Third Order Coulomb Correction to $t\bar{t}$ Threshold Cross Section

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Talk based on M.Beneke, Y. Kiyo, K. Schuller NPB714(2005)67

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Outline

Part I

Introduction

ILC enables us to determine the top quark mass precisely:

- top mass $\delta m_t \leq 50$ MeV Martinez-Miquel(02)
- α_s, Γ_t, y_t can be also determined

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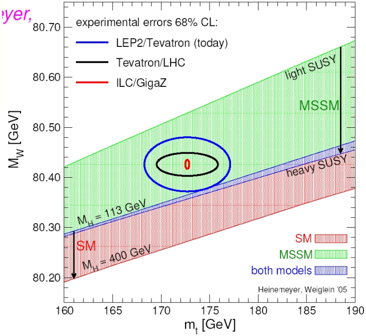
Fig from talk by G. Weiglein at Snowmass 2005



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$$\Downarrow$$

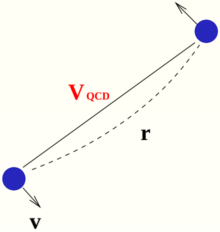
Precise Theory prediction is necessary to match Exp accuracy

$\longrightarrow$ We aim few % accuracy

Threshold production of $t\bar{t}$

- threhold energy $\sqrt{s} \sim 2m_t$
- beam profile is important for threshold scan

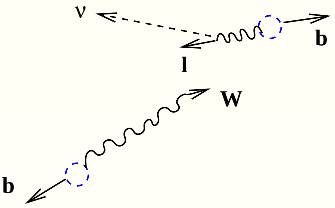
Threshold production of $t\bar{t}$



- nonrelativistic top: $v \sim \alpha_s \sim 0.16$
- $t\bar{t}$ in QCD potential: $V_{\mathrm{QCD}}(r) = -\dfrac{C_F \alpha_s}{r}$
- short distance bounding: $r \sim (m_t \alpha_s)^{-1} \ll \Lambda_{\mathrm{QCD}}^{-1}$

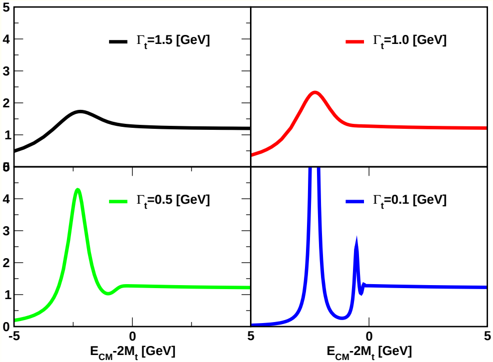
  $\rightarrow$ NR QCD is our tool

Threshold production of $t\bar{t}$



- quick decay: $t \to bW$
  $\to$ hadronization effect does NOT enter
  Bigi-Dokshitzer-Khoze-Kuhn-Zerwas('86)
- effect of unstableness: Γ_t
  $\to$ interplay between QCD and EW corrections
  (naive way is a replacement: $m_t \to m_t + i\Gamma_t$)

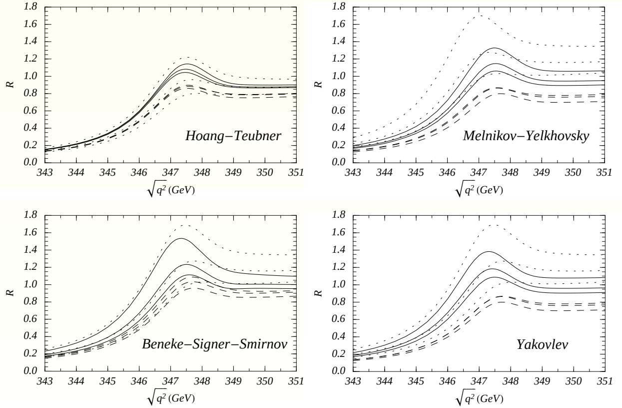
$t\bar{t}$ threshold X section with Γ_t



$R(e^+e^- \to t\bar{t})$ varying Γ_t

- $e^+e^- \to \gamma^* \to t\bar{t}$ (No Z^* for simplicity)
- here and hereafter only the QCD effect is discussed
- finite width effect: $\sqrt{s} \to \sqrt{s} + i\Gamma_t$

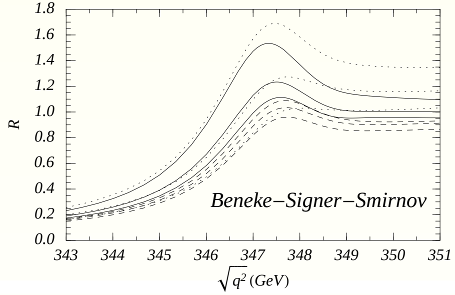
R_{NNLO} from Top Quark Working Group Report



Top Quark WG Report by A.H. Hoang et al.(2000)

- figs by 4 Groups: Scheme difference
- solid/dashed/dotted lines : NNLO/NLO/LO
- $\mu = 15, 30, 60$ GeV

What do we know? What should we do?



