}({\alpha_s}^4)$ contribution to $R^S(s)$ $\longleftarrow$ / $R(s)$ for the scalar quark currents /

4: full 4-loop contribution to the QCD decoupling relation

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* **1–3:** Baikov, Chetyrkin, Kühn, 2004 – 2005; **4:** BChK + Sturm, 2005

OUR RESULTS ($N_F = 3$)

The most demanding five-loop anomalous dimension of the SS-correlator:

$$\gamma_q^S = -6 + a_s - 10a_s^2\left[-\frac{383}{12} + 3\,\zeta_3\right] + a_s^3\left[-\frac{122935}{864} + \frac{2365}{36}\zeta_3 + \frac{21}{4}\zeta_4 - \frac{65}{2}\zeta_5\right]$$

$$+ \; a_s^4\left[-\frac{39620177}{41472} + \frac{1288967}{3456}\zeta_3 - \frac{29425}{288}\zeta_3^2 - \frac{44971}{384}\zeta_4 - \frac{124045}{576}\zeta_5 + \frac{16375}{64}\zeta_6 + \frac{97895}{384}\zeta_7\right]$$

Note the structure of the α_s^4 term: $\zeta(6)$ and $\zeta(7)$ are new irrationalities which appear at five loop level

Some Details about Complexity of the Calculation

- CPU-time consumption (very approximately) $3 \cdot 10^8$ sec (about 10 years) for a 1.5GH PC
- Due to the heavy use of the SGI cluster (of 32 parallel SMP CPU of 1.5 GH frequency each) the calculation took about 15 calendar months
- Vector case: optimistically vector is 3 times more complicated, pessimistically 10 times

Define "Adler function" as ($a_s \equiv \alpha_s/\pi$)

$$D^S(Q^2) = \int_0^\infty \frac{Q^2 \; R^S(s) ds}{(s+Q^2)^2} \quad \Longleftrightarrow \quad R^S(s) = 1 + \frac{17}{3}\alpha_s/\pi + \dots$$

$$D^S(Q^2) = 1 + \sum_{i=1}^{\infty} d_i \; (a_s(Q))^i \quad \Longleftrightarrow \quad R^S(s) = 1 + \sum_{i=1}^{\infty} r_i \; (a_s(s))^i$$

Our results (for $N_f = 3$) read:

$$D^S = 1 + 5.666 \; a_s + 45.846 \; a_s^2 + 465.85 \; a_s^3 + 5588.7 \; a_s^3$$

$$R^S = 1 + 5.667 \; a_s + 31.86 \; a_s^2 + 89.16 \; a_s^3 - 536.84 \; a_s^4$$

Effects of analytic continuation* inside of R^S:

$$1 + 5.6 \; a_s \; + \; a_s^2 \; (46 - 14) + \; a_s^3 \; (466 - 377) + \; a_s^4 \; (5589 - 6126)$$

* come from running + one-loop-less coefficients, basically are trivial

Comparison to PMS[1] |APAP[2] |NNA[3] predictions

Exact: $d_4 = 5588.7$ and $r_4 = -536.84$

$$d_4(\text{FAC/PMS}) = 5180 \longrightarrow r_4(\text{FAC/PMS}) = -945.28$$

$$r_4(\text{direct application of FAC/PMS in Minkowskian region}) = -528$$

$$d_4(\text{APM}) = 6214 \longleftarrow r_4(\text{APM}) = 195$$

$$d_4(\text{NNA}) = 1116 \longrightarrow r_4(\text{NNA}) = -5009$$

[1] Principle of Minimal Sensitivity (PMS): K.Ch., Kniehl, Sirlin, PRB 402 (1997) 359

[2] Asymptotic Padé-Approximant Method (APAM): Chishtie, Elias, Steele, PRD 59 (1999) 105013

[3] ‘Naive NoNabelization (NNA): Grosin,Broadhurst, PRD 52 (1995) 4082

BLM prediction* for R^S

$\mathcal{O}(\alpha_s^3)$, exact: $164.1 - 25.77\, n_f + 0.259\, n_f^2$

BLM: $249 - 24\, n_f + 0.33\, n_f^2$

$\mathcal{O}(\alpha_s^4)$, exact: $39.34 - 220.9\, n_f + 9.685\, n_f^2 - 0.02046\, n_f^3$

BLM: $340 - 260\, n_f + 13\, n_f^2 - 0.046\, n_f^3$

⋆ "We thus estimate that n_f dependent terms above should be correct to about 30%"

⋆ private communication from S.Brodsky and M.Binger (they have not been informed about numerical details of our result; only about its existence)

Application: to Higgs Decay into b quarks

$$\Gamma(H \to \bar{f} f) = \frac{G_F\, M_H}{4\sqrt{2}\pi} m_f^2 R^S(s = M_H^2)$$

$$\begin{aligned} R^S &= 1 + 5.66677\, a_s + 29.147\, a_s^2 + 41.758\, a_s^3 - 825.7\, a_s^4 \\ &= 1 + 0.2075 + 0.0391 + 0.0020 - 0.00148 \end{aligned}$$

where in the second equation we set $a_s = \alpha_s/\pi = 0.0366$ **which corresponds the Higgs mass value** $M_H = 120$ **GeV. The comparable sizes of the** $\mathcal{O}(a_s^3)$ **and the** $\mathcal{O}(a_s^4)$ **terms can be naturally interpreted as a consequence of the accidentally small coefficient for the** a_s^3 **term.**

(preliminary!) result of the analysis *

of the QCD sum rule for the pseudosccalr correlator

$$\mathbf{m}_s(\mathbf{2\ GeV}) = \mathbf{90} \pm \mathbf{5}\Big|_{\mathbf{param}} \pm \mathbf{5}\Big|_{\mathbf{nonp}} \pm \mathbf{5}\Big|_{hadr}$$

$param$: Λ_{QCD}, scale, condensates

$nonpt$: instanton correction

$hadr$: hadronic input

important: new $\mathcal{O}({\alpha_s}^4)$ term amounts to a negligible

$\leq 1\%$ shift of m_s $\Longrightarrow$ good PT stability!

