Techniques for one-loop tensor integrals in many-particle processes

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Contents

1 Introduction

Many-particle ($\# = 5, 6, \ldots$) processes are important at LHC and ILC,
but only very few processes accurately known:

- $2 \to 3$ processes

  $\mathrm{e^+e^-} \to \nu\bar{\nu}\mathrm{H}, \mathrm{t\bar{t}H}, \mathrm{e\bar{e}H}, \mathrm{ZHH}, \nu\bar{\nu}\gamma, \quad \gamma\gamma \to \mathrm{t\bar{t}H}$

  NLO EW/QCD: GRACE-loop, Denner et al., You et al., Chen et al., Zhang et al. '02–'04

  $\mathrm{pp} \to$ 3jets, $\gamma\gamma$+jet, V+2jets, $\mathrm{t\bar{t}H}$, $\mathrm{b\bar{b}H}$, $\mathrm{b\bar{b}V}$

  NLO QCD: Bern et al., Kunszt et al., Kilgore/Giele, Campbell et al., Nagy, Del Duca et al., Campbell/Ellis, Beenakker et al., Dawson et al., S.D. et al. '96–'05

- $1 \to 4$ processes

  $\mathrm{H} \to 4$ fermions: NLO QED $\hookrightarrow$ talk of C. Carloni-Calame; NLO EW $\hookrightarrow$ talk of A. Bredenstein

- $2 \to 4$ processes

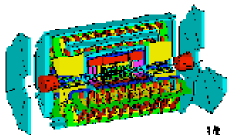
  $\mathrm{e^+e^-} \to \nu\bar{\nu}\mathrm{HH}$: preliminary NLO EW GRACE-loop '04/'05

  $\hookrightarrow$ talk of Y. Kurihara

  $\mathrm{e^+e^-} \to 4$ fermions (CC): complete NLO EW Denner, S.D., Roth, Wieders, '05

  $\hookrightarrow$ talk of A. Denner

Experimenters' wish list for the LHC: (J.Huston, Les Houches 2005)



Experimental priority list

- Note have to specify how inclusive final state is
  - what cuts will be made?
  - how important is b mass for the observables?
- How uncertain is the final state?
  - what does scale uncertainty look like at tree level?
  - new processes coming in at NLO?
- Some information may be available from current processes
  - pp->tT j may tell us something about pp->tTbB?
    - j=g->bB
  - CKKW may tell us something about higher multiplicity final states

1. pp->WW jet
2. pp->tT bB
   1. background to tTH
3. pp->tT + 2 jets
   1. background to tTH
4. pp->WWbB
5. pp->V V + 2 jets
   1. background to WW->H->WW
6. pp->V + 3 jets
   1. beneral background to new physics
7. pp->V V V
   1. background to SUSY trilepton

Beyond the SM Workshop at Columbia

Complications in corrections to many-particle processes

- huge amount of algebra, long final expressions
  - $\hookrightarrow$ computer algebra / automization
- multi-dimensional phase-space integration
  - $\hookrightarrow$ Monte Carlo techniques
- complicated structure of singularities and matching of virtual and real corrections
  - $\hookrightarrow$ subtraction and slicing techniques
- treatment of unstable particles, issue of complex masses
  - $\hookrightarrow$ "complex-mass scheme" recently proposed for higher orders

    Denner, S.D., Roth, Wieders, '05
- numerically stable evaluation of one-loop integrals with up to 5,6,... external legs
  - $\hookrightarrow$ subject of this talk

Some comments on one-loop techniques:

- Reduction techniques: tensors $\to$ scalar integrals = basis integrals

  proposed by Brown/Feynman '52

  systematically worked out by Passarino/Veltman '79

  subsequently modified by many authors

  Stuart et al. '88/'90; v.Oldenborgh/Vermaseren '90; Ezawa et al. '92; Denner '93; Campbell et al. 96; Devaraj/Stuart '97; GRACE-loop '03; del Aguila/Pittau '04; v.Hameren et al. '05; Denner/S.D. '05

- Reduction techniques: tensors $\to$ integrals in $D \neq 4$ dimensions

  proposed by Davydychev '91 and further developed by others

  Bern et al. '93; Tarasov '96; Fleischer et al. '99; Binoth et al. '99/'05; Duplančić/Nižić '03; Giele et al. '04; R.K.Ellis et al. '05

  master integrals for massive case missing

- Reductions for $N \geq 5$ using $D \to 4$

  proposed by Melrose '65 for scalar integrals

  subsequently generalized to tensors by other authors

  v.Neerven/Vermaseren '84; v.Oldenborgh/Vermaseren '90; Campbell et al. 96; Davydychev '91; Bern et al. '93; Denner '93; Suzuki et al. '02; Tramontano '02; Denner/S.D. '02/'05; GRACE-loop '02/'03

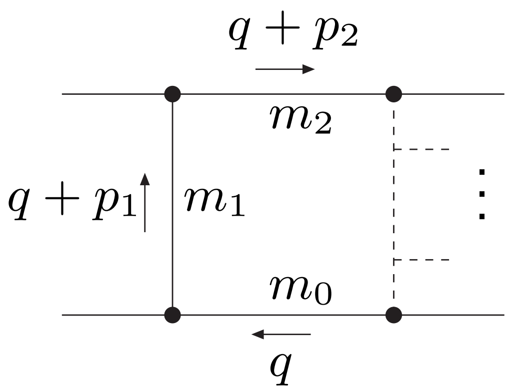
- Numerical techniques

  various proposal by several authors Ferroglia et al. '02; Binoth et al. '02/'05; Nagy/Soper '03 de Doncker et al. 04; Kurihara/Kaneko '05; Denner/S.D. '05

  but not yet successfully applied to complicated physical processes

2 Preliminaries and Passarino–Veltman reduction

General N-point one-loop tensor integrals of rank P



N denominator factors $1/N_k$:

$$N_k = (q+p_k)^2 - m_k^2 + \mathrm{i}\epsilon, \quad p_0 = 0$$
$$k = 0, \dots, N-1$$

Integral definition:

$$T^{N,\mu_1\dots\mu_P} = \frac{(2\pi\mu)^{4-D}}{\mathrm{i}\pi^2} \int \mathrm{d}^D q \, \frac{q^{\mu_1}\cdots q^{\mu_P}}{N_0 N_1 \dots N_{N-1}}$$

($A \equiv T^1$, $B \equiv T^2$, $C \equiv T^3$, etc.)

Decomposition of tensor integral into covariants:

$$T^{N,\mu_1\dots\mu_P} = \sum_{i_1,\dots,i_P=1}^{N-1} p_{i_1}^{\mu_1}\dots p_{i_P}^{\mu_P} T^N_{i_1\dots i_P} + \sum_{i_3,\dots,i_P=1}^{N-1} \{g^{\mu_1\mu_2} p_{i_3}^{\mu_3}\dots p_{i_P}^{\mu_P} + \dots\} T^N_{00 i_3\dots i_P}$$
$$+ \sum_{i_5,\dots,i_P=1}^{N-1} \{ggp\dots p\}^{\mu_1\dots\mu_P}_{i_5\dots i_P} T^N_{0000 i_5\dots i_P} + \dots$$

$\hookrightarrow$ Aim: calculate tensor coefficients $T^N_{i_1\dots i_P}$, etc. in terms of few basis integrals

Basic relations among coefficients upon “contraction and cancellation”:

(i) Contraction with momenta

$$2p_k q = \underbrace{\left[(q+p_k)^2 - m_k^2\right]}_{=N_k} - \underbrace{\left[q^2 - m_0^2\right]}_{=N_0} - \underbrace{\left[p_k^2 - m_k^2 + m_0^2\right]}_{\equiv f_k}$$

$\hookrightarrow$ relation between tensors [“(k)” means propagator denominator N_k omitted]

$$2p_k^{\mu_1} T^N_{\mu_1\ldots\mu_P} = T^{N-1}_{\mu_2\ldots\mu_P}(k) - T^{N-1}_{\mu_2\ldots\mu_P}(0) - f_k T^N_{\mu_2\ldots\mu_P}$$

$\hookrightarrow$ relation between coefficients [“$\hat{i}$” means index i omitted]

$$\sum_{m=1}^{N-1} 2(p_k p_m) T^N_{mi_2\ldots i_P} + 2\sum_{r=2}^{P} \delta_{ki_r} T^N_{00i_2\ldots\hat{i}_r\ldots i_P} + f_k T^N_{i_2\ldots i_P} \;=\; (T^{N-1} \text{ terms})$$

(ii) Contraction with metric tensor

$$q^2 = \underbrace{\left[q^2 - m_0^2\right]}_{=N_0} + m_0^2$$

$\hookrightarrow$ relation between tensors

$$g^{\mu_1\mu_2} T^N_{\mu_1\mu_2\ldots\mu_P} = T^{N-1}_{\mu_3\ldots\mu_P}(0) + m_0^2 T^N_{\mu_3\ldots\mu_P}$$

$\hookrightarrow$ relation between coefficients

$$\sum_{n,m=1}^{N-1} 2(p_n p_m) T^N_{nmi_3\ldots i_P} + \text{const.} \times T^N_{00i_3\ldots i_P} - 2m_0^2 T^N_{i_3\ldots i_P} \;=\; (T^{N-1} \text{ terms})$$

Passarino–Veltman reduction

Basic relations yield recursive solution for tensor coefficients:

$$T^N_{00i_3\ldots i_P} \propto 2m_0^2 \underbrace{T^N_{i_3\ldots i_P}}_{\text{rank } P-2} + \sum_{n=1}^{N-1} f_n \underbrace{T^N_{ni_3\ldots i_P}}_{\text{rank } P-1} + (T^{N-1} \text{ terms}),$$

$$T^N_{i_1\ldots i_P} = \sum_{n=1}^{N-1} (Z^{-1})_{i_1 n} \left[-f_n \underbrace{T^N_{i_2\ldots i_P}}_{\text{rank } P-1} -2 \sum_{r=2}^{P} \delta_{ni_r} \underbrace{T^N_{00i_2\ldots\hat{i}_r\ldots i_P}}_{\text{rank } P-1} + (T^{N-1} \text{ terms}) \right], \qquad i_1 \neq 0$$

$\hookrightarrow$ recursive calculation of $T^N_{i_1\ldots i_P}$ from scalar integral T^N_0 and $T^{N-1}_{i_2\ldots i_P}$:

$$T^N_0 = \text{basis integral} \quad \to \quad T^N_{i_1} \quad \to \quad T^N_{i_1 i_2} \quad \to \quad T^N_{i_1 i_2 i_3} \quad \to \quad \ldots$$

But: relations involve inverse Z^{-1} of Gram matrix $Z = \begin{pmatrix} 2p_1p_1 & \ldots & 2p_1p_N \\ \vdots & \ddots & \vdots \\ 2p_Np_1 & \ldots & 2p_Np_N \end{pmatrix}$

$\hookrightarrow$ potential instabilities for $\det(Z) \to 0$

Example: 4-point tensor integrals

scalar integral $= D_0$

$$D^{\mu} = \sum_{i=1}^{3} p_i^{\mu} D_i$$

$$D^{\mu\nu} = \sum_{i,j=1}^{3} p_i^{\mu} p_j^{\nu} D_{ij} + g^{\mu\nu} D_{00}$$

$$D^{\mu\nu\rho} = \sum_{i,j,k=1}^{3} p_i^{\mu} p_j^{\nu} p_k^{\rho} D_{ijk} + \sum_{i=1}^{3} \{g^{\mu\nu} p_i^{\rho} + \cdots\} D_{00i}$$

$$D^{\mu\nu\rho\sigma} = \sum_{i,j,k,l=1}^{3} p_i^{\mu} p_j^{\nu} p_k^{\rho} p_l^{\sigma} D_{ijkl} + \sum_{i,j=1}^{3} \{g^{\mu\nu} p_i^{\rho} p_j^{\sigma} + \cdots\} D_{00ij} + \{g^{\mu\nu} g^{\rho\sigma} + \cdots\} D_{0000}$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

PV reduction: **step 0**

$\boxed{D_0}$ basis integral

D_i

D_{ij} D_{00}

D_{ijk} D_{00i}

D_{ijkl} D_{00ij} D_{0000}

$\vdots$ $\vdots$ $\vdots$

PV reduction: **step 1**

D_0

$\downarrow$

$\boxed{D_i}$

D_{ij} D_{00}

D_{ijk} D_{00i}

D_{ijkl} D_{00ij} D_{0000}

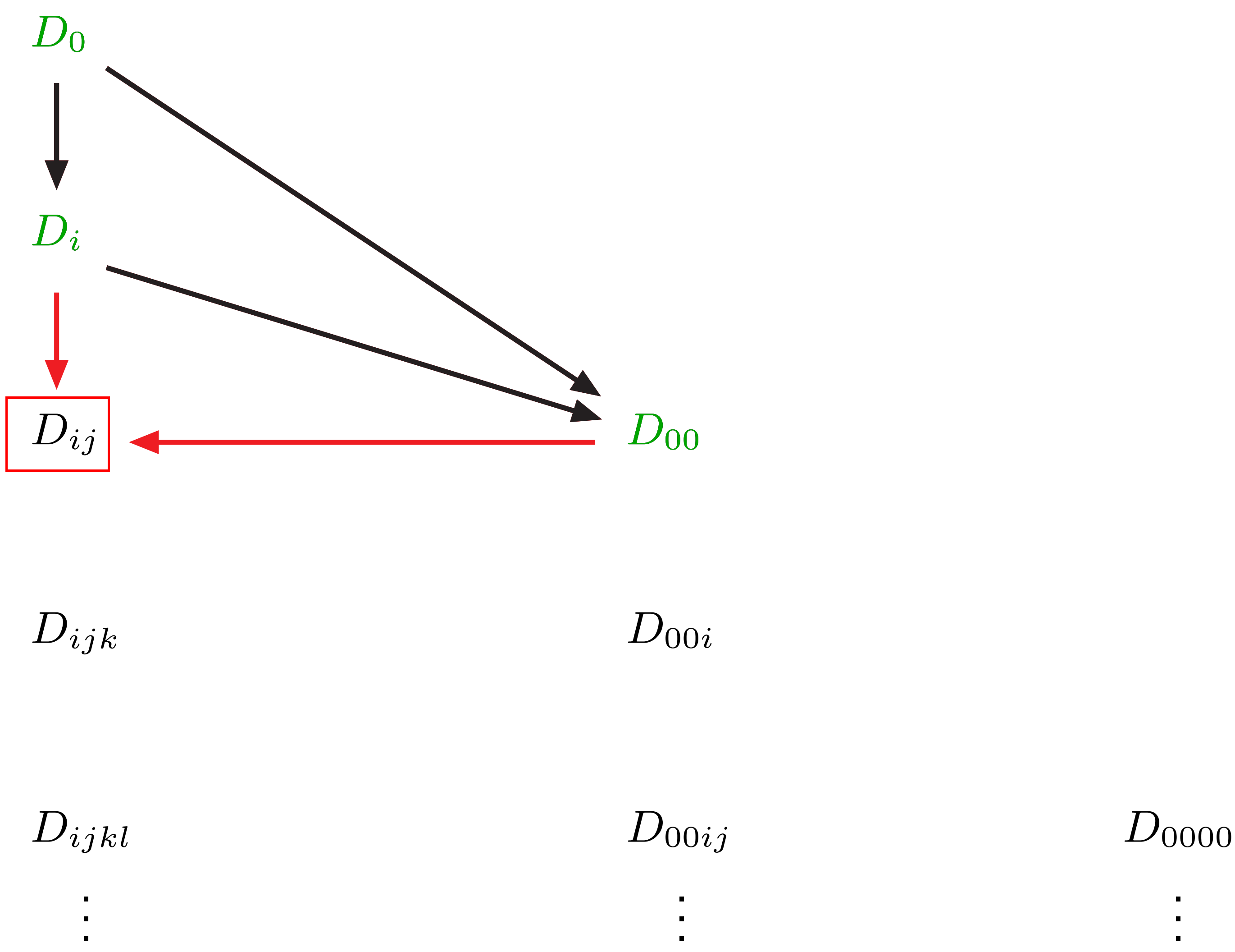
$\vdots$ $\vdots$ $\vdots$

PV reduction:

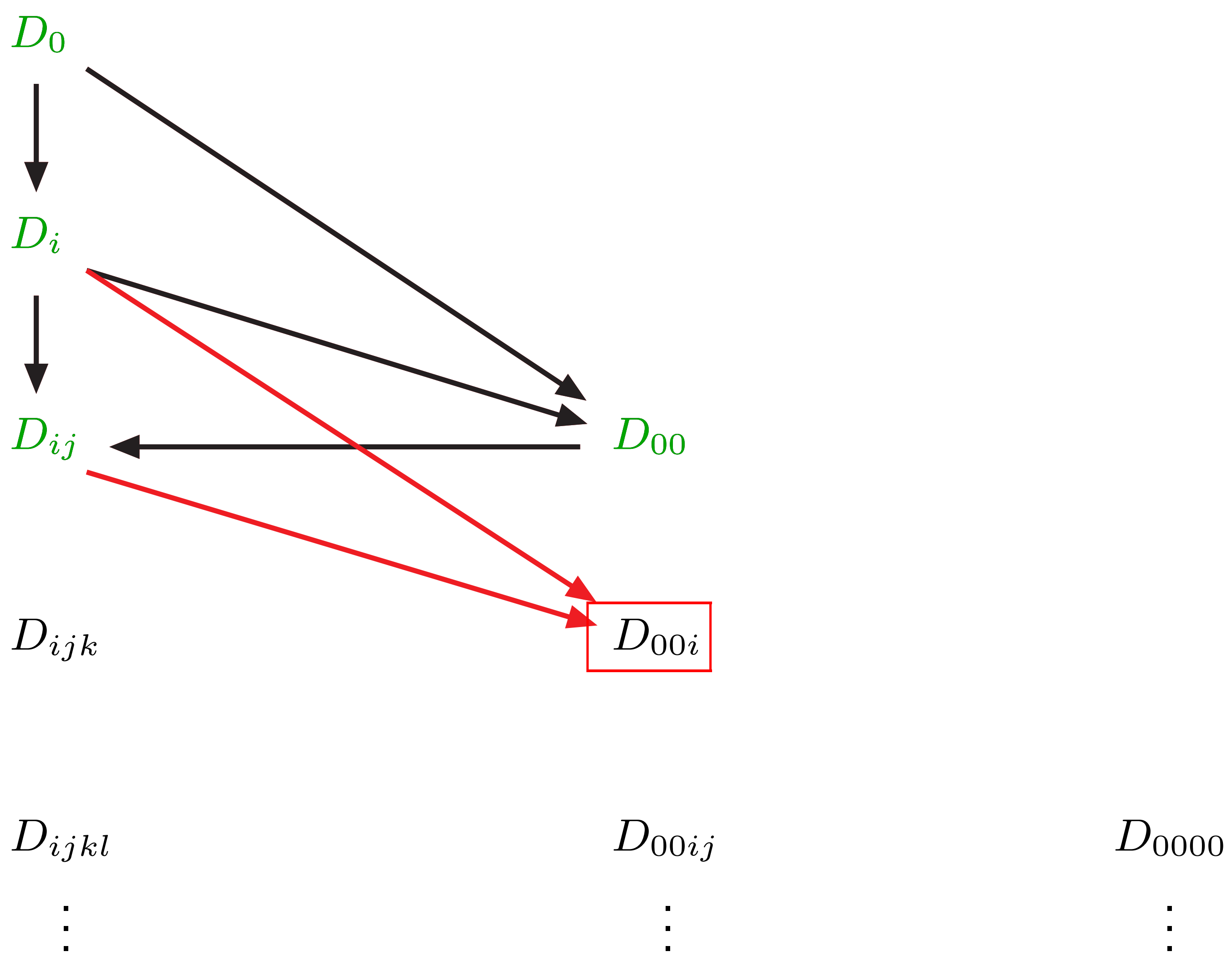
step 2a

$$\begin{tikzcd}
D_0 \arrow[d] \arrow[ddr] & & \\
D_i \arrow[dr] & & \\
D_{ij} & \boxed{D_{00}} & \\
D_{ijk} & D_{00i} & \\
D_{ijkl} & D_{00ij} & D_{0000} \\
\vdots & \vdots & \vdots
\end{tikzcd}$$