So, going to N^3LO corrections is important to achieve few % theoretical prediction of $t\bar{t}$ threshold cross section.

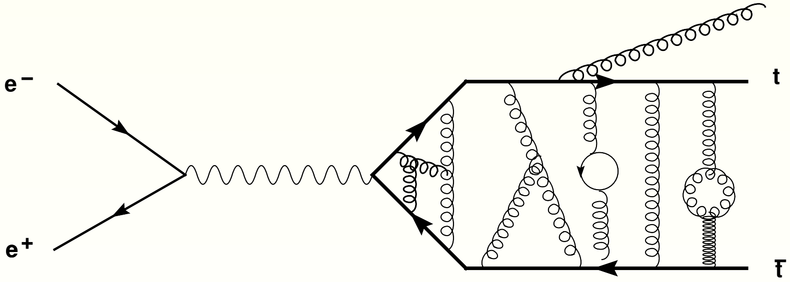
- How large sheme/scale dependence?
- Nonrelativistic 1/c-expansion works ? Necessary to sum up ?
- Do we know the size of third order? $\Leftarrow$ Ultra Soft corrections

Part II

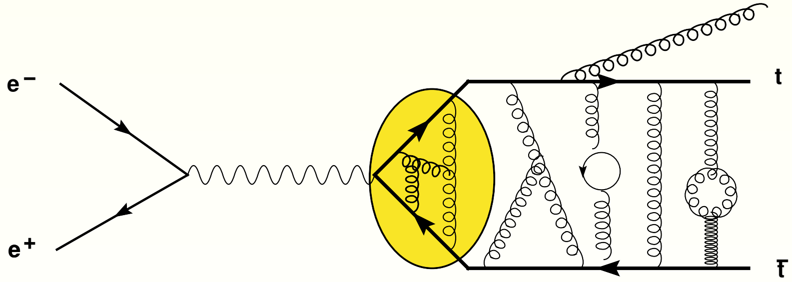
NNNLO Threshold Cross Section

Review of EFT Approach

QCD Corrections



QCD Corrections



- Hard $\gamma - t - \bar{t}$ vertex corrections (3loop=NNNLO)

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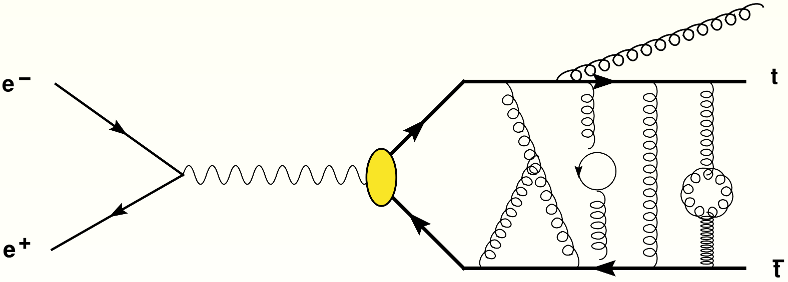

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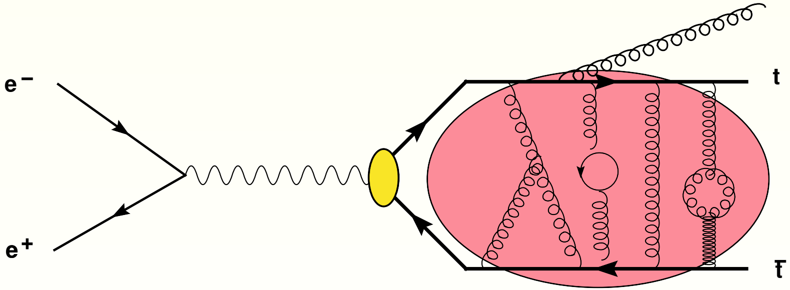

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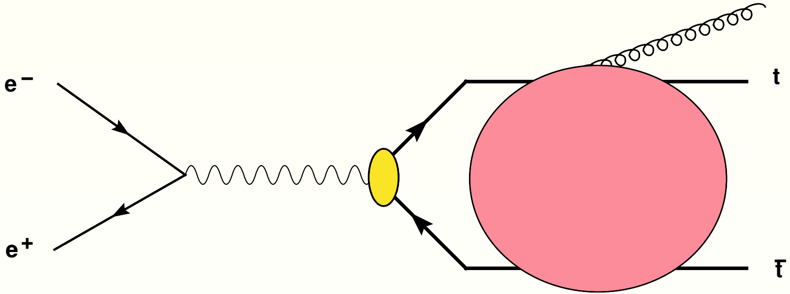