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*preliminary results from K.Ch., A. Khodjamirian (2005)

Application: Quark Mass Bounds

(Lellouch, de Rafael, Taron, 1997)

positivity of the spectral function + known values of K pole contribution imply "rigorous" bounds on the the value of $(m_s + m_u)$

$$[m_s(Q) + m_u(Q)]^2 \geq \frac{16\pi^2}{N_c} \frac{2 f_K^2 M_K^4}{Q^4} \frac{1}{\left(1 + \frac{M_K^2}{Q^2}\right)^3} \frac{1}{\Pi^{(5)}(Q)}$$

For $Q = 1.4$ GeV we get:

$$[(m_s + m_u)](\mu = 2\,\text{GeV}) > 87 \text{ MeV } (\alpha_s^3) \text{ and } 86 \text{ MeV } (\alpha_s^4)$$

good convergency (?) not really: $\frac{1}{\mathbf{\Pi^{(5)}(Q)}} \sim$

$$\left\{1 - 3.67\,a_s - 0.73\,a_s^2 + (54.7 - \underline{77.4})\,a_s^3 + (-190.1 + \underline{567.4} - \boxed{511.8})\,a_s^4\right\}$$

$\Downarrow$

$$[(m_s + m_u)](\mu = 2\,\text{GeV}) > 77 \text{ MeV } (\alpha_s^3) \text{ and } 74 \text{ MeV } (\alpha_s^4)$$

QCD Decoupling Relation in Four Loops

$$n_f = n_\ell + 1, \quad a_s^{(n_\ell)}(\mu) = a_s^{(n_f)}(\mu)\, d(a_s, \mu/m_h)$$

- is used to evolve the $\alpha_s(M_Z)$ to $\alpha_s(M_\tau)$
- the matching function d is known[1] to $\mathcal{O}(\alpha_s^3)$
- technically it is expressible completely through vacuum integrals[2]
- computed using "Laporta" algorithm

[1,2]K. Ch. Kniehl, Steinhauser (97)

New Four Loop $\mathcal{O}(\alpha_s^4)$ Result[1]

$$d(a_s, \mu = m_h) = 1 + 11/72\, a_s^2 + (0.9721 - 0.0847\, n_\ell)\, a_s^3$$

$$+ \quad (5.1703 - 1.001\, n_\ell - 0.0220\, n_\ell^2)\, a_s^4$$

Phenomenology : the new $\mathcal{O}(\alpha_s^4)$ term has almost no effect on resulting $\alpha_s(M_\tau)$ but halves the errors due to the choice of the matching scale and the truncation of the matching function

Interesting: it checks in an independent way all n_f-dependent pieces of the QCD 4-loop β-function!

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[1] K.Ch., Kühn and Sturm (05); (partially confirmed by very recent independent calculation by Schrëder and Steinhauser (05))

SUMMARY

α_s^4 term in $R(s)$ needed for GIGA-Z (and τ decays)

- systematic reduction to master integrals $+$
- Master integrals solved$+$
- ImΠ^{SS} available (H$\longrightarrow b\bar{b}$)$+$
- Mass terms of $\mathcal{O}\left(\alpha_s^4\right)$ available$+$
- $\mathcal{O}\left(\alpha_s^4\right)$ matching function is available$+$
- Im$\Pi^{VV} = R^V$ within reach$+$(?)
- Huge demands on computing

Other possible applications:

- Coefficient functions in OPE at 4-loops (DIS!) and anomalous dimensions at 5-loops /only for few lowest moments/
- QCD running of α_s and m_q at 5-loops
- QCD sum rules

⋮

CHRONOLOGY and FUTURE TIMING of R(s)

1-loop: 1 diagram /BC?/

2-loop: 3 diagrams /1951/

3-loop: 37 diagrams /1979/ (completely by hand)

4-loop: 738 diagrams /1991/ (the first semi-manual calculation /correcrt from the second try only/)
1997 (the first completely automatic calculation)/

$\Downarrow$

5-loop: **19832!** diagrams

1991+ 1997 - 1979 = 2008(?) $\longleftarrow$ looks reasonable

Important "Dots":

- IRR works only for log-divergent integrals, while Π^{jj} is in general quadratically divergent
- The only way to proceede is to use (double!) differentiation w.r.t. he external momentum q to decrease the dimension
- This leas to "dots" $\equiv$ squared propagators which immensely complicates all calcualtions

An important and non-trivial simplification exists for the SS correlator due to the well-known Ward identity:

$$q_\mu q_\nu \Pi^{\mathrm{V/A}}_{\mu\nu,ij}(q) = (m_i \mp m_j)^2 \Pi^{\mathrm{S/P}}_{ij}(q) + (m_i \mp m_j)(\langle \overline{\psi}_{\mathrm{i}} \psi_{\mathrm{i}} \rangle \mp \langle \overline{\psi}_{\mathrm{j}} \psi_{\mathrm{j}} \rangle)$$

basically it means that $\mathcal{O}(m_q^2)$ part of the longitudinal part of VV corelator identical to the massless SS one. This has allowed us to compute the $\mathcal{O}(m_q^2)$ part of the VV correlator instead of the massless SS which resulted to diagram with ONE SQUARED PROPAGATOR LESS and saved us a lot of work!