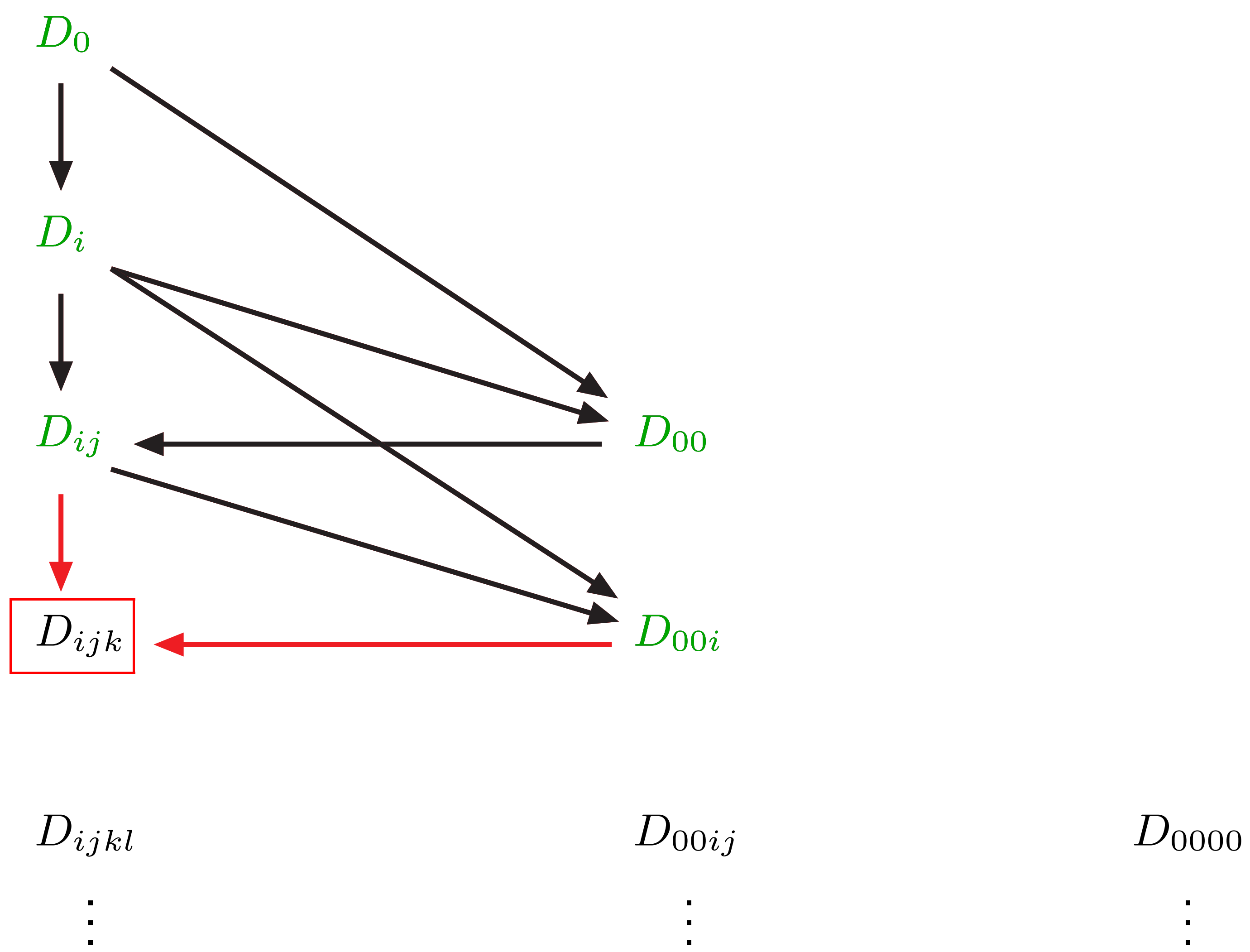
PV reduction: **step 2b**



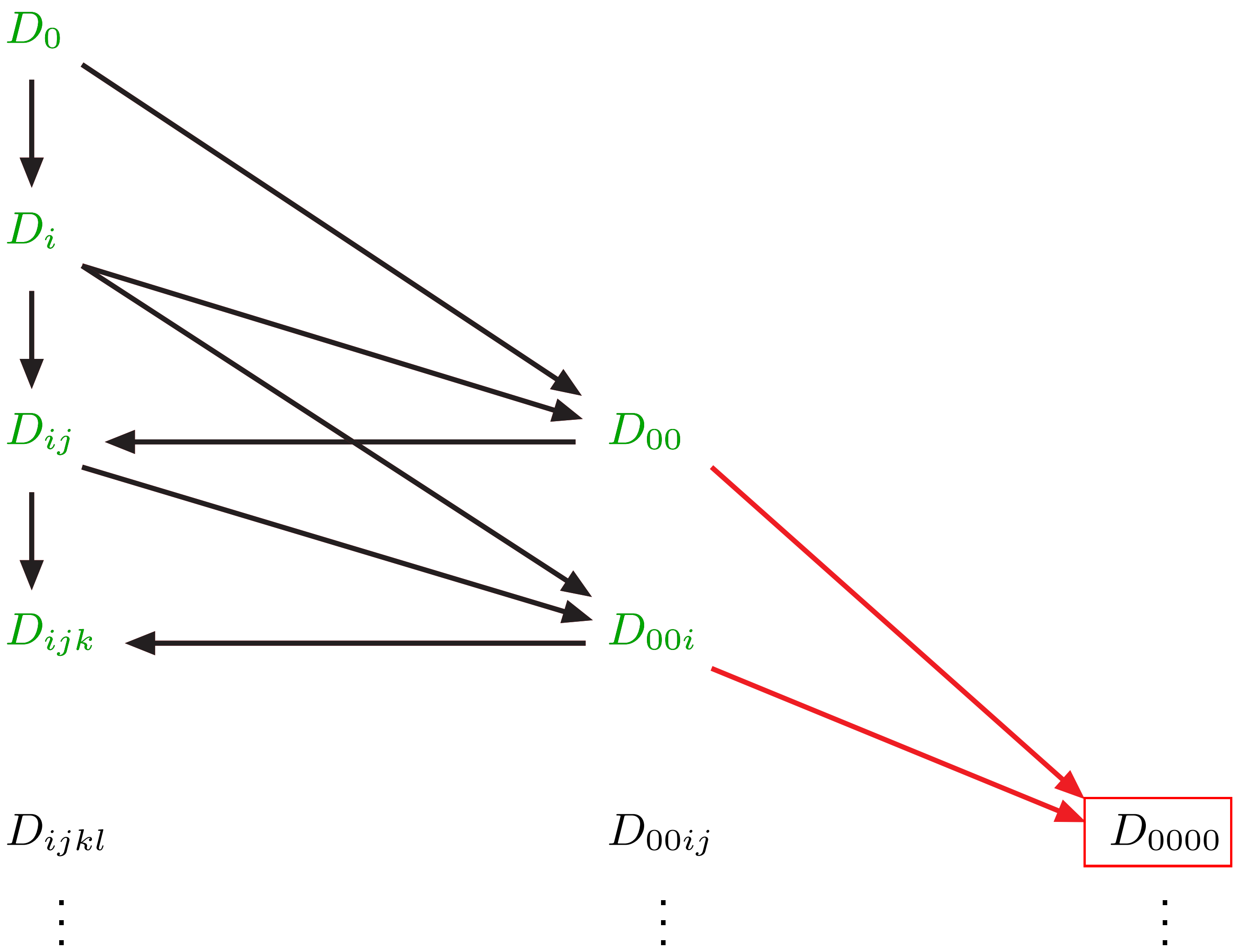
PV reduction: **step 3a**



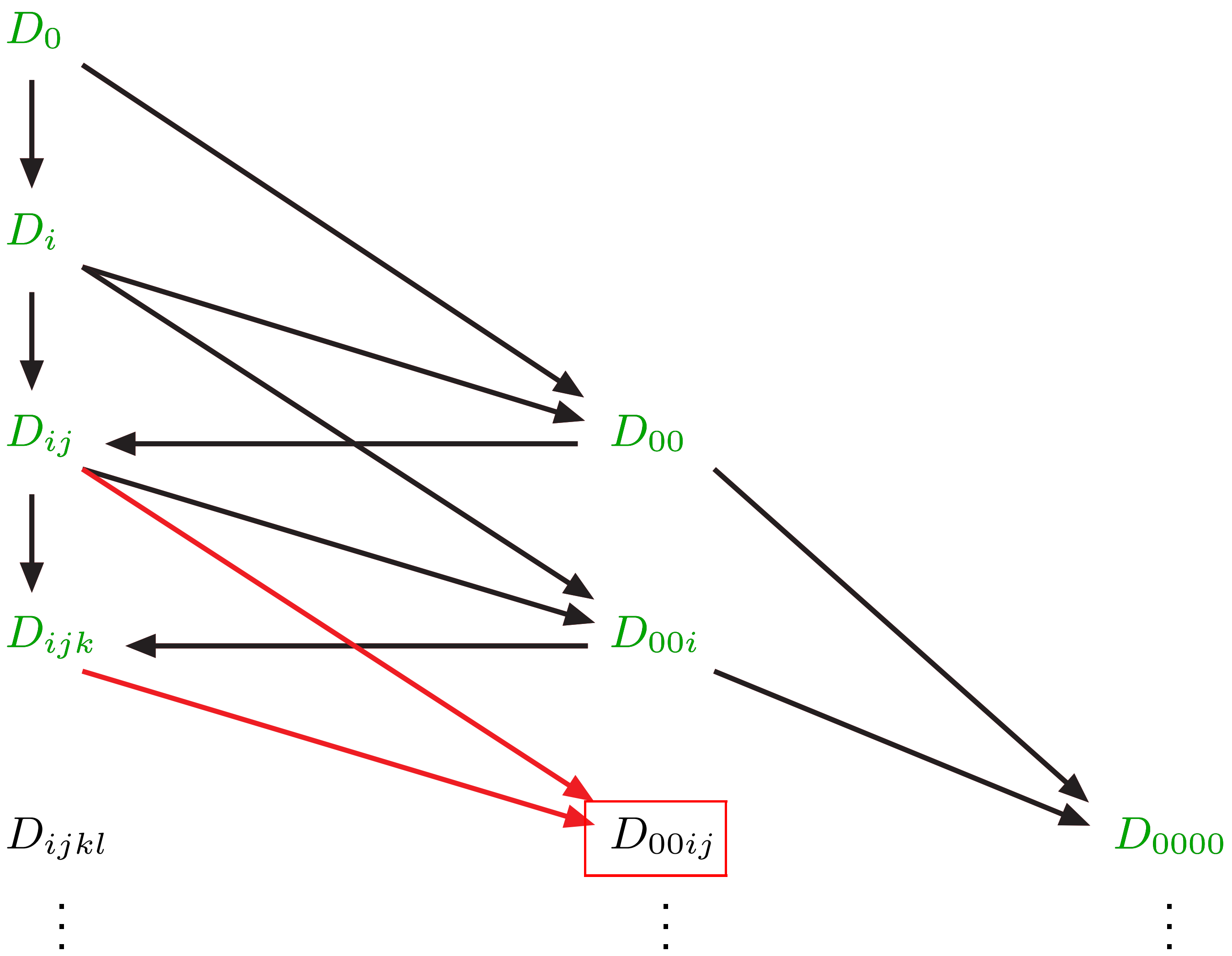
PV reduction: **step 3b**



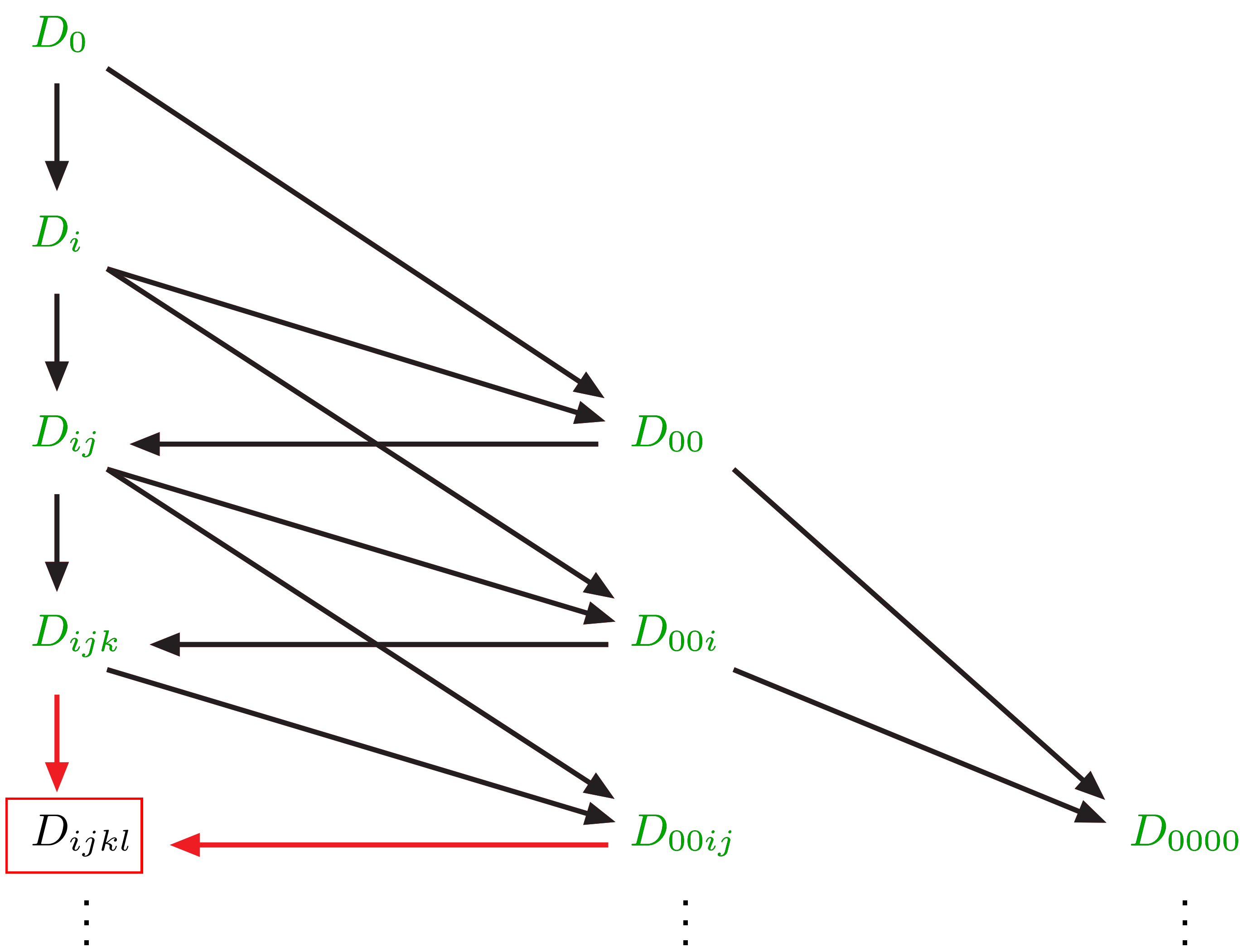
PV reduction: **step 4a**



PV reduction: **step 4b**



PV reduction: **step 4c**



3 A seminumerical method for 3- and 4-point integrals

Alternative form of basic PV relations: example of 4-point functions

$$\underbrace{\begin{pmatrix} 2m_0^2 & f_1 & f_2 & f_3 \\ f_1 & & & \\ f_2 & & Z & \\ f_3 & & & \end{pmatrix}}_{\equiv X} \begin{pmatrix} D_{i_2\dots i_P} \\ D_{1i_2\dots i_P} \\ D_{2i_2\dots i_P} \\ D_{3i_2\dots i_P} \end{pmatrix} = \begin{pmatrix} \text{const.} \times D_{00i_2\dots i_P} \\ -2\sum_{r=2}^{P} \delta_{1i_r} D_{00i_2\dots\hat{i}_r\dots i_P} \\ -2\sum_{r=2}^{P} \delta_{2i_r} D_{00i_2\dots\hat{i}_r\dots i_P} \\ -2\sum_{r=2}^{P} \delta_{3i_r} D_{00i_2\dots\hat{i}_r\dots i_P} \end{pmatrix} + C\text{'s}$$

$\hookrightarrow$ recursive reduction of coefficients $D_{i_1 i_2\dots i_P}$ from $D_{00i_2\dots i_P}$

Basis integral $D_{\underbrace{0\dots0}_{2P>2}}$ can be safely done numerically in Feynman-parameter space:

e.g. for $P=3$: $$D_{000000} = D_{000000}\Big|_{\mathrm{UV-div}} - \frac{1}{8}\int_{\sigma_3} \mathrm{d}^3x\,(A+1)\,\ln(A-\mathrm{i}\epsilon)$$

$$A = \text{quadratic form in Feynman parameters } x_i$$

Reduction with modified Cayley determinants up to rank 2: **step 0**

D_0

D_i

D_{ij} $\quad$ D_{00}

D_{ijk} $\quad$ D_{00i}

basis integral

D_{ijkl} $\quad$ D_{00ij} $\quad$ $\boxed{D_{0000}}$

$\vdots$ $\quad$ $\vdots$ $\quad$ $\vdots$

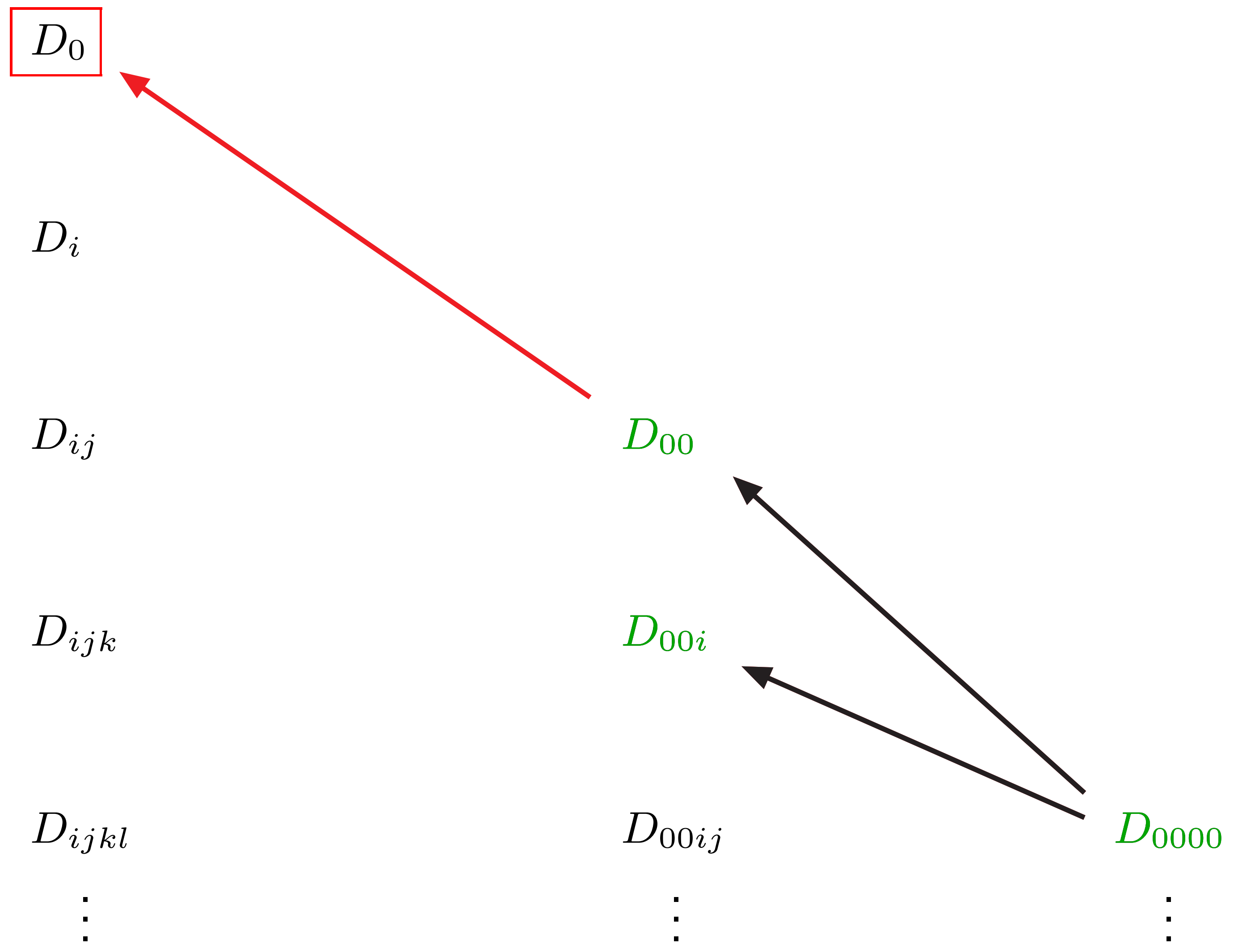
Reduction with modified Cayley determinants up to rank 2: **step 1a**

$$
\begin{array}{ccc}
D_0 & & \\
D_i & & \\
D_{ij} & \boxed{D_{00}} & \\
D_{ijk} & D_{00i} & \\
D_{ijkl} & D_{00ij} & \color{green}{D_{0000}} \\
\vdots & \vdots & \vdots
\end{array}
$$

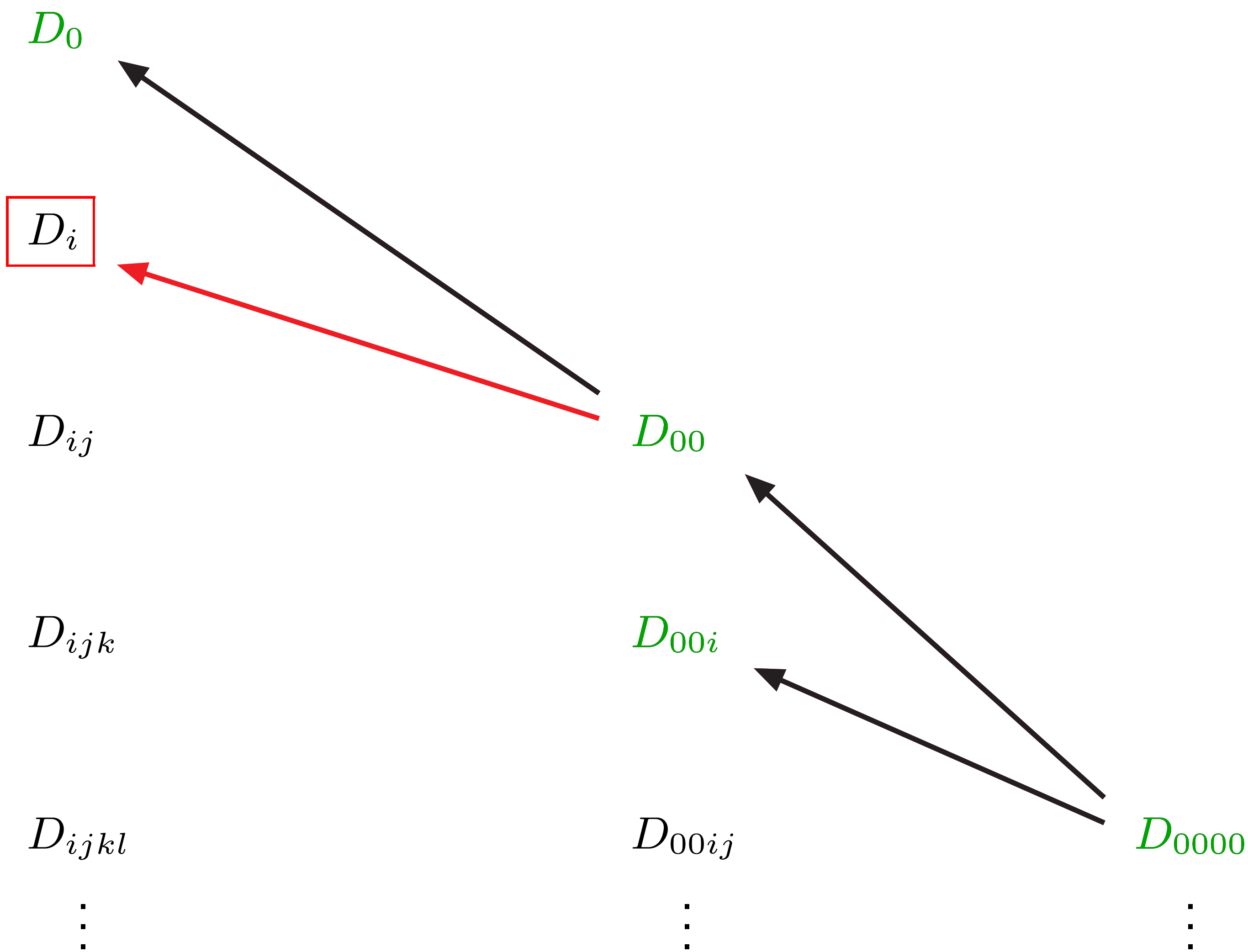
Reduction with modified Cayley determinants up to rank 2: **step 1b**

D_0		
D_i		
D_{ij}	D_{00}	
D_{ijk}	$\boxed{D_{00i}}$	
D_{ijkl}	D_{00ij}	D_{0000}
$\vdots$	$\vdots$	$\vdots$