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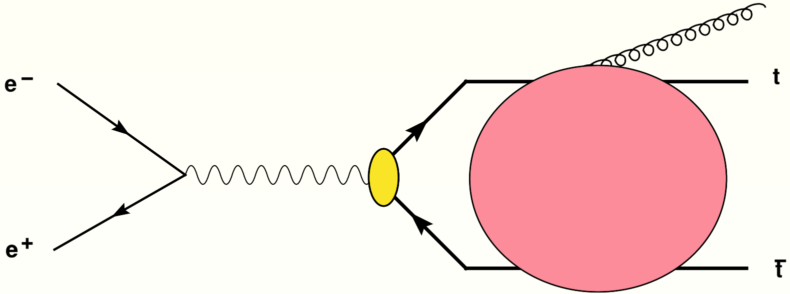

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We discuss $t\bar{t}$ boundstate dynamics

Potential NRQCD

- Integrate out Hard

$$\mathcal{L}_{\rm NRQCD} = \psi^\dagger \left(iD_0 + \frac{\vec{D}^2}{2m_t} \right) \psi + [\psi \to \chi] + \cdots$$

$$\Downarrow$$

- Integrate out Soft/Potentia gluons

(Pineda-Soto('98); Luke-Manohar-Rothstein('99))

$$\delta\mathcal{L}_{\rm pNRQCD} = \int d\vec{r}\, [\psi^\dagger \chi]\, V_{pot}(r)\, [\chi^\dagger \psi]$$

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* EFT $= t\bar{t}$-Potential and low energy gluon(Ultra soft gluon)

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* EFT = $t\bar{t}$-Potential and low energy gluon(Ultra soft gluon)

* $V_{pot} = -\dfrac{C_F \alpha_s(\mu)}{r}$ + Higher Order Corr

Potentials are Wilson Coeff: $V_{pot}(r)\left[\psi^\dagger\chi\right](r)\left[\chi^\dagger\psi\right](0)$

- $V_{pot} = -\frac{C_F\alpha_s}{r} + \frac{C_2}{r^2} + C_3\delta(\mathbf{r}) + \cdots$
- Corr to the Coulomb potential

- Every potential (except a_3) is known to 3rd order
  - a_2 Schröder('99)
  - $a_{3,pade}$ Chishtie-Elias (01)
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$\Rightarrow$ cancels UltraSoft UV divergence

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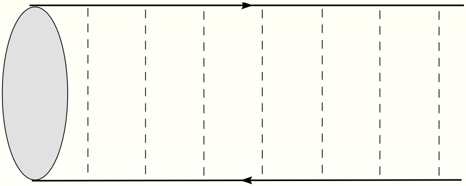

Coulomb gluons to be summed up

Diagrams for NNNLO

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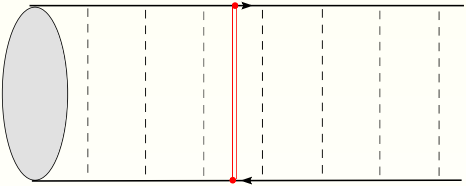

single insertion of δV

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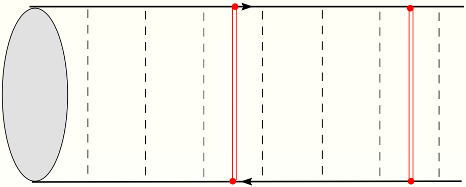

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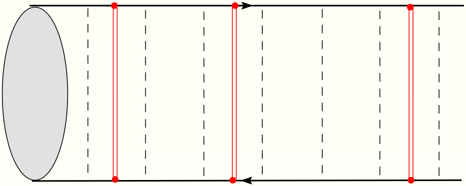

triple insertion of δV

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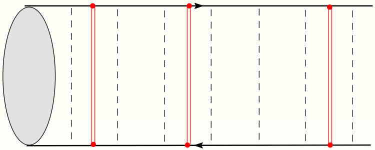

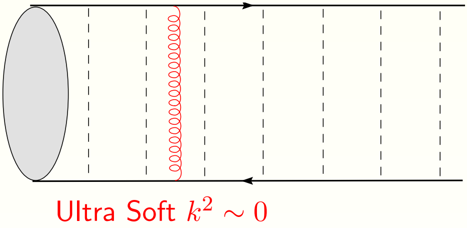

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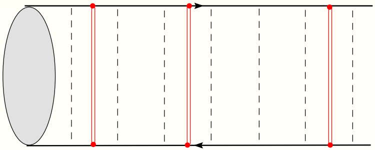

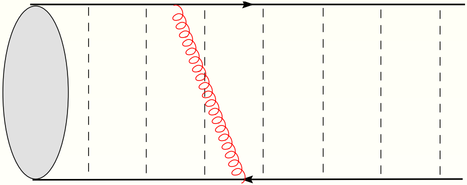

Ultra Soft $k^2 \sim 0$

Diagrams for NNNLO

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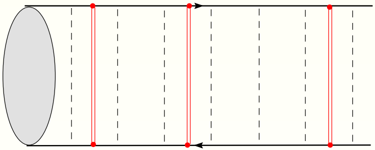

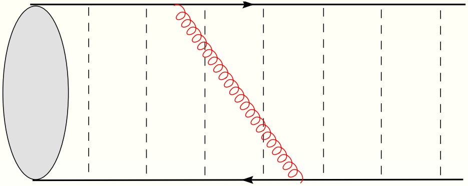

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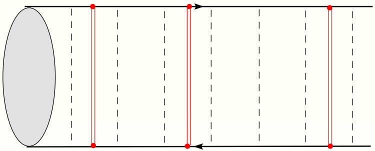

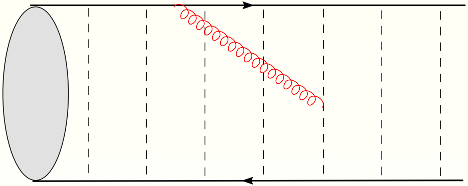

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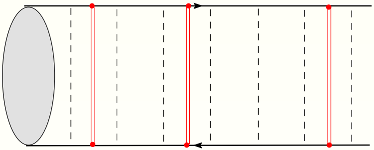

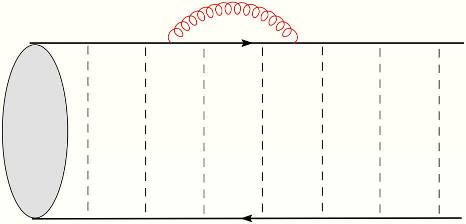

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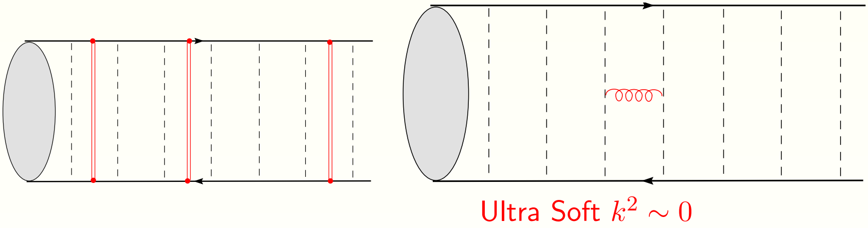

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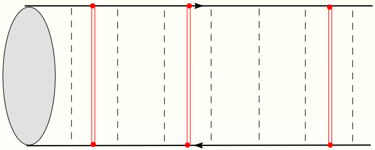

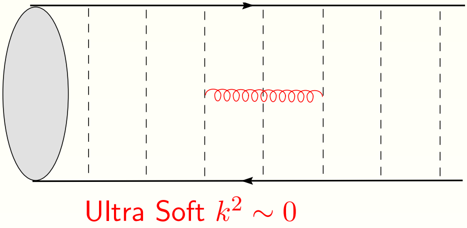

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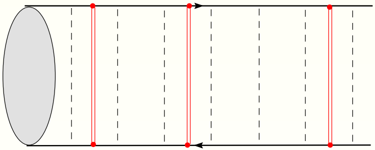

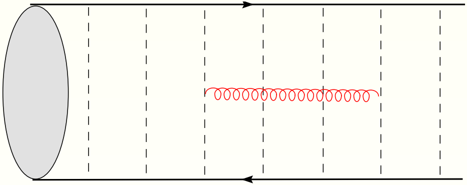

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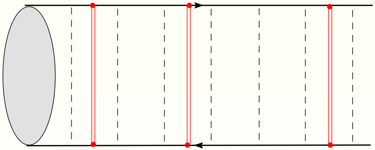

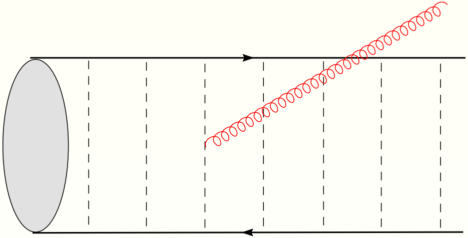

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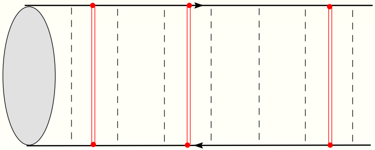