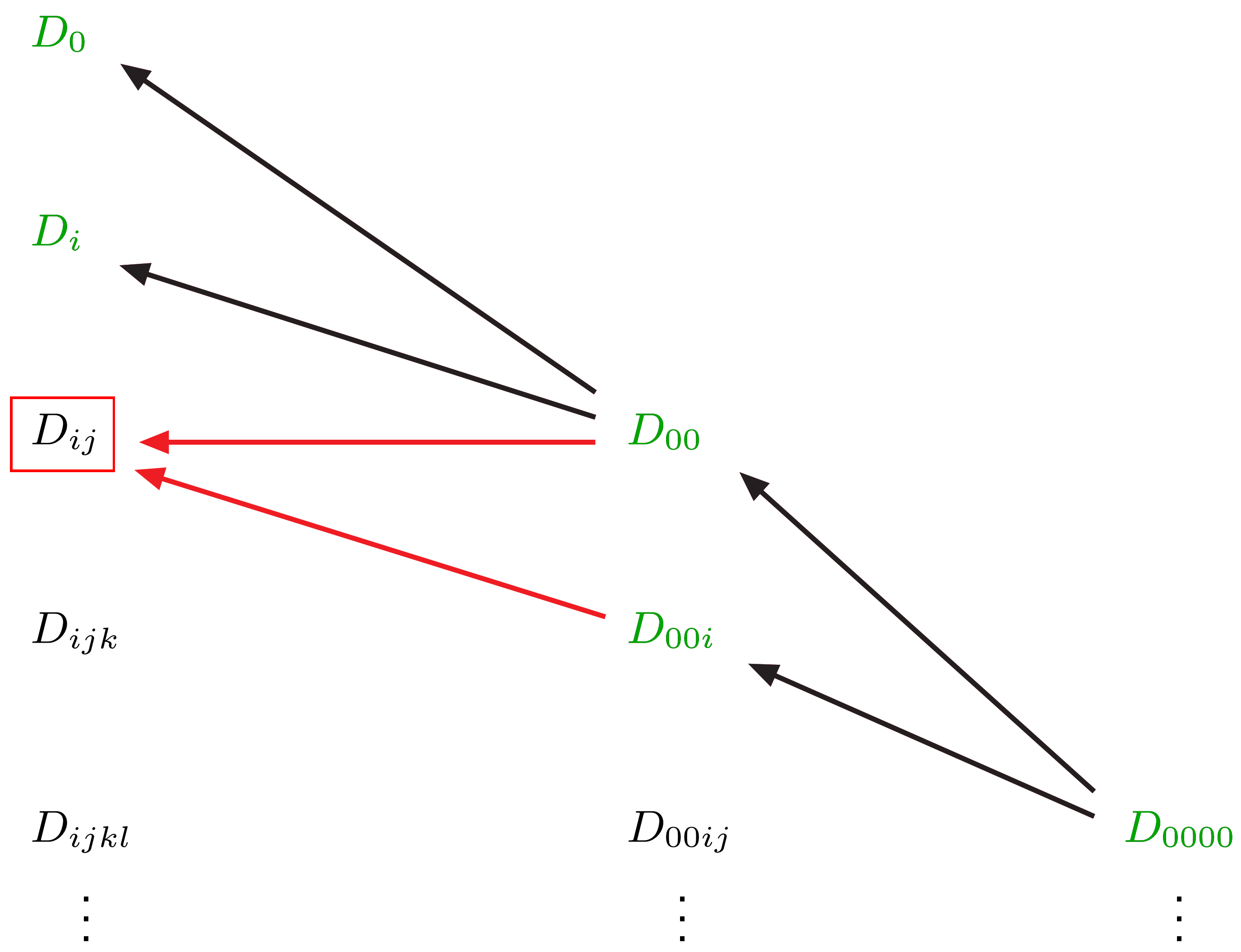
Reduction with modified Cayley determinants up to rank 2: **step 2a**



Reduction with modified Cayley determinants up to rank 2: **step 2b**

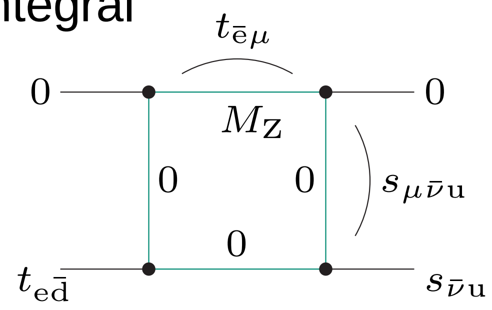


Reduction with modified Cayley determinants up to rank 2: **step 2c**

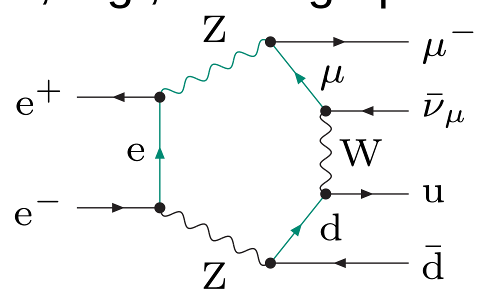


A typical example with small Gram determinant:

Box integral

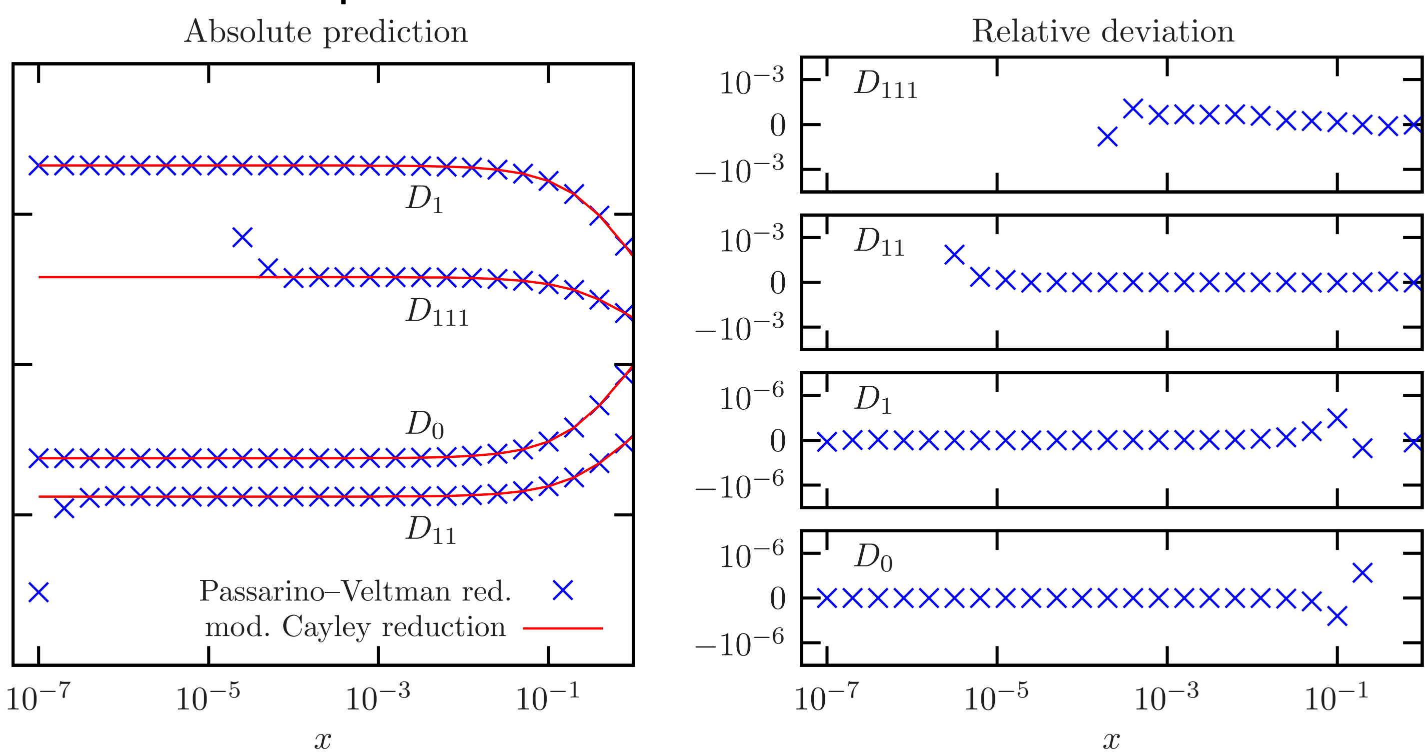


appears, e.g., in subgraph of diagram



Gram det.: $\det(Z) \to 0$ if $t_{e\bar{d}} \to t_{\mathrm{crit}} \equiv \dfrac{s_{\mu\bar{\nu}u}(s_{\mu\bar{\nu}u} - s_{\bar{\nu}u} + t_{\bar{e}\mu})}{s_{\mu\bar{\nu}u} - s_{\bar{\nu}u}}$

Numerical comparison:



$$x \equiv \frac{t_{e\bar{d}}}{t_{\mathrm{crit}}} - 1$$

$$\begin{aligned} s_{\mu\bar{\nu}u} &= +2\times 10^4\ \mathrm{GeV}^2 \\ s_{\bar{\nu}u} &= +1\times 10^4\ \mathrm{GeV}^2 \\ t_{\bar{e}\mu} &= -4\times 10^4\ \mathrm{GeV}^2 \\ t_{\mathrm{crit}} &= -6\times 10^4\ \mathrm{GeV}^2 \end{aligned}$$

PV reduction breaks down,
but mod. Cayley red. stable
for $\det(Z) \to 0$!

Comments:

- Basis integral $D_{0\dots0}$ enters with prefactor $\propto \dfrac{\det(Z)}{\det(X)}$

  $\hookrightarrow$ impact of $D_{0\dots0}$ and of its numerical error suppressed for small $\det(Z)$

- Limitation: reduction involves inverse matrix X^{-1}

  $\hookrightarrow$ potential instability if modified Cayley determinant $\det X \to 0$

  ($\det X = 0$ is necessary condition for Landau singularity)

- If appropriate [for small $\det(X)$] more coefficients (D_{0000}, D_{0000i}, etc.) can be evaluated numerically

  $\hookrightarrow$ accummulation of instabilities can be somewhat suppressed

4 Expansion methods for 3- and 4-point integrals

4.1 Expansion for small Gram determinant

PV relations rewritten again:

$$\tilde{X}_{0j} D_{i_1 \dots i_P} \;=\; 2 \sum_{n=1}^{N-1} \tilde{Z}_{jn} \sum_{r=1}^{P} \delta_{n i_r} D_{00 i_1 \dots \hat{i}_r \dots i_P} + \det(Z) D_{j i_1 \dots i_P} \;+\; C\text{'s}$$

$$\tilde{Z}_{kl} D_{00 i_1 \dots i_P} \;\propto\; \sum_{n,m=1}^{N-1} \tilde{\tilde{Z}}_{(kn)(lm)} \Bigg[f_n f_m D_{i_1 \dots i_P} + 2 \sum_{r=1}^{P} (f_n \delta_{m i_r} + f_m \delta_{n i_r}) D_{00 i_1 \dots \hat{i}_r \dots i_P}$$
$$+ 4 \sum_{\substack{r,s=1 \\ r \neq s}}^{P} \delta_{n i_r} \delta_{m i_s} D_{0000 i_1 \dots \hat{i}_r \dots \hat{i}_s \dots i_P} \Bigg] + 2 m_0^2 \tilde{Z}_{kl} D_{i_1 \dots i_P} - \det(Z) D_{kl i_1 \dots i_P} \;+\; C\text{'s}$$

$\tilde{X}$, $\tilde{Z}$, $\tilde{\tilde{Z}}$ = minors (subdeterminants) of X and Z (j, k, l chosen appropriately)

$\hookrightarrow$ Coefficients $\underbrace{D_{ij\dots}}_{\text{rank } P}$ and $\underbrace{D_{00i\dots}}_{\text{rank } P+1}$ from lower-rank terms $\underbrace{D_{ij\dots}}_{\text{rank } P-1}$ and $\underbrace{D_{00i\dots}}_{\text{rank } P}$

up to suppressed higher-rank terms $\det(Z)$ $\underbrace{D_{ij\dots}}_{\text{rank } P+1}$

$\hookrightarrow$ Equations suited for iteration for small $\det(Z)$:

D's directly from C's up to terms suppressed by $\det(Z)$

Expansion for small Gram determinant: step 0

$\boxed{D_0^{(0)}}$

starting point: all D tensors = 0

$$D_i$$

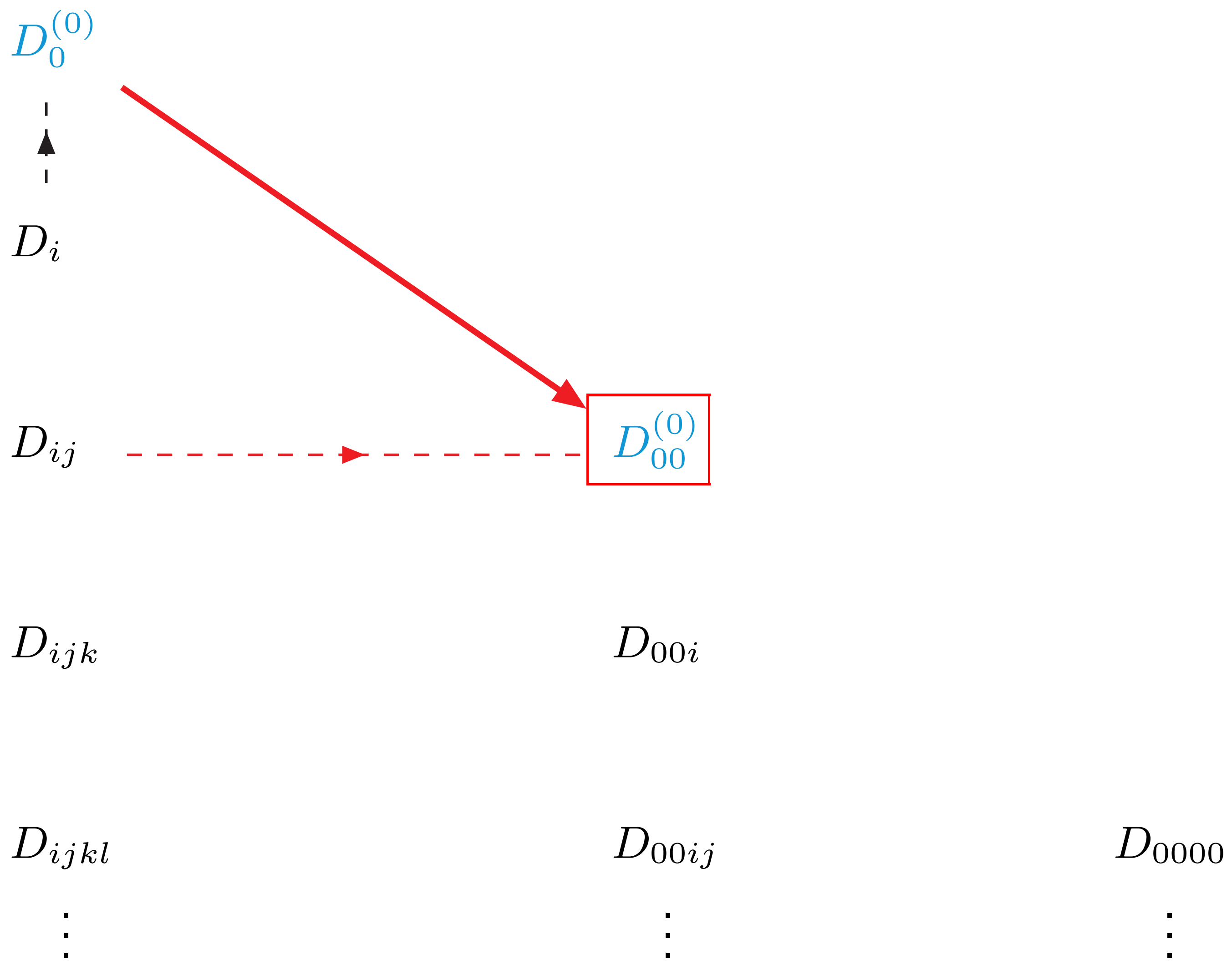
$$D_{ij} \qquad D_{00}$$

$$D_{ijk} \qquad D_{00i}$$

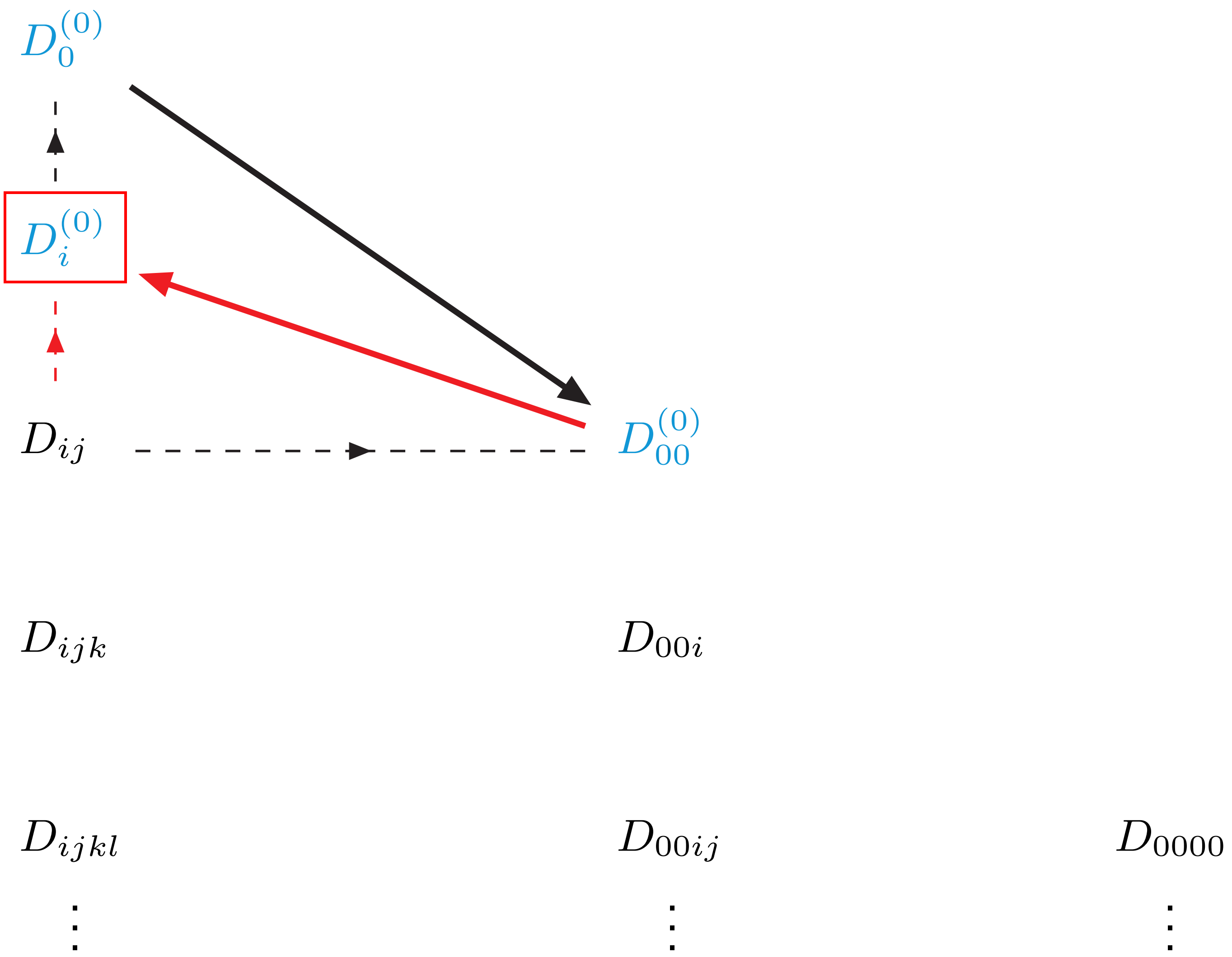
$$D_{ijkl} \qquad D_{00ij} \qquad D_{0000}$$

$$\vdots \qquad \vdots \qquad \vdots$$

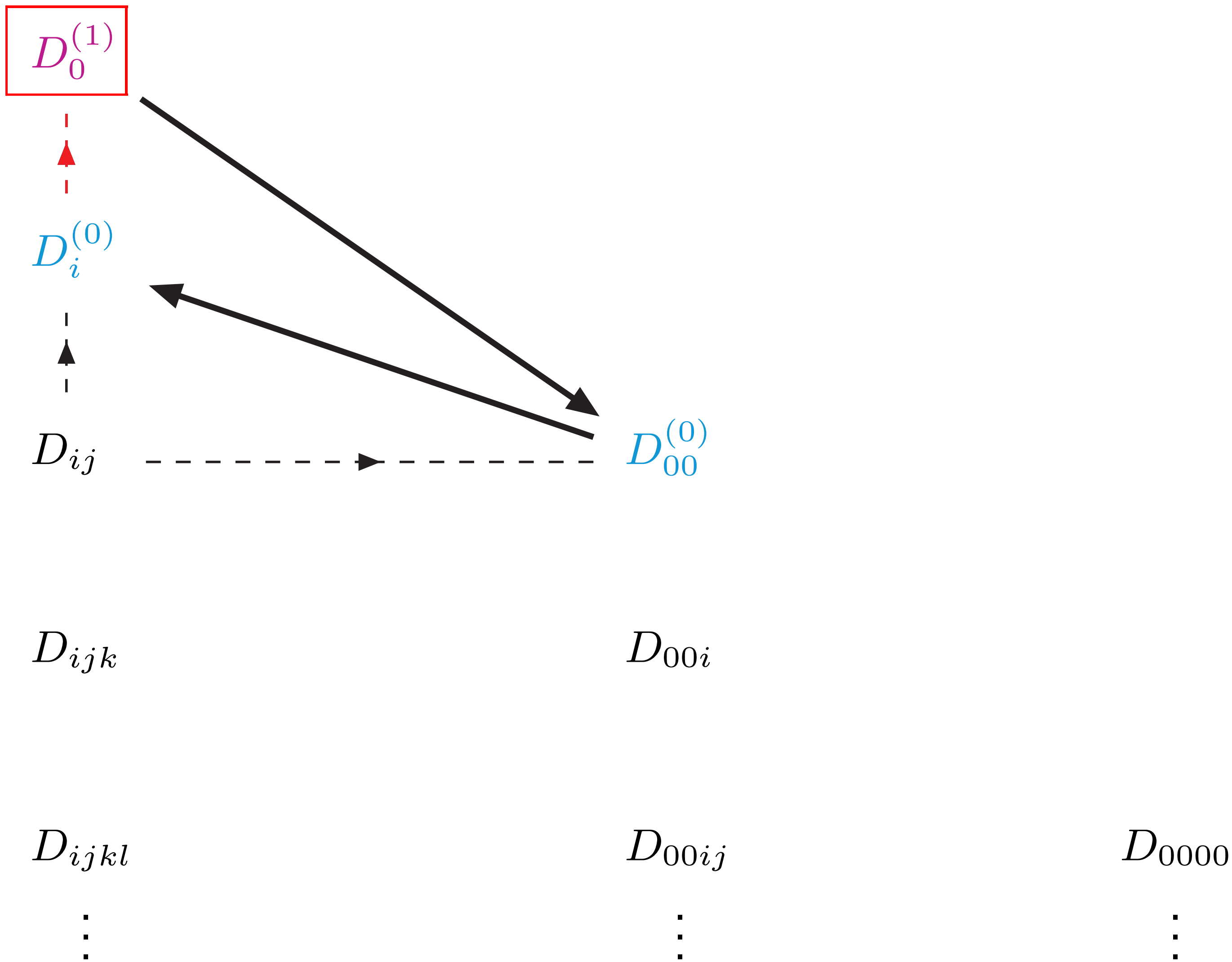
Expansion for small Gram determinant: step 1a