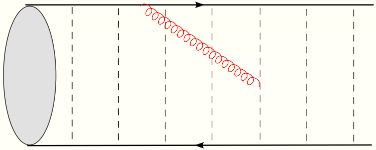

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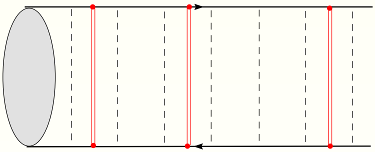

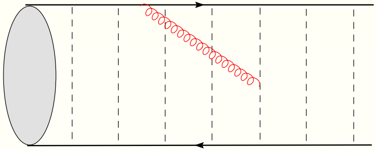

Ultra Soft $k^2 \sim 0$

- Coulombic pot insertions $\Rightarrow$ Complete (Beneke-YK-Schuller(05))
- Singular pot insertions $\Rightarrow$ UV renormalization is necessary (work in progress with Beneke-Schuller)
- Ultrasoft Corr $\Rightarrow$ UV renormalization is necessary (work in progress with Beneke-Penin)

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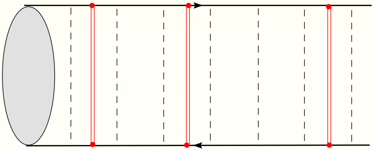

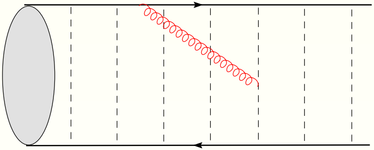

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Part III

Coulomb Corrections

Based on Beneke-Kiyo-Schuller, NPB714(2005)67.

(Coulomb) X Section near Threshold; $E = \sqrt{s} - 2m_t \sim 0$

$$\sigma_{t\bar{t}} \sim Im\langle 0|\frac{1}{H - E - i\Gamma_t}|0\rangle + \text{non-potential}(\mu_{US})$$

$$H = \frac{p^2}{m_t} - \frac{C_F \alpha_s}{r} + \delta V_C(r) + \frac{\#}{r^2} + \#\delta(\vec{r}) + \cdots,$$

- We do NOT discuss singular potentials

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- Coulomb pot: $\delta\tilde{V}_C = -\frac{C_F\alpha_s}{r}$

$$\times\Big[\frac{\alpha_s}{4\pi}\left(\beta_0 \ln(\mu^2/q^2) + a_1\right) + \cdots$$

$$+\left(\frac{\alpha_s}{4\pi}\right)^3 \left(\clubsuit \ln^3 + \clubsuit \ln^2 + \clubsuit \ln + a_3 + 8\pi^2 C_A^3 \ln(\mu_{US}^2/q^2)\right)\Big]$$

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  $\times\Big[\frac{\alpha_s}{4\pi}\left(\beta_0 \ln(\mu^2/q^2) + a_1\right) + \cdots$
  $+\left(\frac{\alpha_s}{4\pi}\right)^3\left(\clubsuit \ln^3 + \clubsuit \ln^2 + \clubsuit \ln + a_3 + 8\pi^2 C_A^3 \ln(\mu_{US}^2/q^2)\right)\Big]$
- δV_C is renormalization free $\Rightarrow$ Num evaluation is possible

Perturbation in QM: $H_0 = \frac{p^2}{m_t} - \frac{C_F \alpha_s}{r}$

Iteration of the Coulomb potentials

$$\langle \frac{1}{H_0 + \delta V_C - E} \rangle = \langle G_0 \rangle - \langle G_0 \delta V_1 G_0 \rangle + \langle G_0 \delta V_1 G_0 \delta V_1 G_0 \rangle + \cdots$$

with $\langle G_0 \rangle = \langle 0 | \frac{1}{H_0 - E} | 0 \rangle$ and $\delta V_C = \delta V_1 + \delta V_2 + \delta V_3$,

- N^3LO Corrections are given by:
  - Single: $\langle G_0 \delta V_3 G_0 \rangle$
  - Double: $\langle G_0 \delta V_1 G_0 \delta V_2 G_0 \rangle$
  - Triple: $\langle G_0 \delta V_1 G_0 \delta V_1 G_0 \delta V_1 G_0 \rangle$

Results

- We have calculated the Green function $\langle \frac{1}{H_0+\delta V_C-E-i\Gamma_t} \rangle$ treating δV_C as iterative interaction
  $\rightarrow$ Result is too lengthy to show in a slide

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  * Expressed by well known special functions

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- Pole structure $\Leftrightarrow$ bound-state: $\sum_n \frac{|\Psi_n(0)|^2}{E_n - E - i\Gamma_t}$

  * Pole and residue were obtained Analytically
  * Expressed by multi-ξ and nested $S_{i,j,k}(n)$
    $\longrightarrow$ This result fit in a page

Coulomb Energy

$$E_n^{(3)} = E_n^{(0)} \left(\frac{\alpha_s}{4\pi}\right)^3 \{[32\beta_0^3 L^3 + L^2 (-56\beta_0^3 + 28\beta_0\beta_1 + 24\beta_0^2 c_{E,1}) + L (16\beta_0^3 - 16\beta_0\beta_1 + 4\beta_2$$

$$- 12\beta_0^2 c_{E,1} + 6\beta_1 c_{E,1} + 8\beta_0 c_{E,2})] + c_{E,3} + 32\pi^2 C_A^3 [\ln\left(\frac{n\mu_{US}}{mC_F\alpha_s}\right) + S_1(n)]\},$$

$S_i = S_i(n) = \sum_k^n \frac{1}{k^i}$ is harmonic-sum

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$$- 12\beta_0^2 c_{E,1} + 6\beta_1 c_{E,1} + 8\beta_0 c_{E,2})] + c_{E,3} + 32\pi^2 C_A^3 [\ln\left(\frac{n\mu_{US}}{mC_F\alpha_s}\right) + S_1(n)]\},$$

Every coefficient of $L = \ln\left(\frac{n\mu}{mC_F\alpha_s}\right)$ is determined by RG

we used this to check the result

Coulomb Energy

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$$c_{E,\,1} = 2a_1 + 4S_1\beta_0,$$

1st Corr

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$$c_{E,\,1} = 2a_1 + 4S_1\beta_0,$$

$$c_{E,\,2} = a_1^2 + 2a_2 + 4S_1\beta_1 + 4a_1\beta_0[3S_1 - 1] + \beta_0^2[S_1(12S_1 - 8 - \frac{8}{n}) + 16S_2 - 8nS_3 + \frac{2\pi^2}{3} + 8n\xi(3)],$$

2nd Corr

Coulomb Energy

$$E_n^{(3)} = E_n^{(0)} \left(\frac{\alpha_s}{4\pi}\right)^3 \{ [32\beta_0^3 L^3 + L^2 (-56\beta_0^3 + 28\beta_0\beta_1 + 24\beta_0^2 c_{E,1}) + L (16\beta_0^3 - 16\beta_0\beta_1 + 4\beta_2$$
$$- 12\beta_0^2 c_{E,1} + 6\beta_1 c_{E,1} + 8\beta_0 c_{E,2})] + c_{E,3} + 32\pi^2 C_A^3 [\ln \left(\frac{n\mu_{US}}{mC_F\alpha_s}\right) + S_1(n)]\},$$

$$c_{E,1} = 2a_1 + 4S_1\beta_0,$$

$$c_{E,2} = a_1^2 + 2a_2 + 4S_1\beta_1 + 4a_1\beta_0[3S_1 - 1] + \beta_0^2[S_1(12S_1 - 8 - \frac{8}{n}) + 16S_2 - 8nS_3 + \frac{2\pi^2}{3} + 8n\xi(3)],$$

$$c_{E,3} = 2a_1a_2 + 2a_3 + 2a_1^2\beta_0[4S_1 - 5] + 4a_2\beta_0[4S_1 - 1] + 4a_1\beta_1[3S_1 - 1] + 4S_1\beta_2 + \beta_0\beta_1[S_1(28S_1$$
$$-16 - \frac{24}{n}) + 36S_2 - 16nS_3 + \frac{7\pi^2}{3} + 16n\xi(3)] + a_1\beta_0^2[S_1(48S_1 - 56 - \frac{32}{n}) + 64S_2 - 32nS_3 + 8 + \frac{8\pi^2}{3}$$
$$+32n\xi(3)] + \beta_0^3[S_1(S_1(32S_1 - 56 - \frac{32}{n}) + 96S_2 - 64nS_3 + 16 + \frac{16}{n} + \frac{32\pi^2}{3} + 64n\xi(3)) + S_2(8nS_2$$
$$+16n^2S_3 - 32 - \frac{16}{n} - \frac{40n\pi^2}{3} - 16n^2\xi(3)) + S_3(96 + 16n + 8n^2\pi^2) - 104nS_4 + 48n^2S_5 - 144S_{2,1}$$
$$+224nS_{3,1} - 32n^2S_{3,2} - 96n^2S_{4,1} - \frac{4\pi^2}{3} + \frac{2n\pi^4}{45} + \xi(3)(32 - 16n - 8n^2\pi^2) + 96n^2\xi(5)]$$

3rd Corr: All $\xi(i, j, k)$ were reduced to $\xi(i)$, $S_{i,j,k} = S_{i,j,k}(n)$ is nested harmonic-sum

Coulomb Wave Func(constant part)