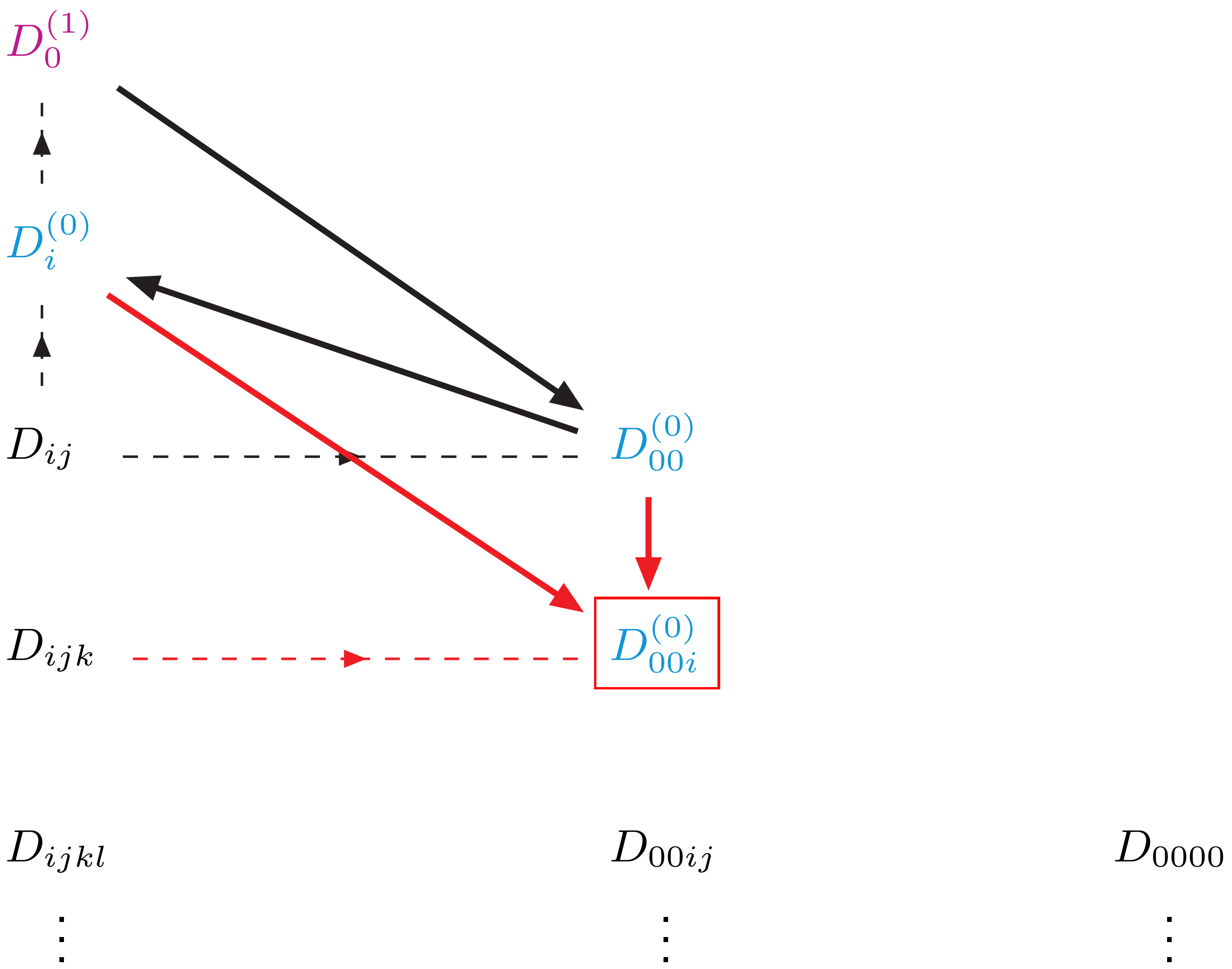
Expansion for small Gram determinant: step 1b



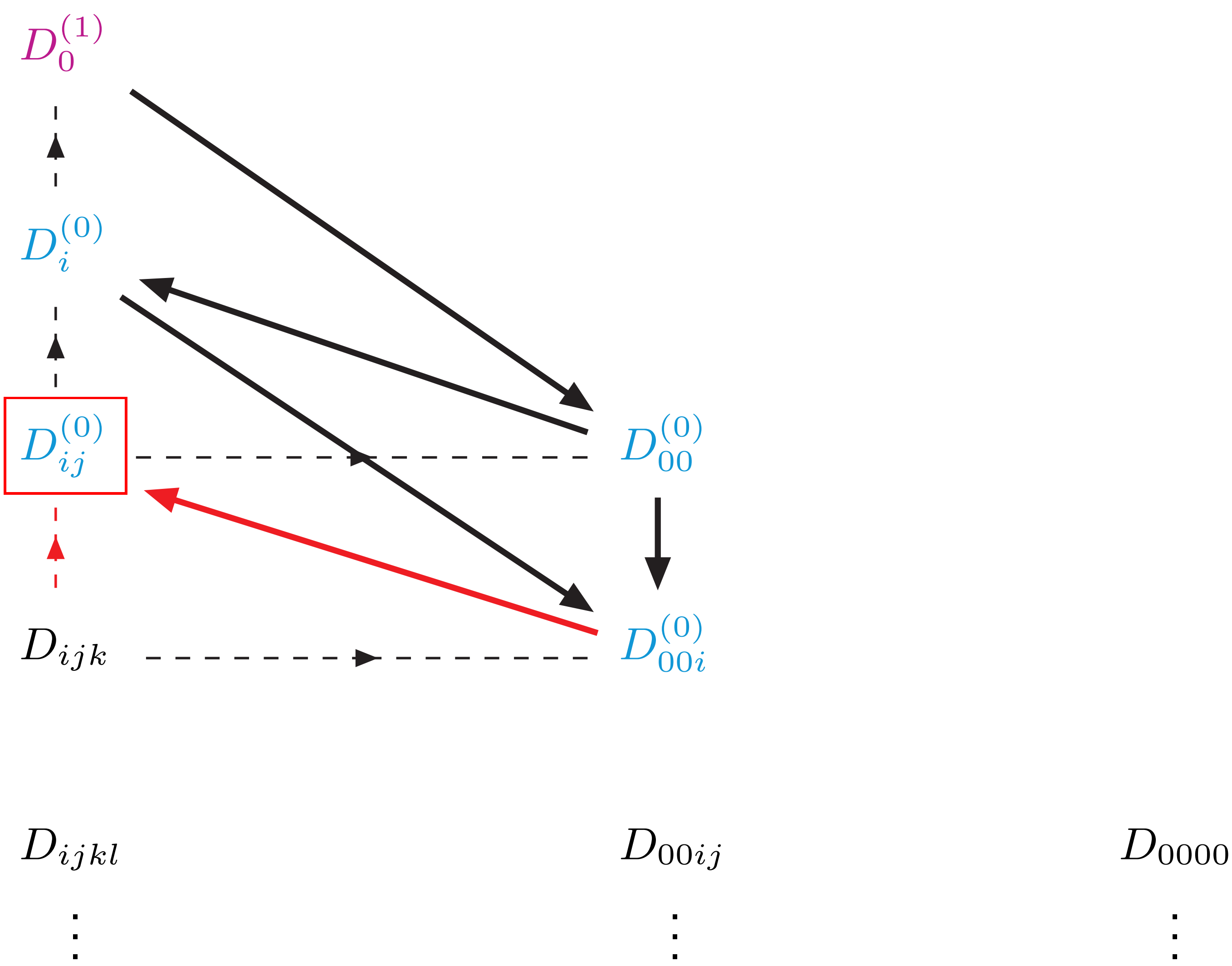
Expansion for small Gram determinant: step 1.0



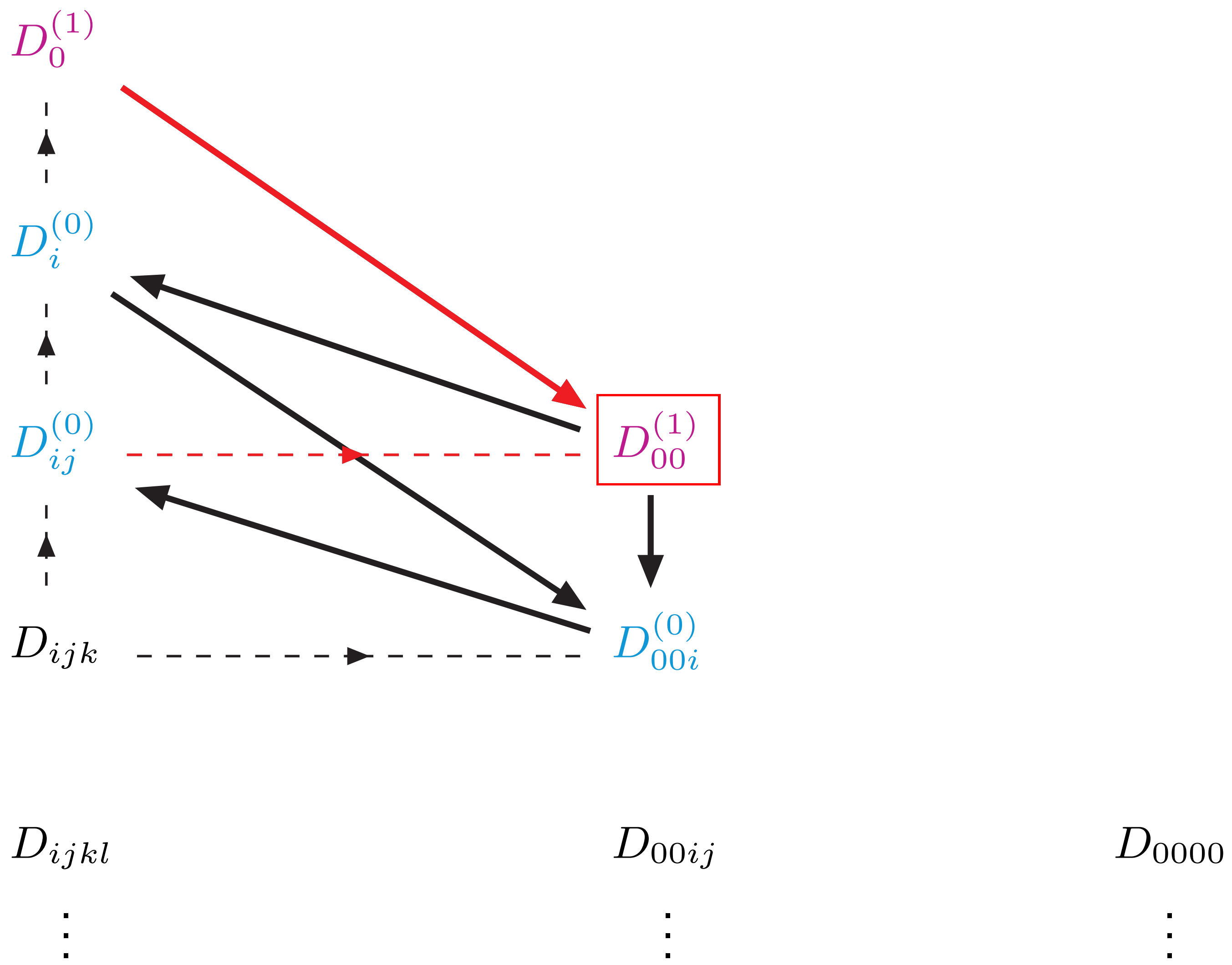
Expansion for small Gram determinant: step 2a



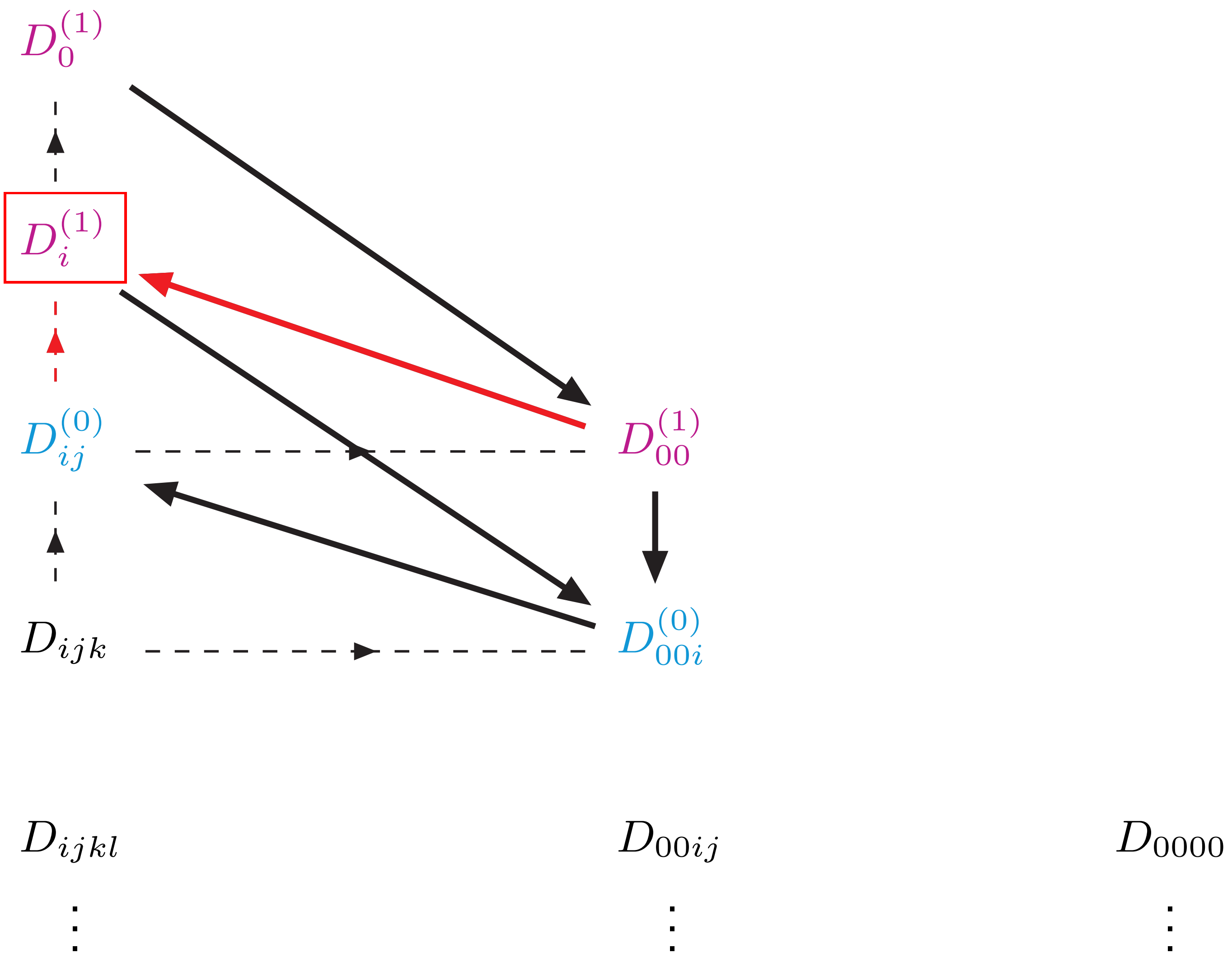
Expansion for small Gram determinant: step 2b



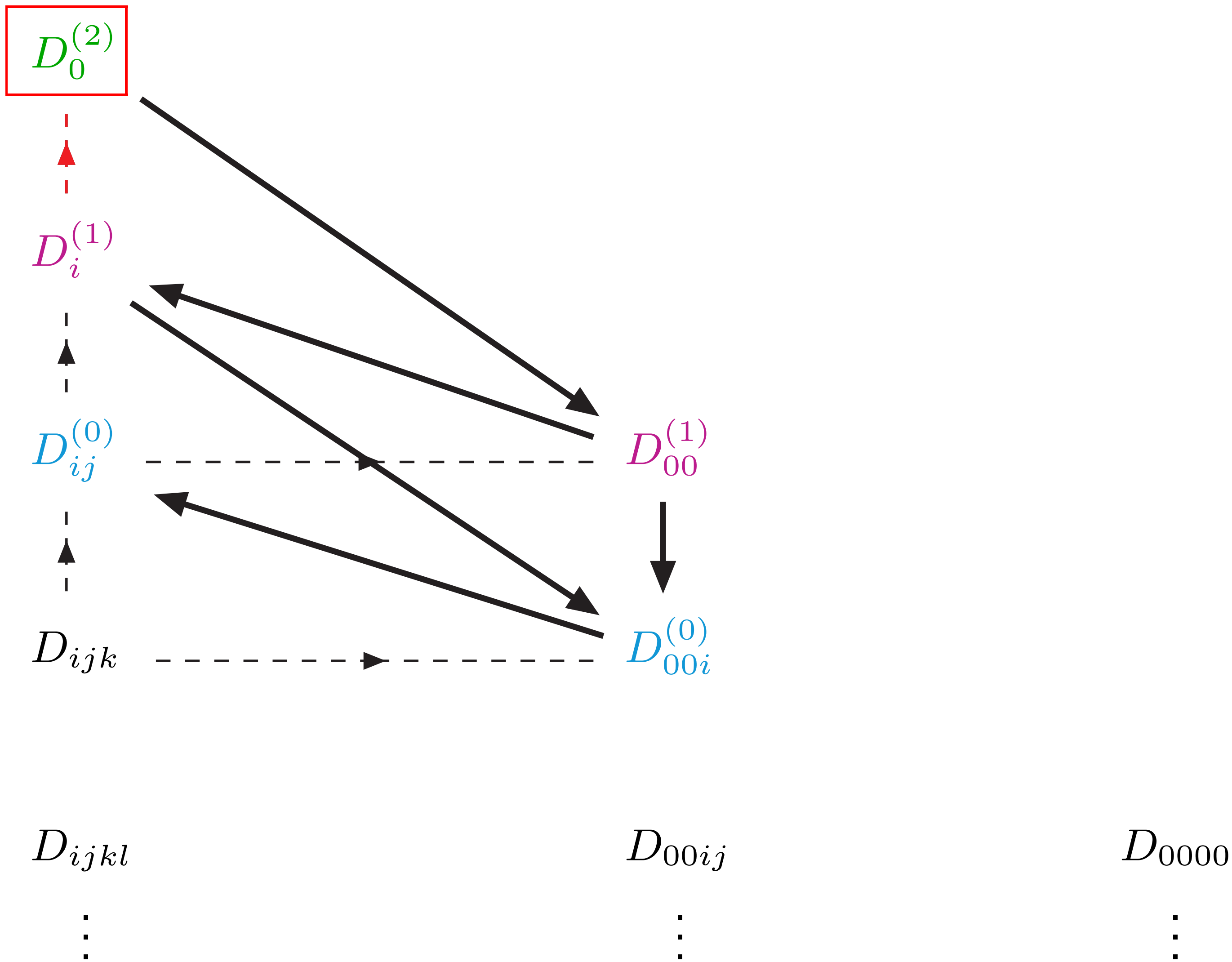
Expansion for small Gram determinant: step 2.1a



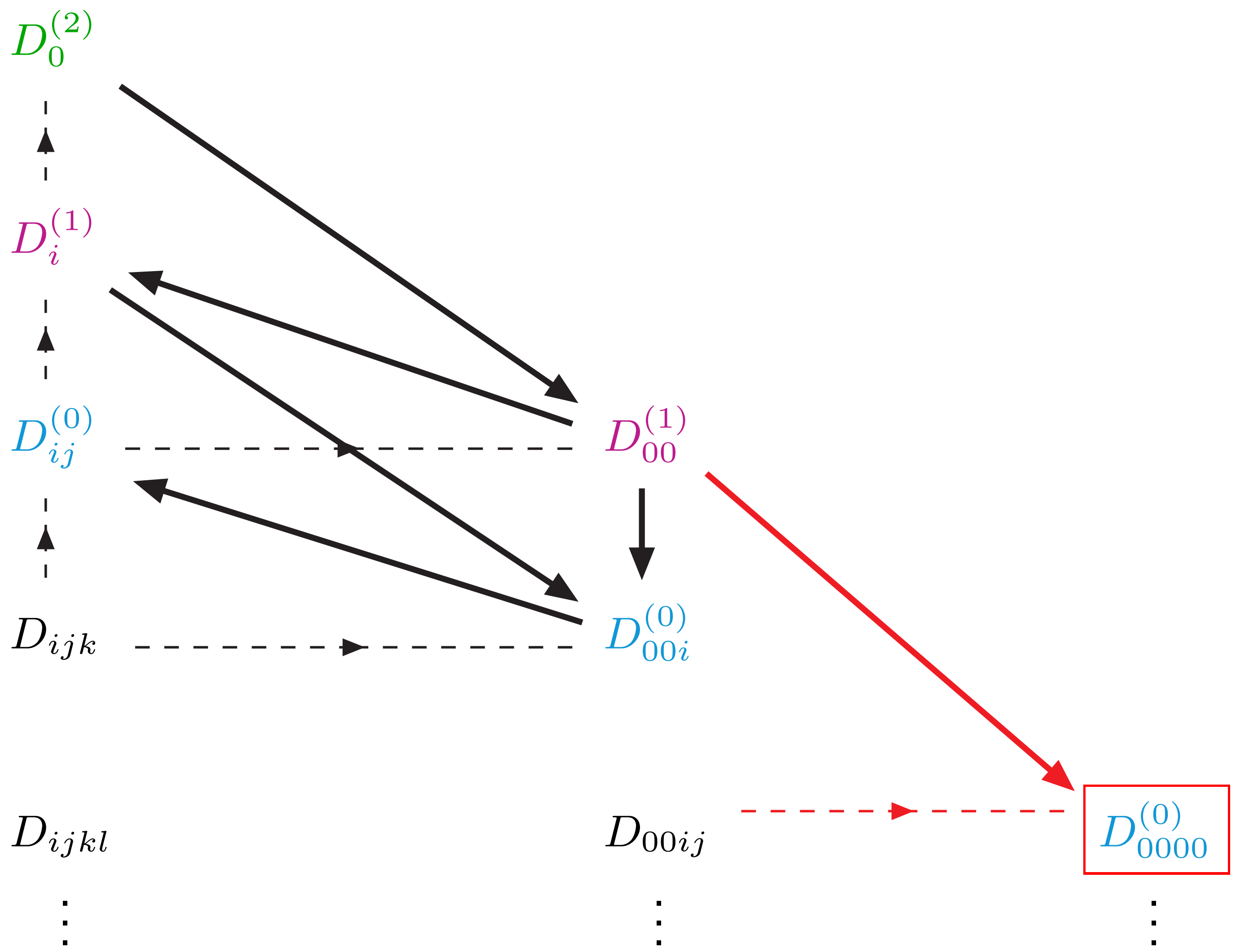
Expansion for small Gram determinant: step 2.1b