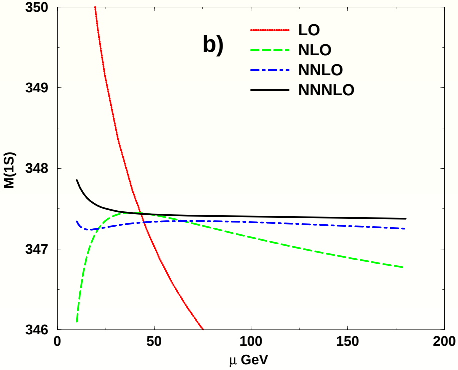
$$
\begin{aligned}
c_{\psi,3} \; = \; & a_1^3 + 6a_1a_2 + 3a_3 + 10a_1^2\beta_0[\,S_1 + 2nS_2 - \frac{31}{10} - \frac{n\pi^2}{3}\,] + 10a_2\beta_0[\,S_1 + 2nS_2 \\
& - \frac{8}{5} - \frac{n\pi^2}{3}\,] + 8a_1\beta_1[\,S_1 + 2nS_2 - \frac{7}{4} - \frac{n\pi^2}{3}\,] + 2\beta_2[\,S_1 + 2nS_2 - 1 - \frac{n\pi^2}{3}\,] \\
& + \beta_0\beta_1[\,S_1(22S_1 + 40nS_2 - 44 - \frac{36}{n} - \frac{20n\pi^2}{3}) + S_2(8n^2S_2 + 14 - 16n - \frac{8n^2\pi^2}{3}) \\
& + 64nS_3 - 40n^2S_4 - 56nS_{2,1} + 32n^2S_{3,1} + 8 + \frac{(21+16n)\pi^2}{6} + \frac{2n^2\pi^4}{9} + 48n\xi(3)\,] \\
& + a_1\beta_0^2[\,S_1(40S_1 + 80nS_2 - 116 - \frac{60}{n} - \frac{40n\pi^2}{3}) + S_2(20n^2S_2 + 40 - 72n - \frac{20n^2\pi^2}{3}) \\
& + 140nS_3 - 100n^2S_4 - 120nS_{2,1} + 80n^2S_{3,1} + 48 + (5+12n)\pi^2 + \frac{5n^2\pi^4}{9} + 100n\xi(3)] \\
& + \beta_0^3[\,S_1(4S_1(4S_1 + 16nS_2 - 19 - \frac{6}{n} - \frac{8n\pi^2}{3}) + 8S_2(3n^2S_2 + 2 - 14n - n^2\pi^2) \\
& + 104nS_3 - 120n^2S_4 - 112nS_{2,1} + 96n^2S_{3,1} + 80 + \frac{64}{n} + \frac{(58+56n)\pi^2}{3} + \frac{2n^2\pi^4}{3} \\
& + 120n\xi(3)) + S_2(-4n(17+2n)S_2 + 72n^2S_3 - 96n^2S_{2,1} + 64n^3S_{3,1} - 96 + 16n \\
& - \frac{24}{n} - \frac{8(5-n)n\pi^2}{3} - 8n^2\xi(3)) + S_3(-16n^3S_3 + 64 - 16n - 20n^2\pi^2 + 32n^3\xi(3))
\end{aligned}
$$

More........

$$+S_4(68n + 40n^2 + \frac{64n^3\pi^2}{3}) - 312n^2 S_5 + 144n^3 S_6 + S_{2,1}(48n - 120 + 16n^2\pi^2)$$
$$-32S_{3,1}(\frac{15n}{2} + n^2 + \frac{n^3\pi^2}{3}) + 384n^2 S_{3,2} + 576n^2 S_{4,1} - 224n^3 S_{4,2} - 256n^3 S_{5,1}$$
$$+256nS_{2,1,1} + 64n^2 S_{2,2,1} - 64n^3 S_{2,3,1} - 448n^2 S_{3,1,1} + 192n^3 S_{4,1,1} - 8 - \frac{8(2+n)\pi^2}{3}$$
$$-\frac{(83+10n)n\pi^4}{45} + \frac{4n^3\pi^6}{105} + \xi(3)(48 - 80n - 12n^2\pi^2 - 16n^3\xi(3)) - 40n^2\xi(5)\,] \,.$$

Toponiumn mass

- Complete NNNLO 1S mass is known by Penin-Steinhauser (2002)
- Extraction of $m_{\overline{\rm MS}}$ was discussed by Sumino-Y.K.(2002).