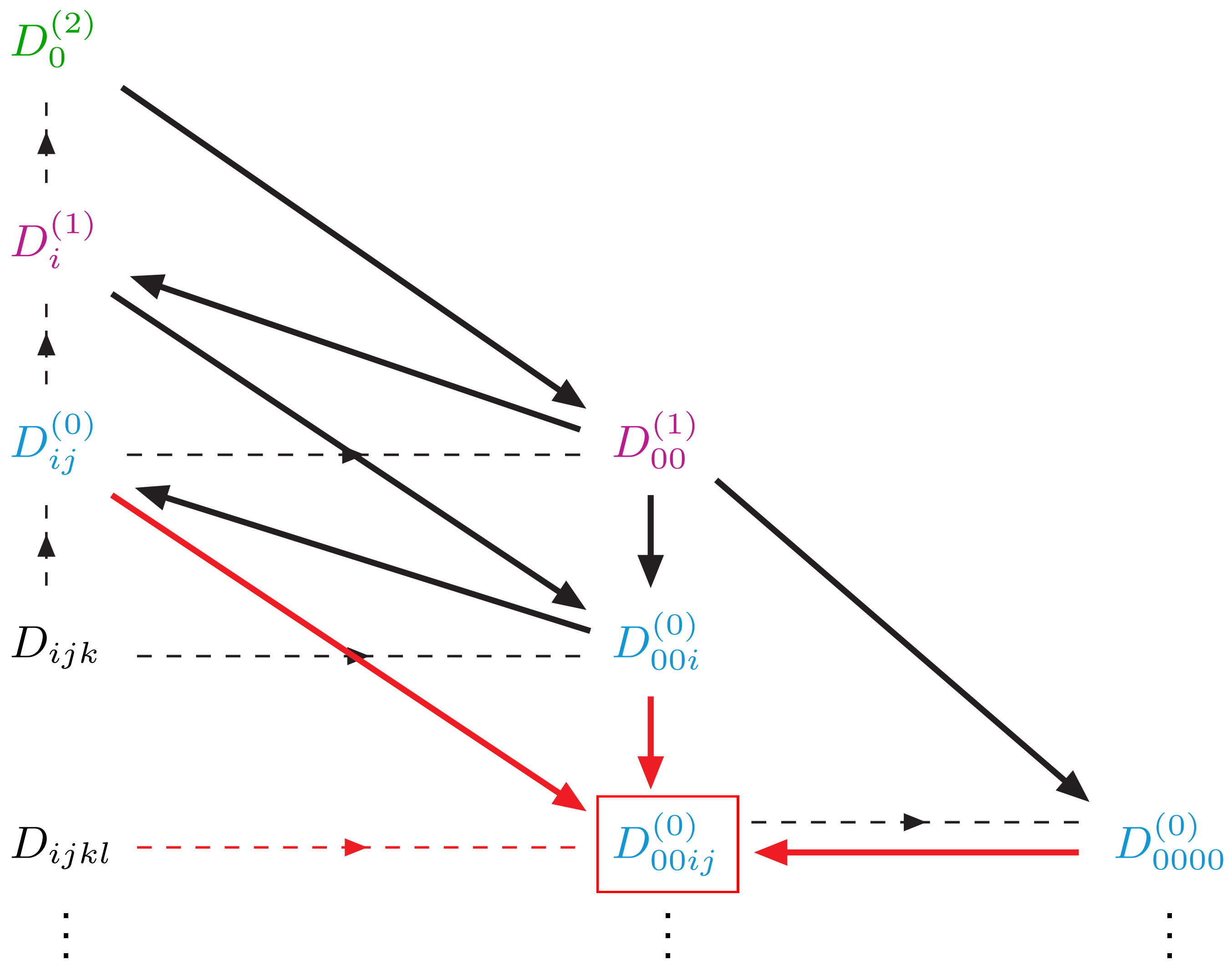
Expansion for small Gram determinant: step 2.0



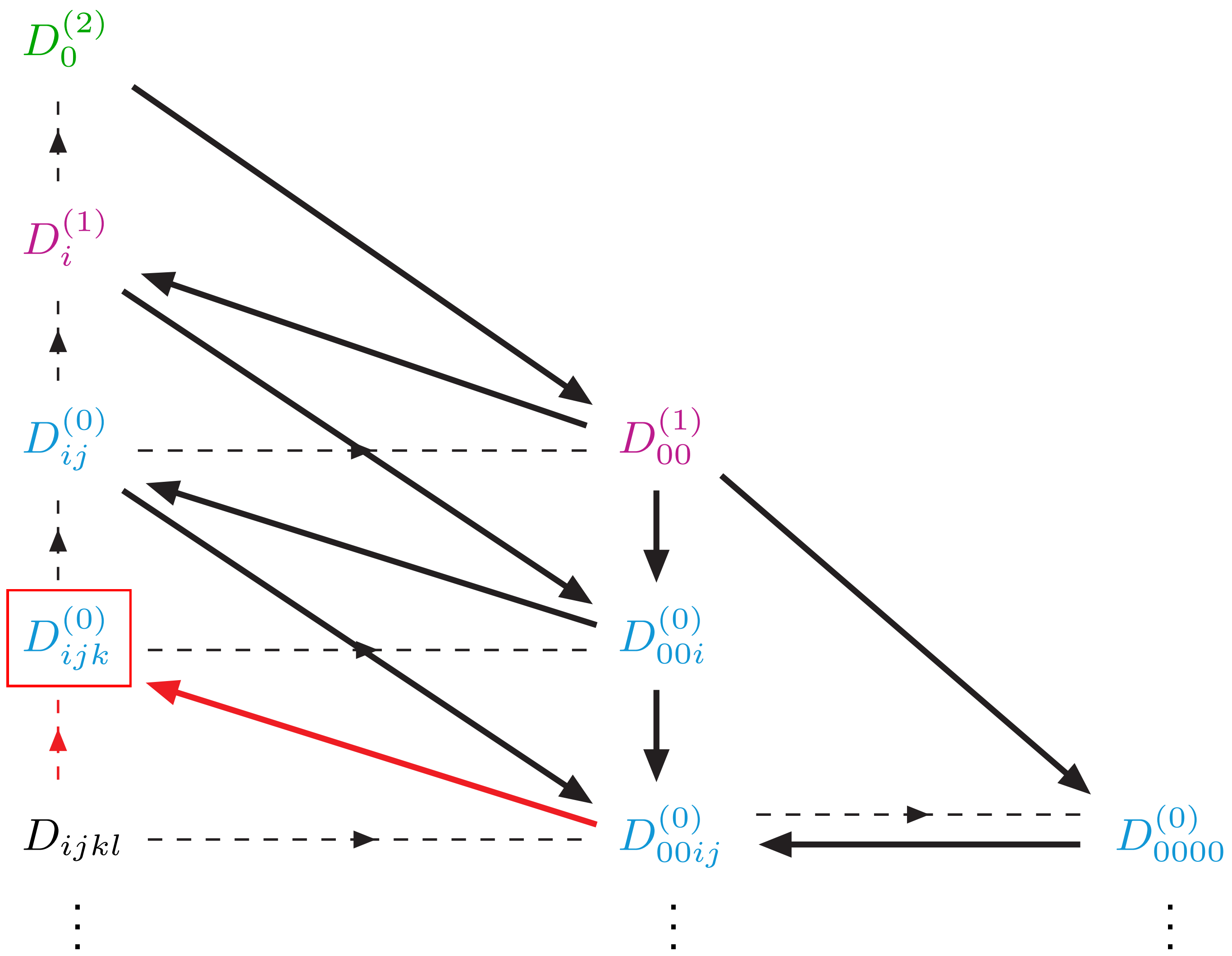
Expansion for small Gram determinant: step 3a



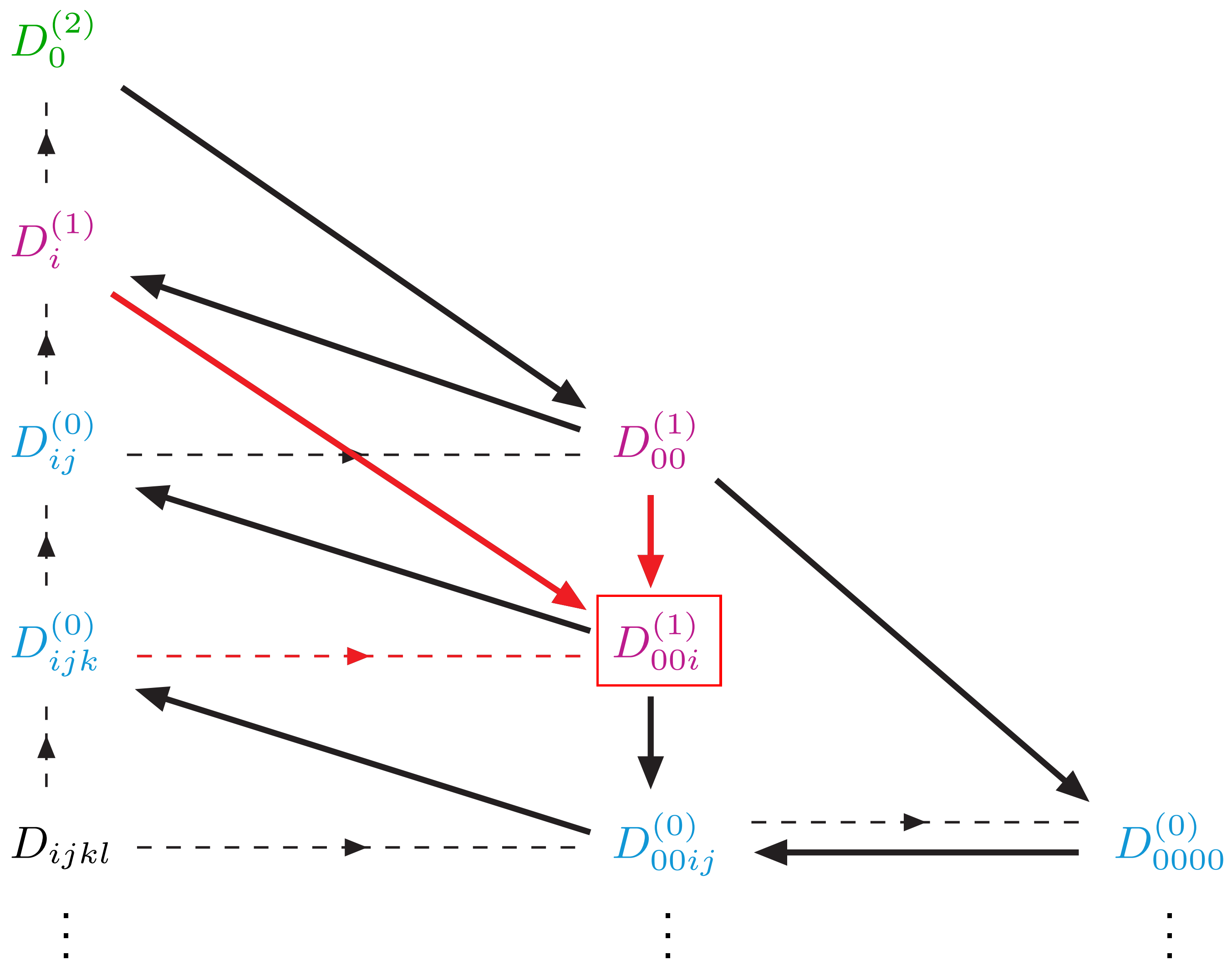
Expansion for small Gram determinant: step 3b



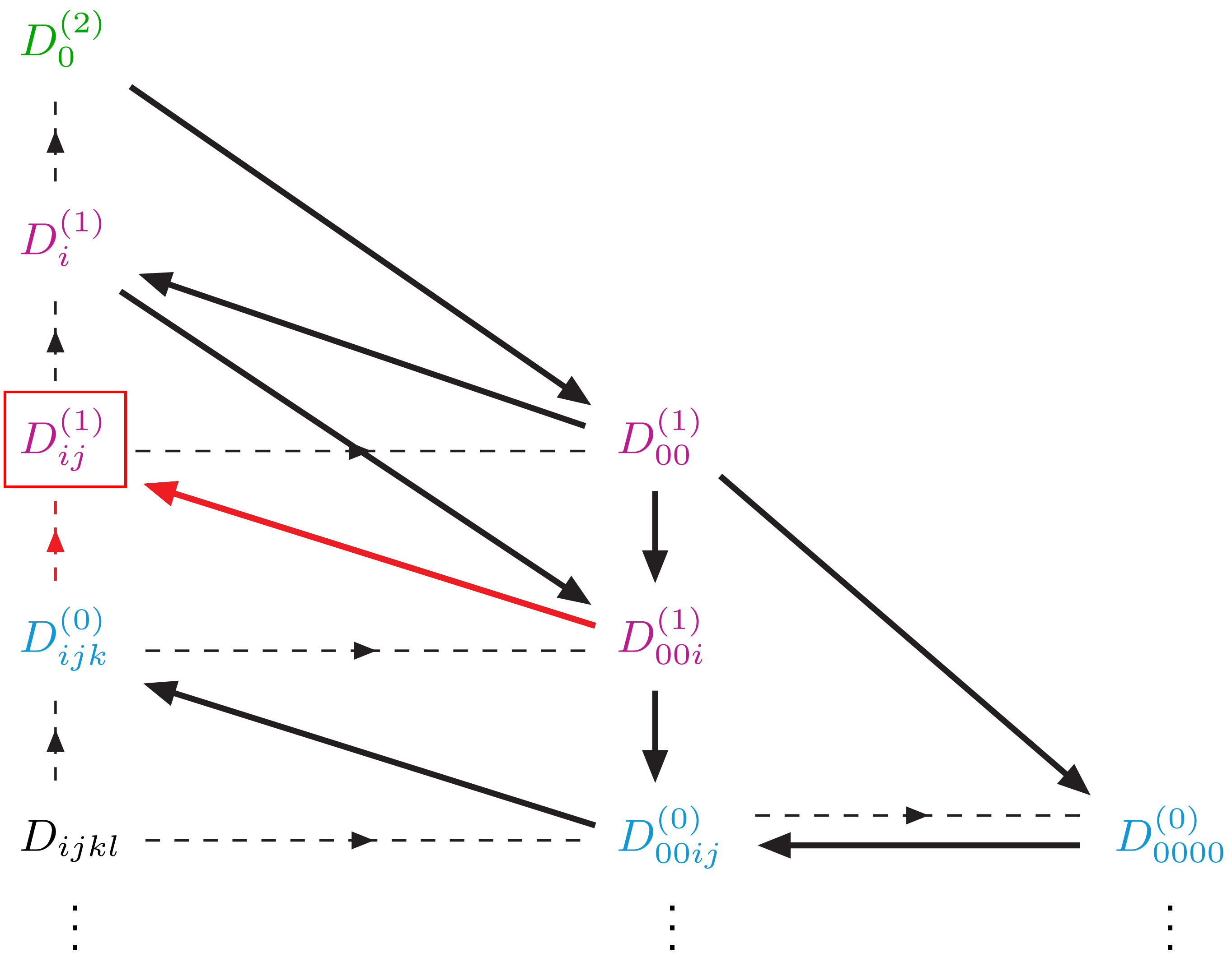
Expansion for small Gram determinant: step 3c



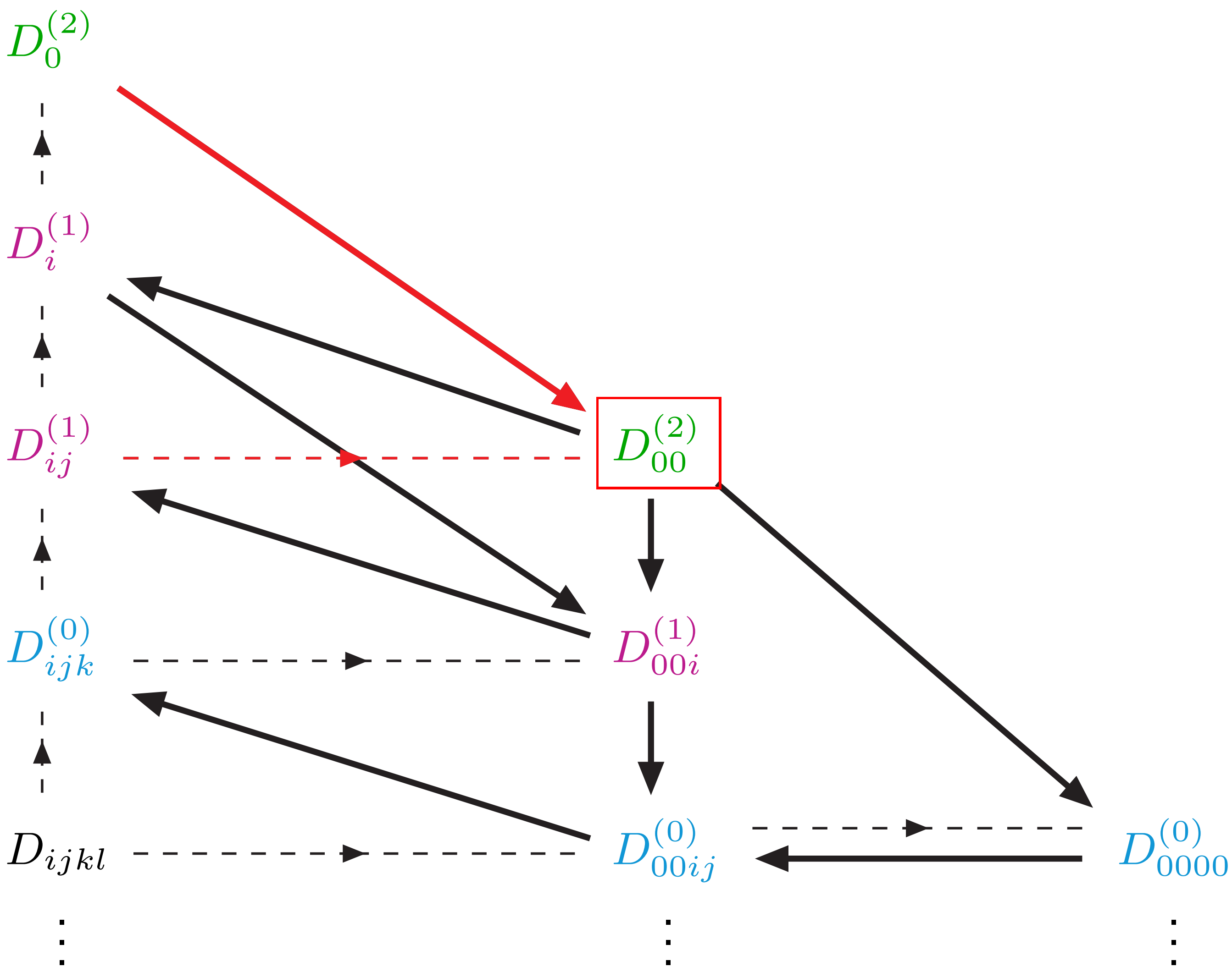
Expansion for small Gram determinant: step 3.2a



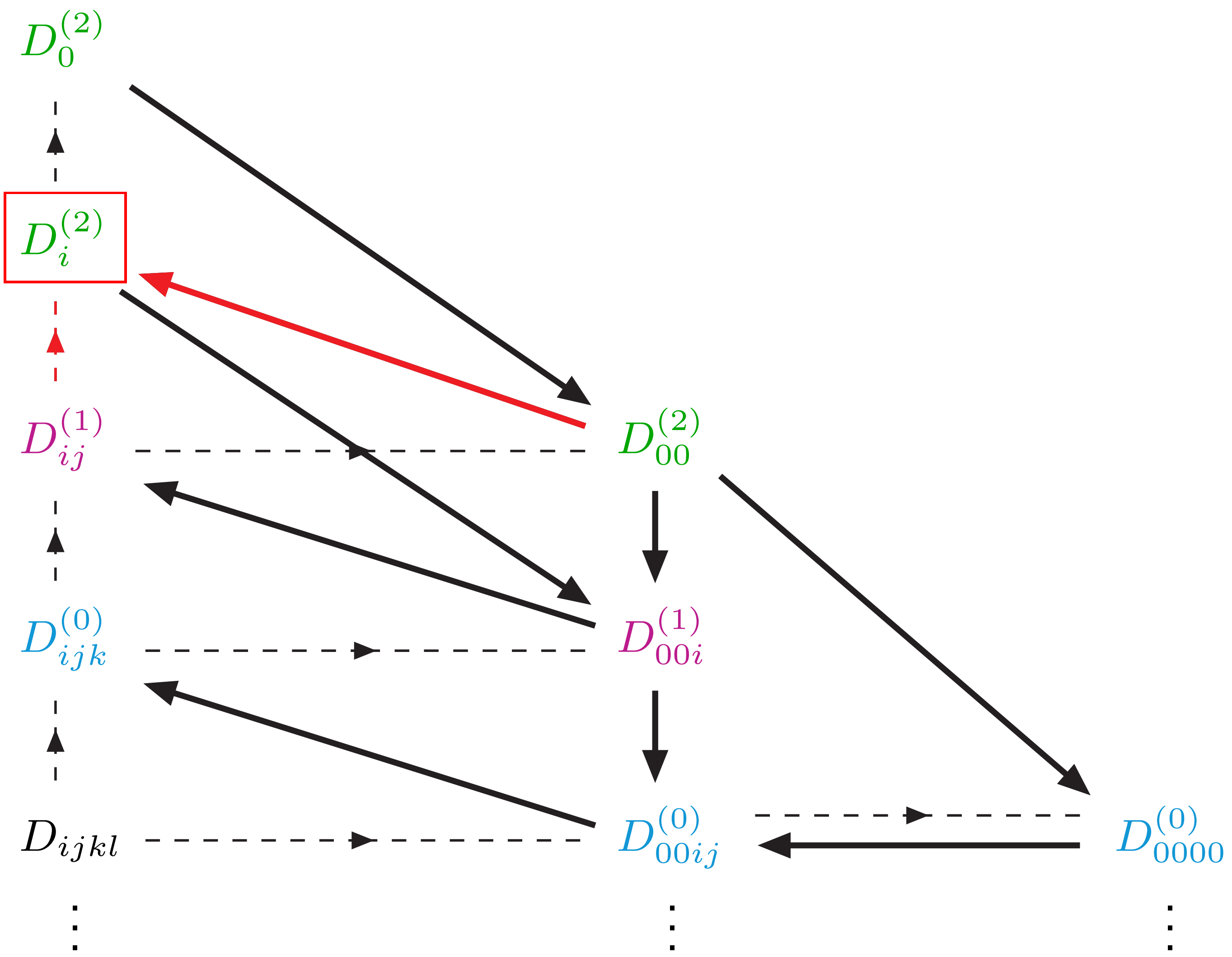
Expansion for small Gram determinant: step 3.2b