Sumino-YK(02)

Summary of toponium mass

- N^{3}LO Corr to E_n is complete;
  Beneke-YK-Schuller&Penin-Smirnov-Steinhauser&Kniehl-Penin
- QCD corr to the peak position is under control
  $M_{t\bar{t}(1S)} =$
  $(350 + 0.85_{LO} + 0.05_{NLO} - 0.13_{N^2LO} + 0.01_{N^3LO})\mathrm{GeV}$
  using $m_{t,PS} = 175$, $\mu = 32.6\mathrm{GeV}$, $\Lambda^{(5)}_{\mathrm{QCD}} = 208\mathrm{MeV}$
- Extraction of $\overline{\mathrm{MS}}$ mass needs 4-loop relation between $\overline{\mathrm{MS}}$ pole masses to keep the N^{3}LO accuracy.

Wave function and σ_C

$$\sigma_C \sim \mathrm{Im}\, G(E + i\Gamma_t)$$

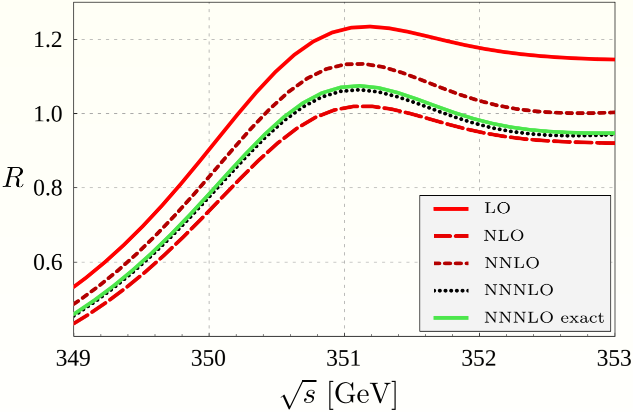
We performed two analyses for threshold cross section taking NNNLO Coulomb corr into account,

- Analytical cal at NNNLO taking $\delta V_{1,2,3}$ as ITERATIVE interaction, dropping followings;

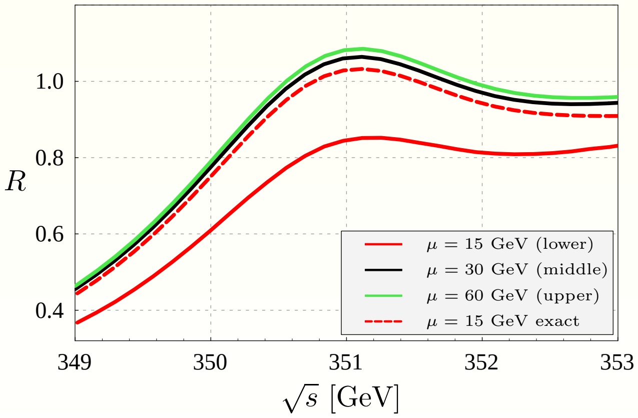
$$\langle \delta V_1 \times \delta V_1 \times \delta V_1 \times \delta V_1 \rangle \sim \text{4th order}$$
$$\langle \delta V_1 \times \delta V_1 \times \delta V_2 \rangle \sim \text{4th order}$$
$$\langle \delta V_1 \times \delta V_3 \rangle \sim \text{4th order}$$
$$\vdots$$

- NUMERICALLY solved schrödinger Eq. with $V = V_C + \delta V_1 + \delta V_2 + \delta V_3 \qquad \equiv$ " N^3LO Exact"

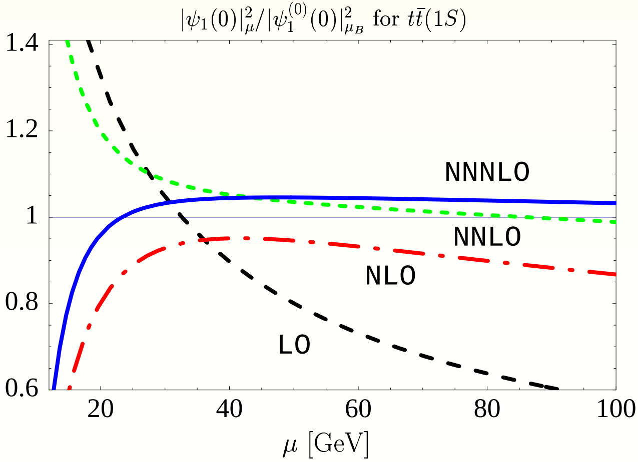
X section; LO→ NLO→ NNLO→NNNLO



X section; X section; scale dependence



Wave function and μ dependence



Summary of Wave func and σ

- N^{3}LO Coulomb Corr to σ, E_n, $|\Psi_n(0)|^2$
- Comparison between analytic and numerical method for σ_C

  * Naively taking $\mu \sim mC_F\alpha_s/n$ is not working.
    Large correction and large μ-dependence
- The strong μ-dependence, $\mu \leq mC_F\alpha_s/n$ is an artifact of iterative method

  $\Leftrightarrow$ "Num Exact" result does NOT have this behavior
- We estimated error; $\Delta\sigma_C \leq 5\%$