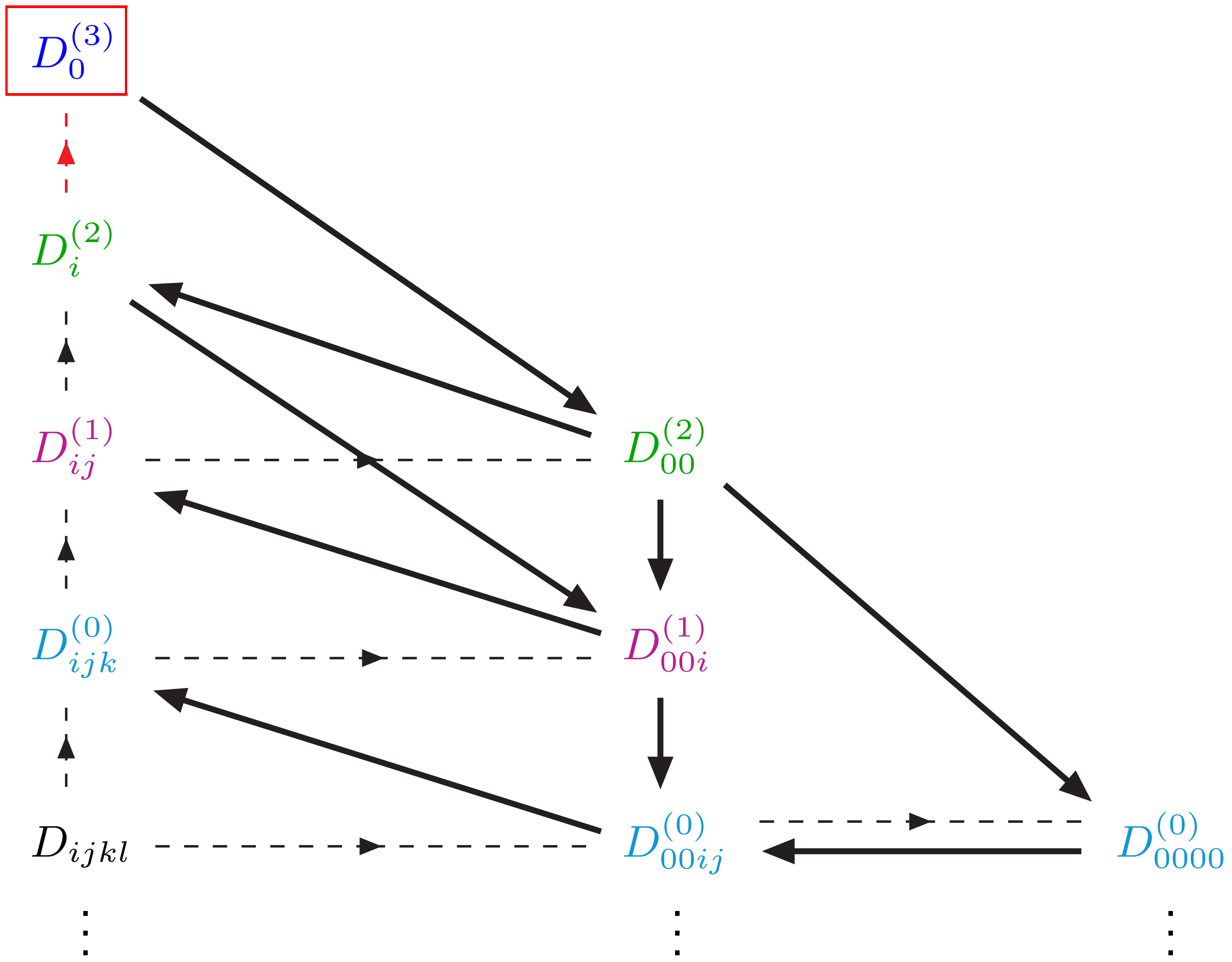
Expansion for small Gram determinant: step 3.1a



Expansion for small Gram determinant: step 3.1b

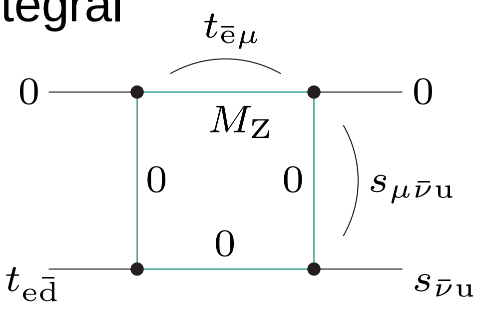


Expansion for small Gram determinant: step 3.0

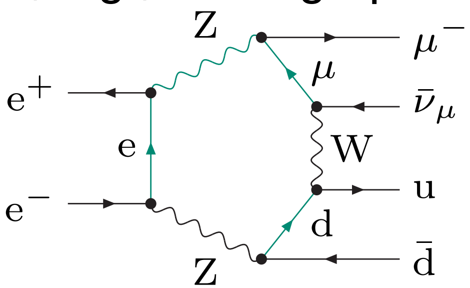


A typical example with small Gram determinant: (same as above)

Box integral

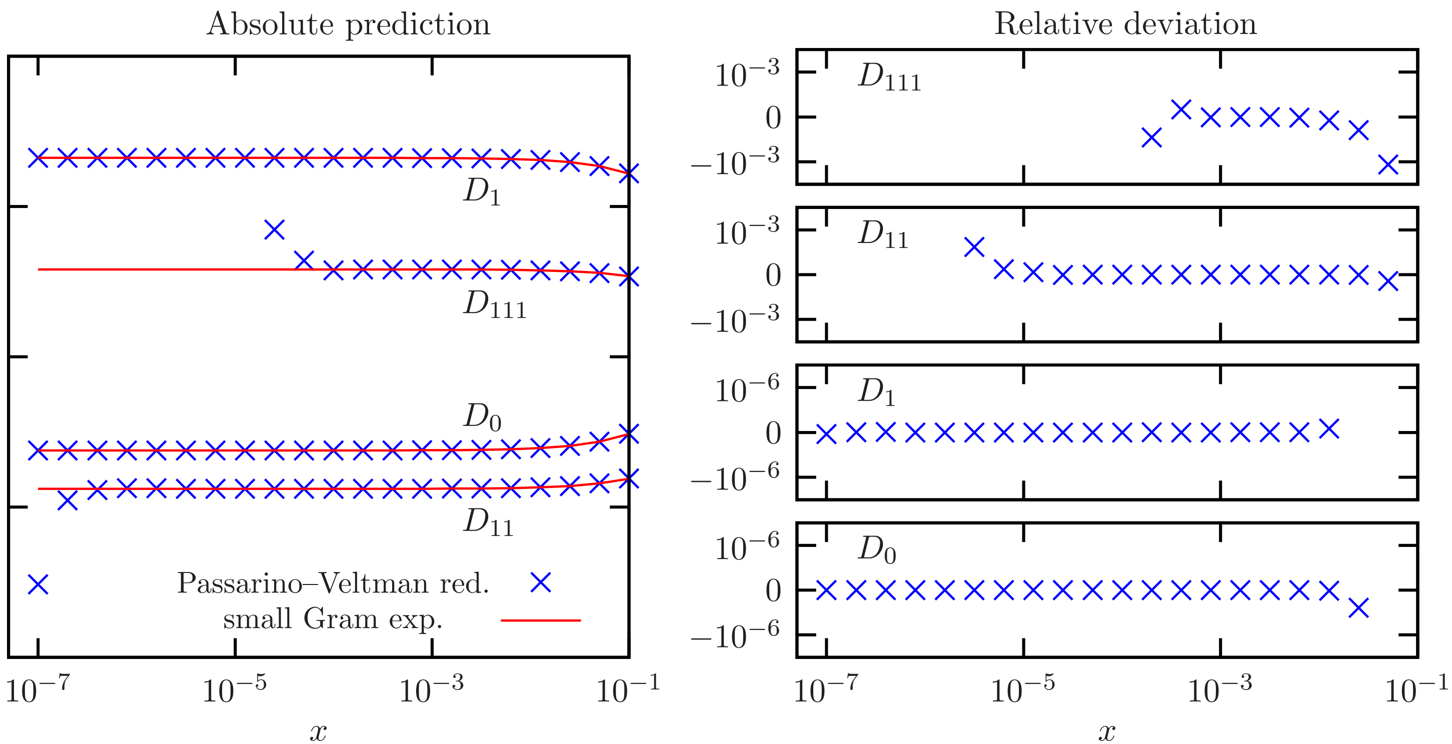


appears, e.g., in subgraph of diagram



Gram det.: $\det(Z) \to 0$ if $t_{e\bar{d}} \to t_{\text{crit}} \equiv \dfrac{s_{\mu\bar{\nu}u}(s_{\mu\bar{\nu}u} - s_{\bar{\nu}u} + t_{\bar{e}\mu})}{s_{\mu\bar{\nu}u} - s_{\bar{\nu}u}}$

Numerical comparison:



$$x \equiv \frac{t_{e\bar{d}}}{t_{\text{crit}}} - 1$$

$$\begin{aligned} s_{\mu\bar{\nu}u} &= +2\times10^4\ \text{GeV}^2 \\ s_{\bar{\nu}u} &= +1\times10^4\ \text{GeV}^2 \\ t_{\bar{e}\mu} &= -4\times10^4\ \text{GeV}^2 \\ t_{\text{crit}} &= -6\times10^4\ \text{GeV}^2 \end{aligned}$$

PV reduction breaks down,
but Gram exp. stable
for $\det(Z) \to 0$!

Comments:

- Indices j, k, l of $\tilde{X}_{0j}$ and $\tilde{Z}_{kl}$ can be chosen to optimize stability
- For $\mathrm{e}^+\mathrm{e}^- \to 4f$ iteration up to step 5 was sufficient,
  but number of iterations not restricted
- Iteration does not converge if
  - ◇ either all $\tilde{Z}_{kl}$ are small
    happens most frequently if $Z \to 0$

    $\hookrightarrow$ new expansion about $Z \to 0$ possible (worked out!)
  - ◇ or all $\tilde{X}_{0j}$ are small [which implies $\det(X) \to 0$]

    $\hookrightarrow$ new double expansion in small $\det(Z)$ and $\tilde{X}_{0j}$ (described next!)

4.2 Expansion for small Gram and modified Cayley determinants

PV relations rewritten again:

$$\sum_{n=1}^{N-1} \tilde{Z}_{kn} \sum_{r=1}^{P} \delta_{ni_r} D_{00i_1\dots\hat{i}_r\dots i_P} \propto \tilde{X}_{k0} D_{i_1\dots i_P} - \det(Z) D_{ki_1\dots i_P} + C\text{'s},$$

$$\tilde{X}_{ij} D_{i_1\dots i_P} = \text{const.} \times \tilde{Z}_{ij} D_{00i_1\dots i_P} - 2 \sum_{m,n=1}^{N-1} \tilde{\tilde{Z}}_{(in)(jm)} f_n \sum_{r=1}^{P} \delta_{mi_r} D_{00i_1\dots\hat{i}_r\dots i_P}$$
$$+ \tilde{X}_{0j} D_{ii_1\dots i_P} + C\text{'s}$$

$\hookrightarrow$ Coefficients $\underbrace{D_{00ij\dots}}_{\text{rank } P}$ from C's up to suppressed terms $\underbrace{D_{ij\dots}}_{\text{rank } P}$ and $\underbrace{D_{j\dots}}_{\text{rank } P-1}$

Coefficients $\underbrace{D_{ij\dots}}_{\text{rank } P}$ from $\underbrace{D_{00j\dots}}_{\text{rank } P+1}$ and $\underbrace{D_{00ij\dots}}_{\text{rank } P+2}$

up to suppressed higher-rank terms $\underbrace{D_{ijk\dots}}_{\text{rank } P+1}$

$\hookrightarrow$ Equations suited for iteration for small $\det(Z)$ and $\tilde{X}_{0j}$:
D's directly from C's up to terms suppressed by $\det(Z)$ or $\tilde{X}_{0j}$

Expansion for small Gram and modified Cayley determinants: step 0a

D_0

D_i

starting point: all D tensors = 0

D_{ij} $\boxed{D_{00}^{(0)}}$

D_{ijk} D_{00i}

D_{ijkl} D_{00ij} D_{0000}

D_{ijklm} D_{00ijk} D_{0000i}

$\vdots$ $\vdots$ $\vdots$

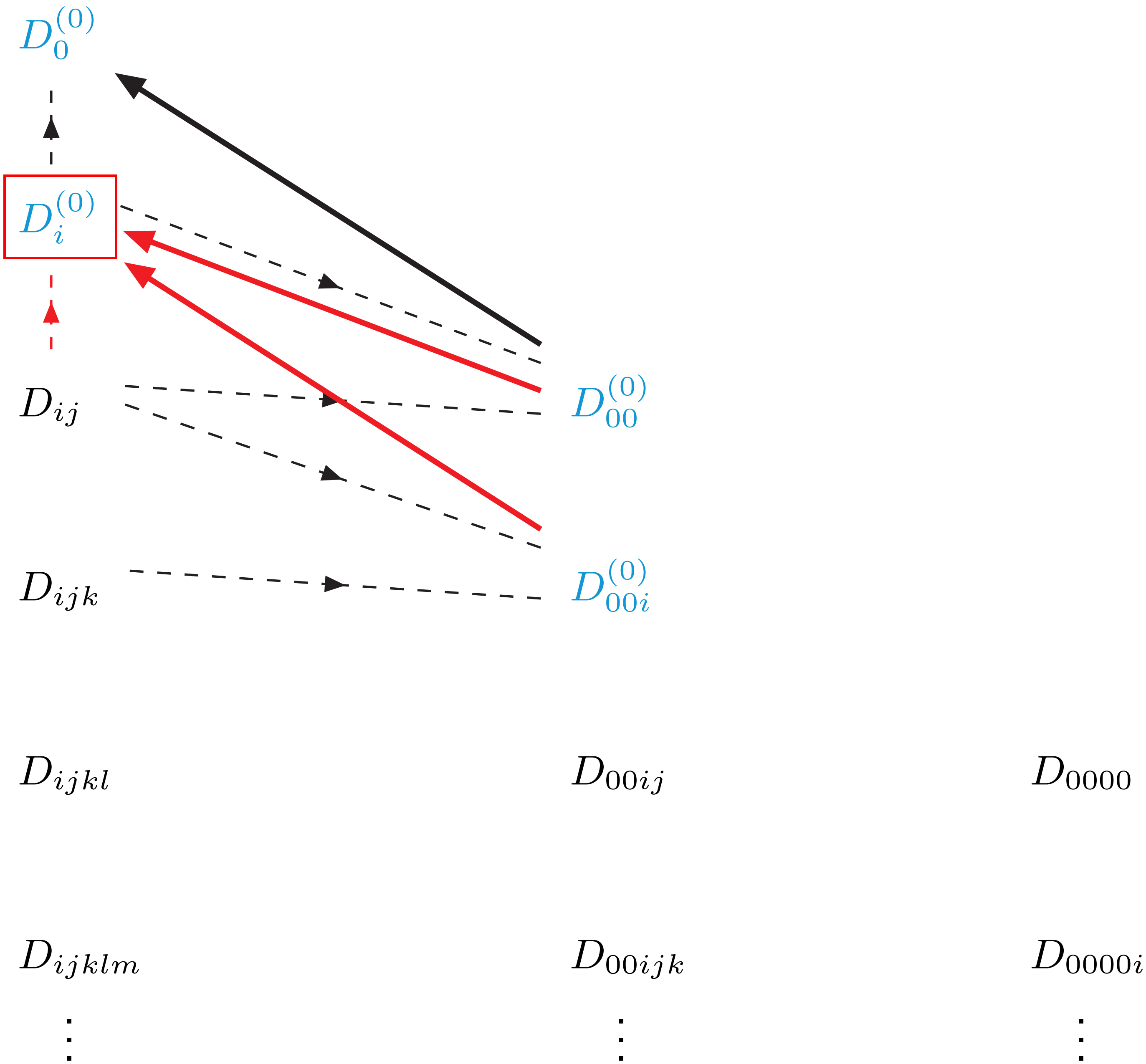
Expansion for small Gram and modified Cayley determinants: step 0b

$$
\begin{tikzcd}
D_0 & & \\
D_i \arrow[dr, dashed] & & \\
D_{ij} \arrow[r, dashed] \arrow[dr, dashed, red] & D^{(0)}_{00} & \\
D_{ijk} \arrow[r, dashed, red] & \boxed{D^{(0)}_{00i}} & \\
D_{ijkl} & D_{00ij} & D_{0000} \\
D_{ijklm} & D_{00ijk} & D_{0000i} \\
\vdots & \vdots & \vdots
\end{tikzcd}
$$

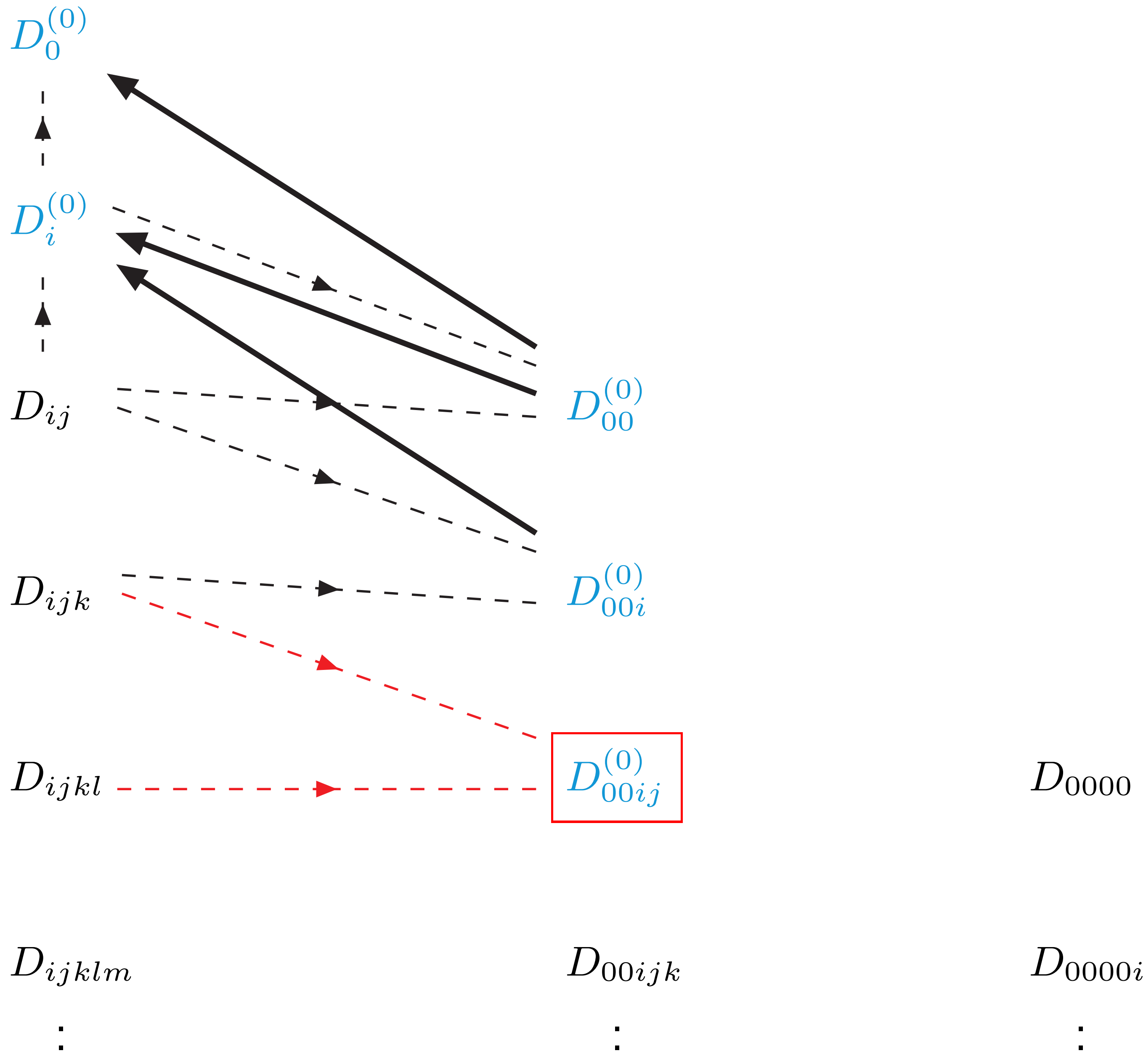
Expansion for small Gram and modified Cayley determinants: step 0c

$\boxed{D_0^{(0)}}$		
D_i		
D_{ij}	$D_{00}^{(0)}$	
D_{ijk}	$D_{00i}^{(0)}$	
D_{ijkl}	D_{00ij}	D_{0000}
D_{ijklm}	D_{00ijk}	D_{0000i}
$\vdots$	$\vdots$	$\vdots$

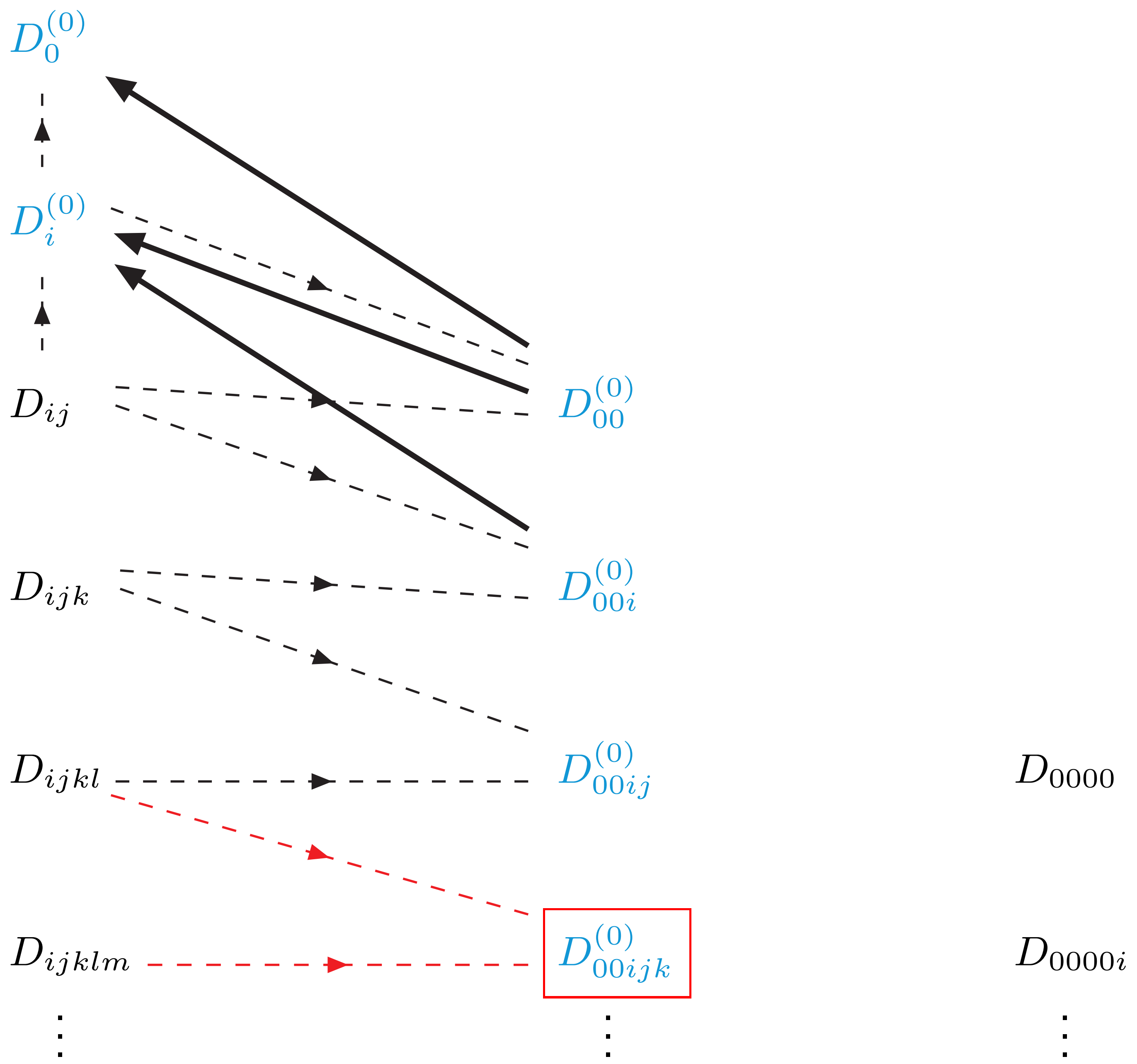
Expansion for small Gram and modified Cayley determinants: step 0d



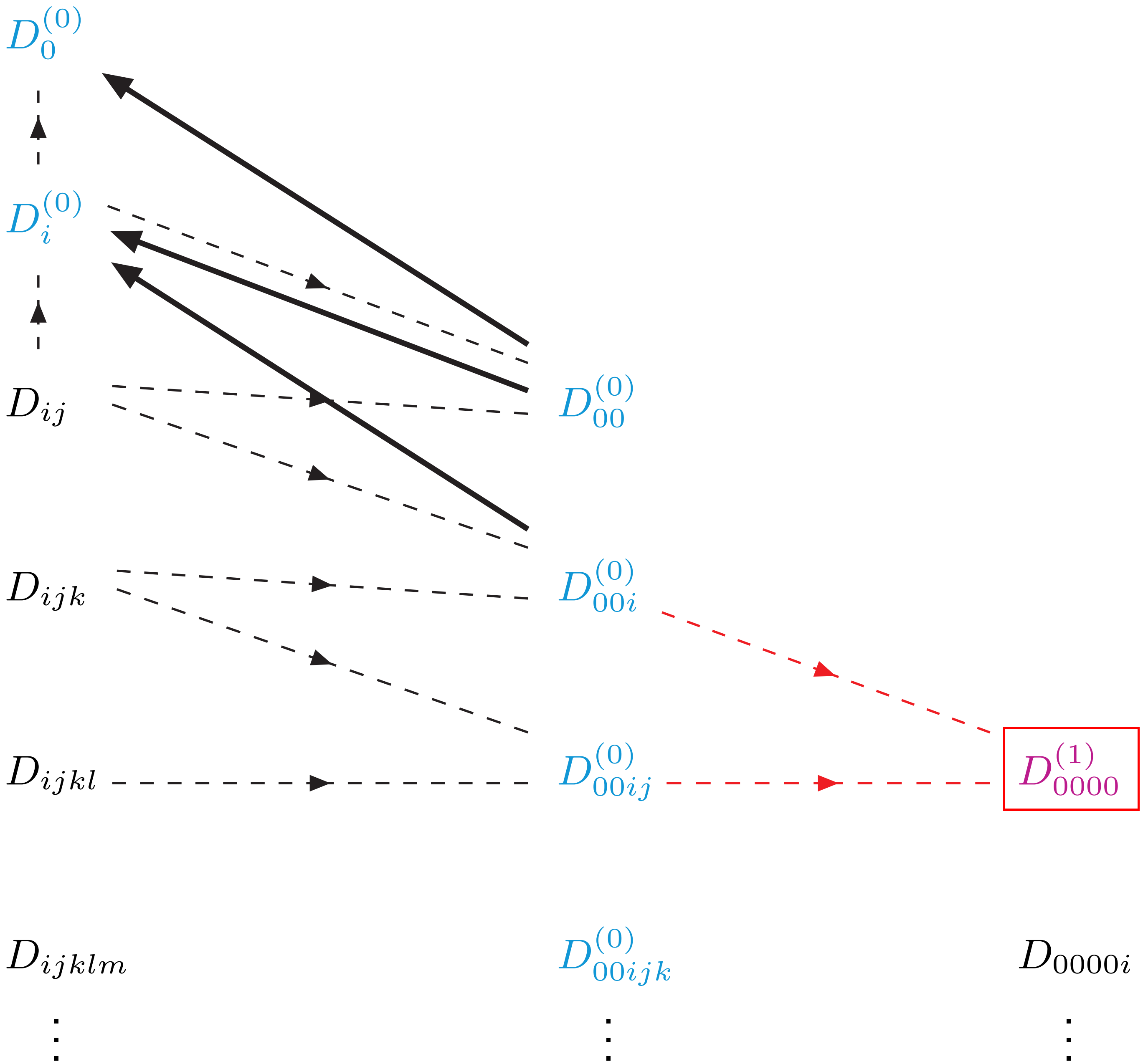
Expansion for small Gram and modified Cayley determinants: step 1a



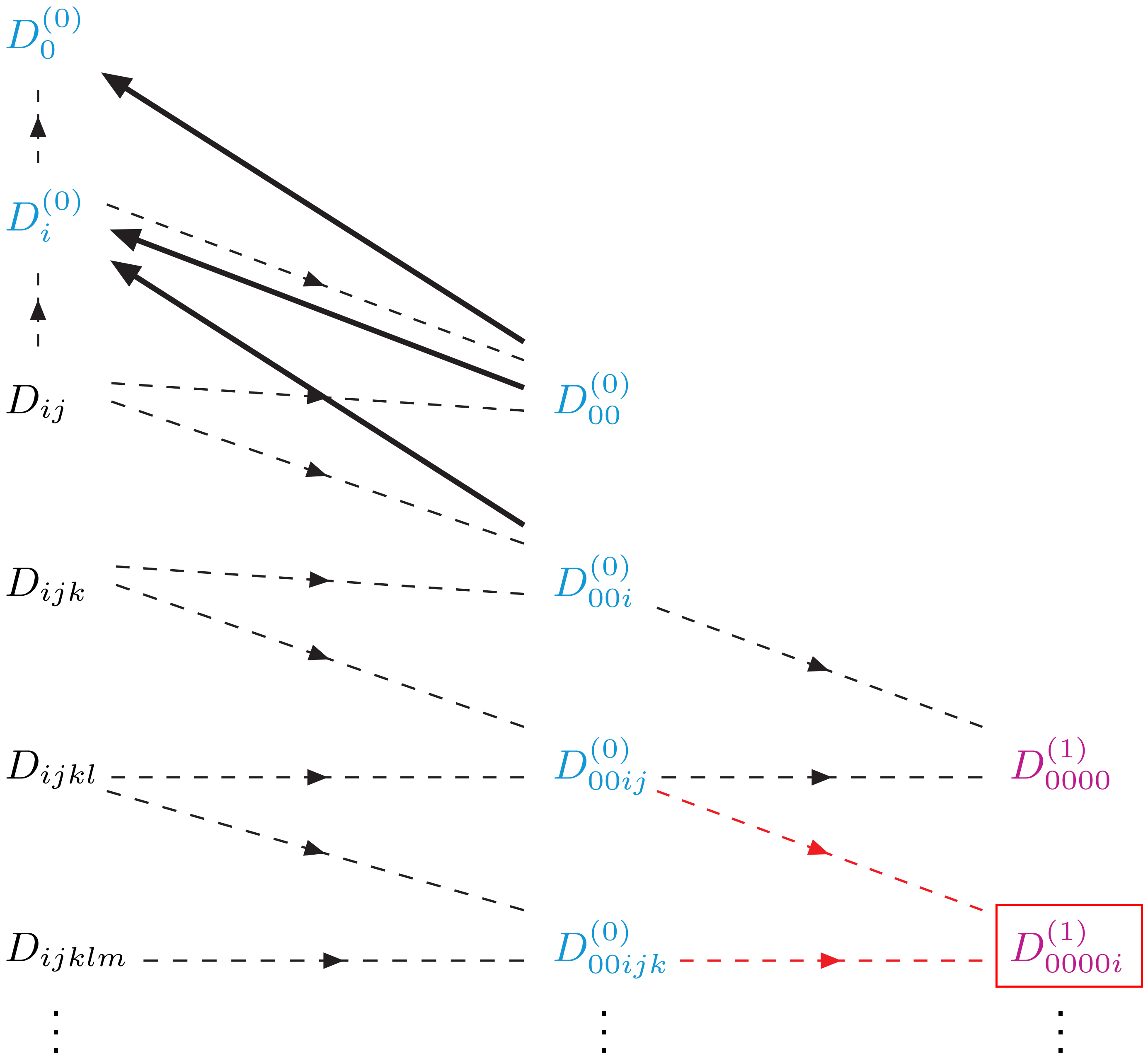
Expansion for small Gram and modified Cayley determinants: step 1b



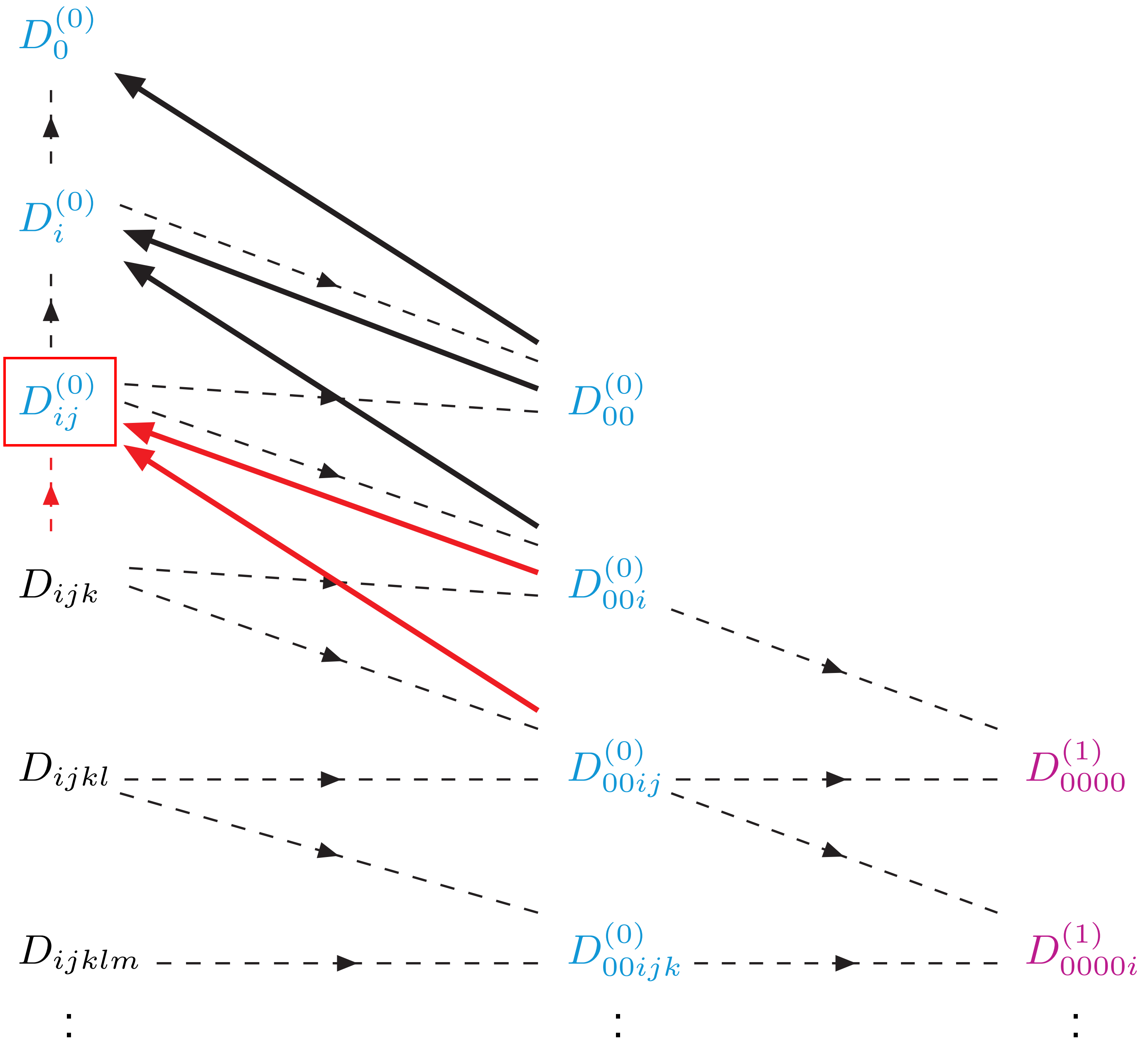
Expansion for small Gram and modified Cayley determinants: step 1c



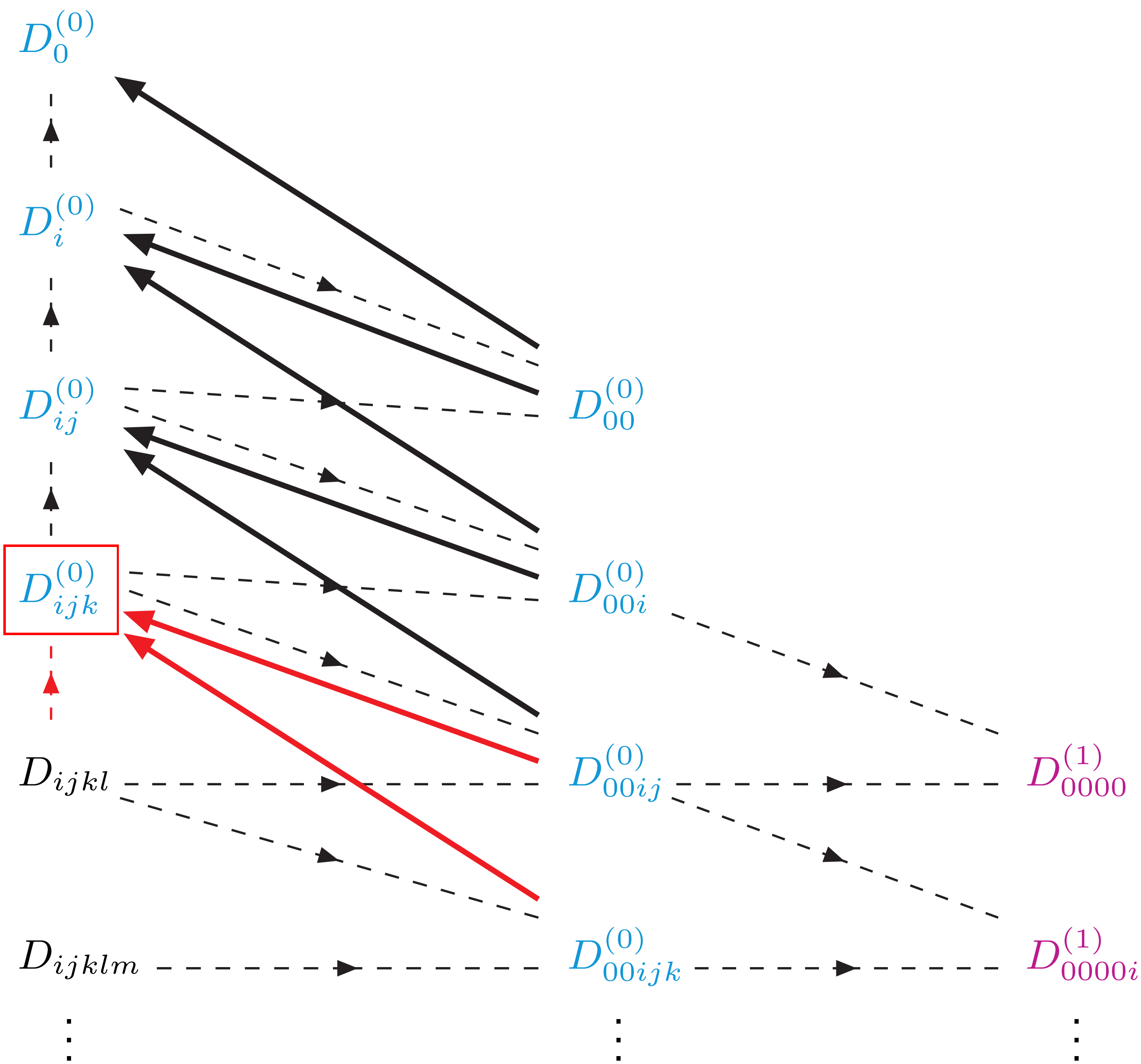
Expansion for small Gram and modified Cayley determinants: step 1d



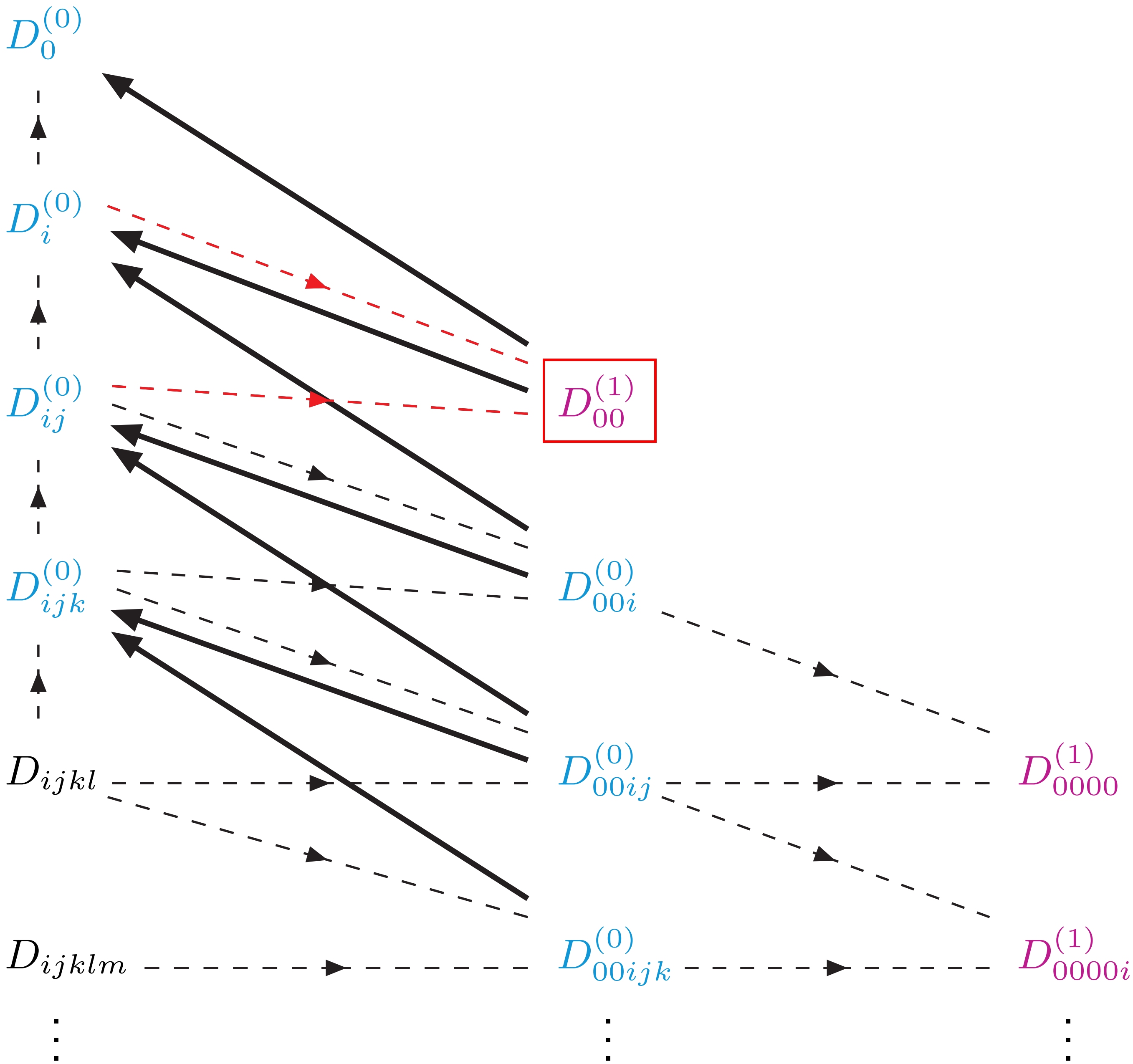
Expansion for small Gram and modified Cayley determinants: step 1e



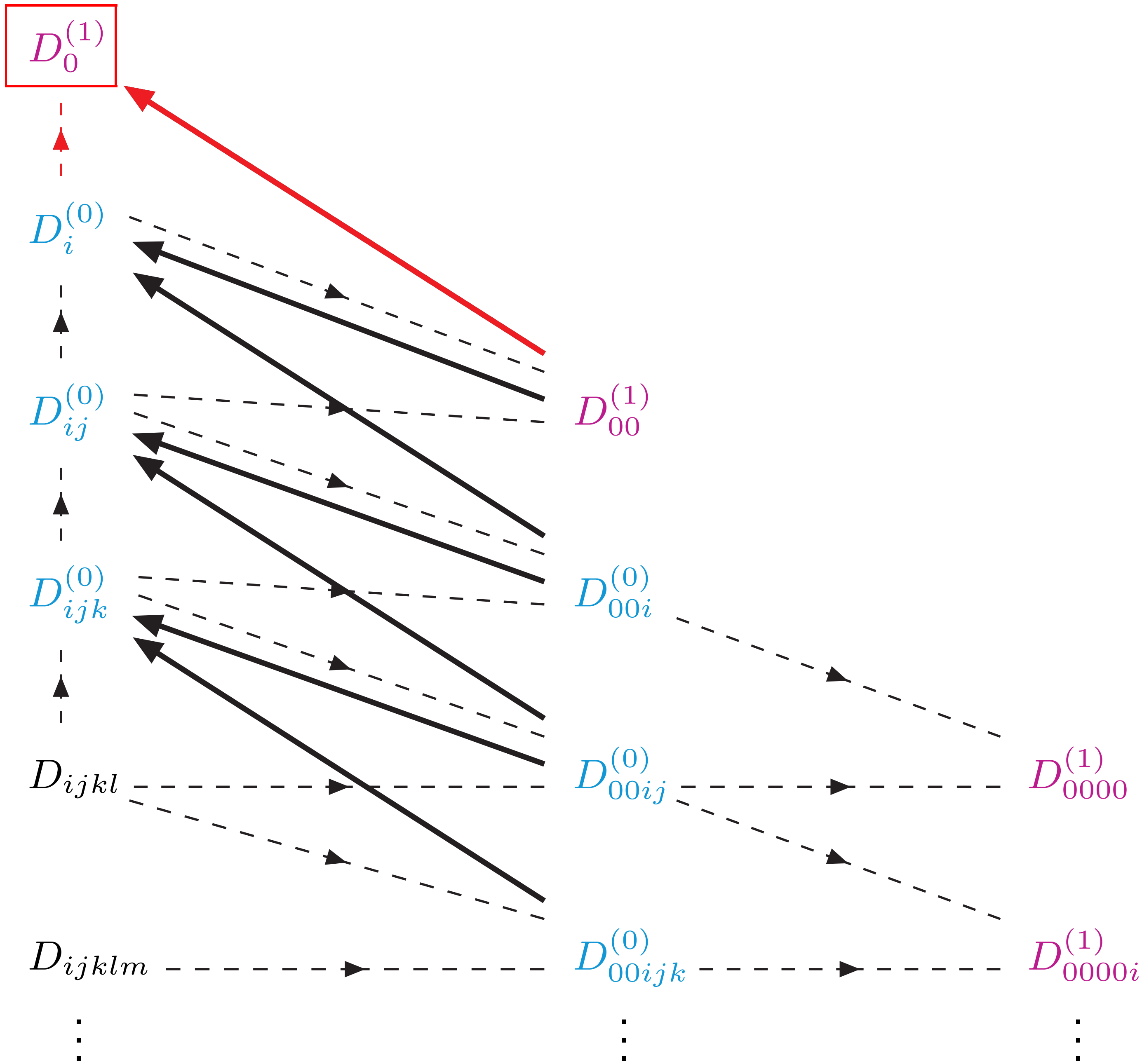
Expansion for small Gram and modified Cayley determinants: step 1f



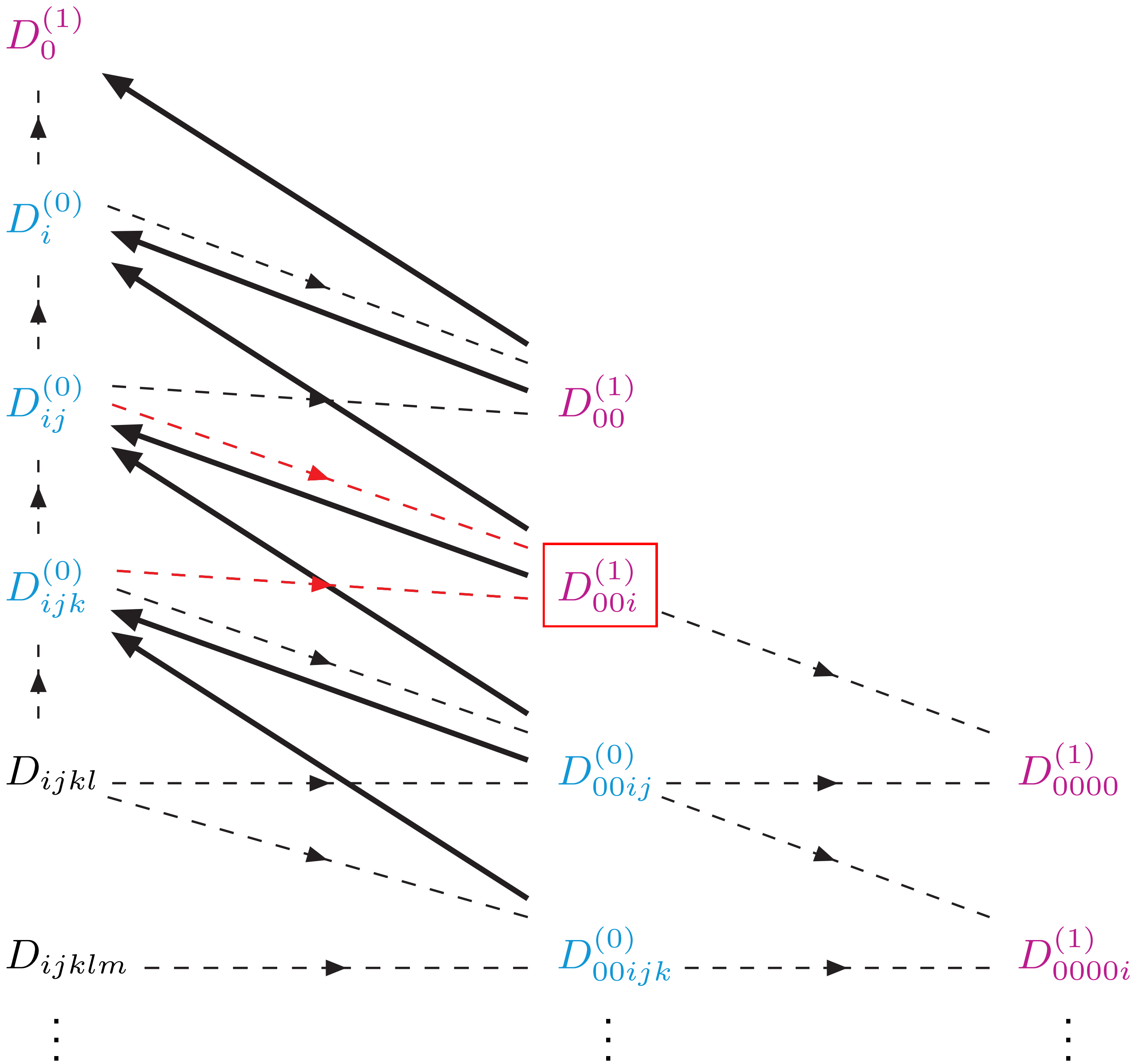
Expansion for small Gram and modified Cayley determinants: step 1.0a



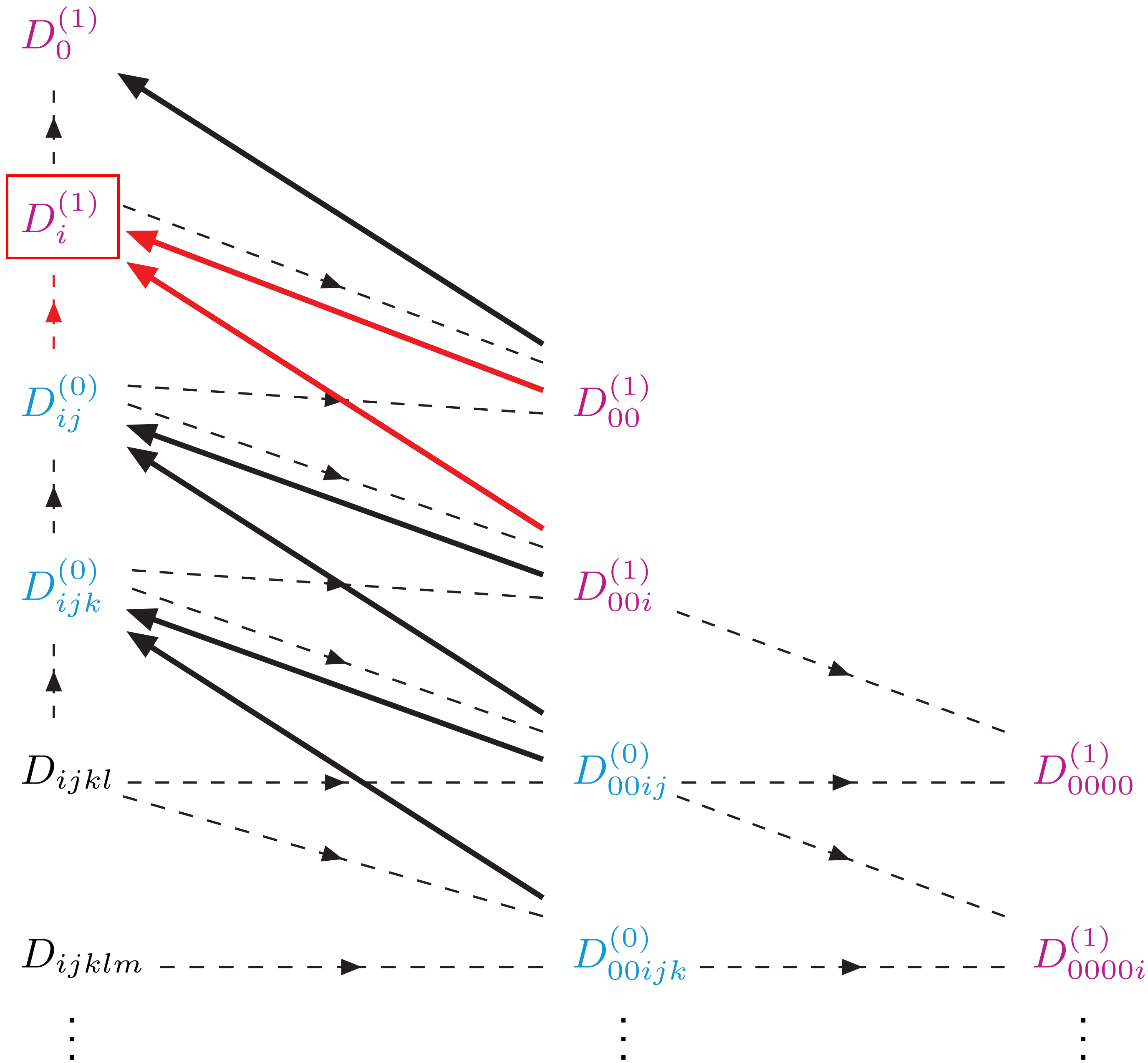
Expansion for small Gram and modified Cayley determinants: step 1.0b



Expansion for small Gram and modified Cayley determinants: step 1.0c

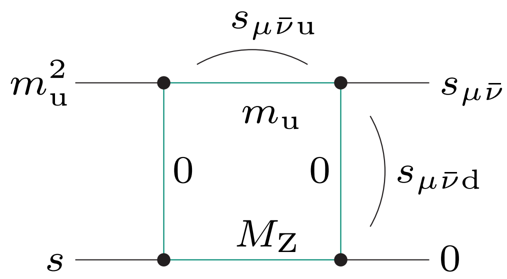


Expansion for small Gram and modified Cayley determinants: step 1.0d

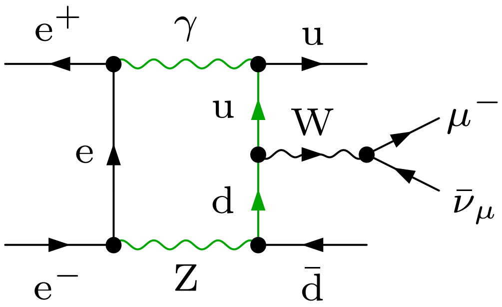


A very delicate example with small Gram and mod. Cayley determinants:

Box integral

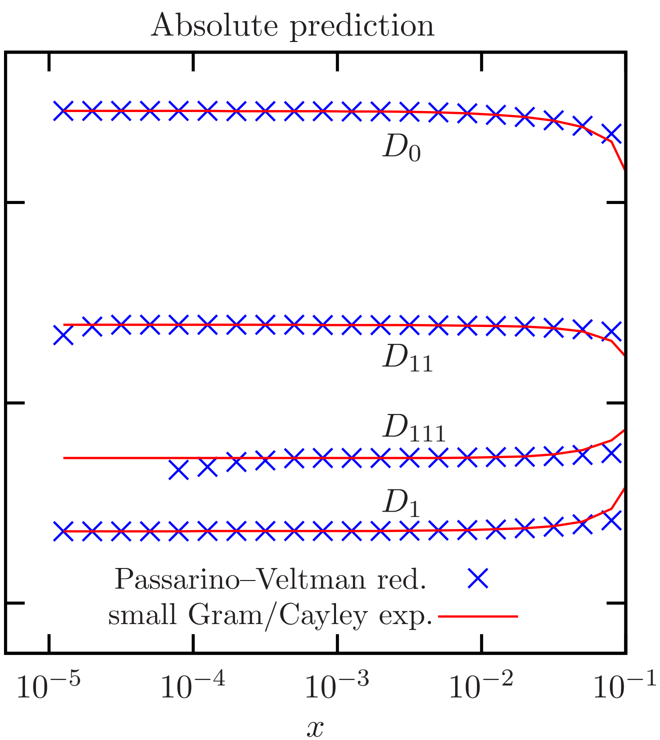


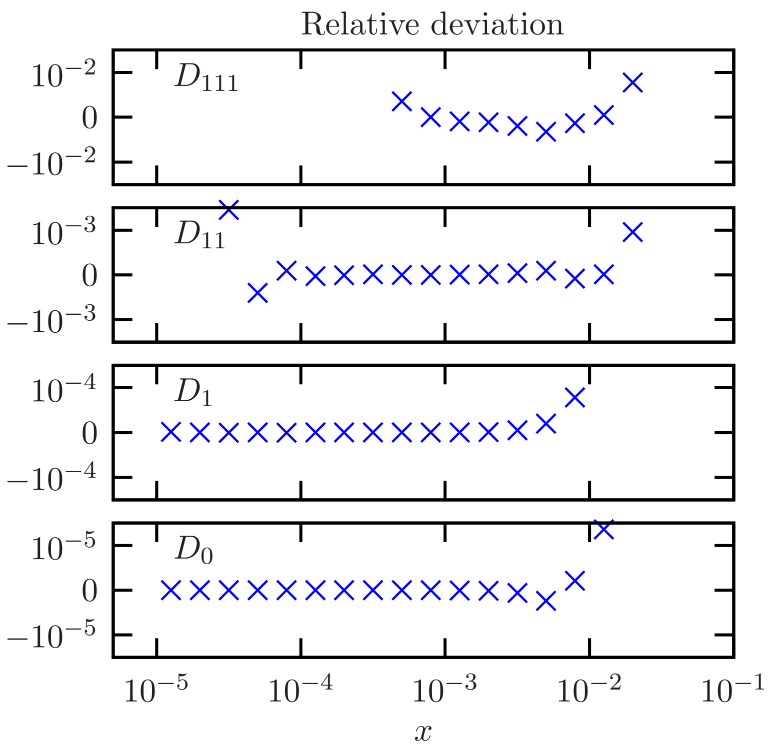
appears, e.g., in subgraph of diagram



Gram det.: $\det(Z), \det(X) \to 0$ if $s_{\mu\bar{\nu}\mathrm{d}} \to s$ and $s_{\mu\bar{\nu}\mathrm{u}} \to s_{\mu\bar{\nu}}$

Numerical comparison:





$$x \equiv \frac{s_{\mu\bar{\nu}\mathrm{d}}}{s} - 1 \equiv \frac{s_{\mu\bar{\nu}\mathrm{u}}}{s_{\mu\bar{\nu}}} - 1$$

$$s = 4\times10^4\ \mathrm{GeV}^2$$
$$s_{\mu\bar{\nu}} = 64\times10^2\ \mathrm{GeV}^2$$

PV reduction breaks down, but Gram/Cayley exp. stable for $\det(Z), \det(X) \to 0$!

5 Reduction schemes for 5- and 6-point integrals

General idea: reduce a determinant that is zero in 4 space–time dimensions

$\hookrightarrow$ relation between 5-(6-)point and 4-(5-)point integrals

5.1 5-point integrals

Starting point:

$$\int \mathcal{E} = \frac{(2\pi\mu)^{4-D}}{\mathrm{i}\pi^2} \int \mathrm{d}^D q \, \frac{q^{\mu_1} \cdots q^{\mu_P}}{N_0 N_1 \dots N_4} \begin{vmatrix} q^\mu & -2q^2 & 2qp_1 & \dots & 2qp_4 \\ 0 & 2m_0^2 & f_1 & \dots & f_4 \\ p_1^\mu & -2p_1 q & 2p_1 p_1 & \dots & 2p_1 p_4 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ p_4^\mu & -2p_4 q & 2p_4 p_1 & \dots & 2p_4 p_4 \end{vmatrix}$$

$$= 2m_0^2 \det(Z) (g^\mu_\alpha - {g_{(4)}}^\mu_\alpha) E^{\alpha\mu_1\dots\mu_P}$$

$$+ 2\sum_{n=1}^{4} \tilde{X}_{n0} \Big[p_n^\mu (\underbrace{g_{\alpha\beta}}_{D\text{-dim}} - \underbrace{{g_{(4)}}_{\alpha\beta}}_{\text{4-dim (built of momenta)}}) - p_{n,\beta} (g^\mu_\alpha - {g_{(4)}}^\mu_\alpha) \Big] E^{\alpha\beta\mu_1\dots\mu_P}$$

$$= \begin{cases} \mathcal{O}(D-4) & \text{for} \quad P \le 3 \\ \text{finite, but simply calculable} & \text{for} \quad P > 3 \end{cases}$$

Reduction of determinant in momentum space (simple manipulations)

$\hookrightarrow$ propagator cancellations lead to 4-point integrals

$$\int \mathcal{E} = \det(X) E^{\mu\mu_1\cdots\mu_P} - \sum_{n,m=1}^{4} \tilde{X}_{mn} p_m^\mu \Big[D^{\mu_1\cdots\mu_P}(n) - D^{\mu_1\cdots\mu_P}(0) \Big]$$
$$- \sum_{n=1}^{4} \tilde{X}_{n0} \Big[-p_n^\mu D^{\mu_1\cdots\mu_P}(0) + \mathcal{D}^{\mu\mu_1\cdots\mu_P}(n) \Big]$$
$$+ \sum_{n=1}^{4} \mathcal{D}^{\alpha\mu_1\cdots\mu_P}(n) \sum_{m,l=1}^{4} 2p_{m,\alpha} p_l^\mu \tilde{\tilde{X}}_{(ln)(0m)}$$

(similar result recently obtained by Binoth et al. '05)

$\hookrightarrow$ Tensor coefficients $E_{\dots}$ read off upon comparing coefficients of covariants

Comments:

- reduction of rank: $E^{\mu\mu_1\cdots\mu_P}$ $\to$ rank $(P+1)$
  D's / $\mathcal{D}$'s $\to$ directly obtained from rank-P integrals
- no inverse Gram determinant, but $E^{\mu\mu_1\cdots\mu_P} = [\dots]/\det(X)$
- reduction works for massive/massless case in any IR regularization

Explicit results for 5-point tensor coefficients:

$$\det(X)E_{i_1} = \sum_{n=1}^{4} \tilde{X}_{i_1 n}\Big[D_0(n) - D_0(0)\Big] - \tilde{X}_{i_1 0} D_0(0),$$

$$\det(X)E_{00} = \sum_{n=1}^{4} \tilde{X}_{n0}\Big[D_{00}(n) - D_{00}(0)\Big],$$

$$2\det(X)E_{i_1 i_2} = \Bigg\{\sum_{n=1}^{4} \tilde{X}_{i_1 n}\Big[D_{(i_2)_n}(n)\bar{\delta}_{i_2 n} - D_{i_2}(0)\Big] - \tilde{X}_{i_1 0} D_{i_2}(0)$$
$$- 2\sum_{n=1}^{4} \tilde{\tilde{X}}_{(i_1 n)(0 i_2)}\Big[D_{00}(n) - D_{00}(0)\Big]\Bigg\} + (i_1 \leftrightarrow i_2),$$

etc., explicitly worked out up to rank 5

Scalar integral via method of Melrose '65 (Denner '93; Denner, S.D. '02):

$$\det(X)E_0 = -\sum_{n=0}^{4} \det(Y_n) D_0(n), \qquad Y_n = \text{kinematical matrices related to } X$$

5.2 6-point integrals

Starting point:

$$\int \mathcal{F} = \frac{(2\pi\mu)^{4-D}}{\mathrm{i}\pi^2} \int \mathrm{d}^D q \, \frac{q^{\mu_1} \cdots q^{\mu_P}}{N_0 N_1 \ldots N_5} \begin{vmatrix} q^\mu & 2qp_1 & \ldots & 2qp_5 \\ p_1^\mu & 2p_1p_1 & \ldots & 2p_1p_5 \\ \vdots & \vdots & \ddots & \vdots \\ p_{k-1}^\mu & 2p_{k-1}p_1 & \ldots & 2p_{k-1}p_5 \\ 0 & f_1 & \ldots & f_5 \\ p_{k+1}^\mu & 2p_{k+1}p_1 & \ldots & 2p_{k+1}p_5 \\ \vdots & \vdots & \ddots & \vdots \\ p_5^\mu & 2p_5p_1 & \ldots & 2p_5p_5 \end{vmatrix}$$

$$= -\tilde{X}_{k0} F^{\alpha\mu_1\ldots\mu_P} \left(g_\alpha^\mu - g_{(4)}{}_\alpha^\mu\right)$$

$$= \begin{cases} \mathcal{O}(D-4) & \text{for} \quad P \le 6 \\ \text{finite, but simply calculable} & \text{for} \quad P > 6 \end{cases}$$

Reduction of determinant in momentum space

$$\int \mathcal{F} = -\tilde{X}_{k0} F^{\mu\mu_1\ldots\mu_P} - \sum_{n,m=1}^{5} \tilde{\tilde{X}}_{(km)(0n)} p_m^\mu \Big[E^{\mu_1\ldots\mu_P}(n) - E^{\mu_1\ldots\mu_P}(0) \Big]$$

(index k can be chosen to optimize stability)

$\hookrightarrow$ reduction of rank and applicability for arbitrary IR regularization as in 5-pt case

Explicit results for 6-point tensor coefficients:

$$
\begin{aligned}
F_{i_1} &= \sum_{n=1}^{5} c_{i_1 n} \Big[E_0(n) - E_0(0) \Big], \\
F_{00} &= 0, \\
F_{i_1 i_2} &= \frac{1}{2} \sum_{n=1}^{5} \Big\{ c_{i_1 n} \Big[E_{(i_2)_n}(n) \bar{\delta}_{i_2 n} - E_{i_2}(0) \Big] + (i_1 \leftrightarrow i_2) \Big\}, \\
F_{00 i_1} &= \frac{1}{3} \sum_{n=1}^{5} c_{i_1 n} \Big[E_{00}(n) - E_{00}(0) \Big], \\
F_{i_1 i_2 i_3} &= \frac{1}{3} \sum_{n=1}^{5} \Big\{ c_{i_1 n} \Big[E_{(i_2)_n (i_3)_n}(n) \bar{\delta}_{i_2 n} \bar{\delta}_{i_3 n} - E_{i_2 i_3}(0) \Big] + (i_1 \leftrightarrow i_2) + (i_1 \leftrightarrow i_3) \Big\},
\end{aligned}
$$

etc., $\qquad c_{i_1 n} =$ related to inverse matrix X^{-1}

Scalar integral via method of Melrose '65 (Denner '93):

$$
\det(X) F_0 = - \sum_{n=0}^{5} \det(Y_n) E_0(n), \qquad Y_n = \text{kinematical matrices related to } X
$$

6 Conclusions

NLO corrections to $2 \to 4$ processes are now feasible

- preliminary NLO EW results for $\mathrm{e}^+\mathrm{e}^- \to \nu\bar{\nu}\mathrm{HH}$ — GRACE-loop '04/'05
- complete NLO EW results for $\mathrm{e}^+\mathrm{e}^- \to 4$ fermions (CC) — Denner, S.D., Roth, Wieders, '05

Techniques described in this talk successfully applied to $\mathrm{e}^+\mathrm{e}^- \to 4f$

- 1- and 2-point integrals $\to$ stable direct calculation
- 3- and 4-point integrals $\to$ two hybrid methods

  (i) Passarino–Veltman $\oplus$ seminumerical method $\oplus$ analytical special cases

  (ii) Passarino–Veltman $\oplus$ expansions in small Gram and other kin. determinants
- 5- and 6-point integrals

  $\hookrightarrow$ stable reduction to lower-point integrals without Gram determinants

$\Rightarrow$ Techniques ready for further applications

(dim. regularization for IR singularities possible; complex masses supported)

Practical experience

- Phase-space integration reveals weaknesses of methods.
- Power + reliability of techniques can only be assessed via non-trivial applications !