Precision Predictions for Deep-Inelastic Scattering

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- Two loops (Methods)

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- Three loops (Results)

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- Three loops (Results)
- Four loops (Applications)

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Introduction

Basic concepts of perturbative QCD

- QCD theory predictions at high energies rely on few basic concepts
 - infrared safety
 - factorization
 - evolution

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Infrared safety

- Small class of cross sections at high energies directly calculable in perturbation theory
- Infrared safe quantities
 - free of long range dependencies at leading power in large momentum scale *Q* Kinoshita '62; Lee, Nauenberg '64
- General structure of cross section
 - large momentum scale Q, renormalization scale μ

$$Q^2 \hat{\sigma} \left(Q^2, \mu^2, \alpha_s(\mu^2) \right) = \sum_n \alpha_s^n c_n(Q^2/\mu^2)$$

Factorization

- Large class of hard-scattering reactions (e.g. initial state hadrons)
 - sensitive to dynamics form different scales (e.g. soft and collinear)
- Structure of factorized cross section
 - large momentum scale Q, factorization scale μ , soft scale m

 $Q^2 \sigma_{\text{phys}} (Q, m) = \hat{\sigma}_{\text{pt}} (Q/\mu, \alpha_s(\mu)) \otimes \phi(\mu, m)$

- \bullet convolution \otimes in suitable kinematical variables
- generalization of operator product expansion

Evolution

- Dependence of cross sections for observable on momentum transfer
- Physical cross section in factorization ansatz cannot depend on µ
 $\mu \frac{d\sigma_{\rm phys}}{d\mu} = 0$ (factorization scale µ arbitrary)
- Classic example: QCD corrections to deep-inelastic scattering
 - scaling violations Gross, Wilczek '73; Politzer '73
 - evolution of parton densities Altarelli, Parisi '77



- Kinematic variables
 - momentum transfer $Q^2 = -q^2$
 - Bjorken variable $x = Q^2/(2p \cdot q)$



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 - coefficient functions $c_{a,i}$



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- coefficient functions $c_{a,i}$
- Evolution equations
 - non-singlet $(2n_f 1 \text{ scalar})$ and singlet $(2 \times 2 \text{ matrix})$ equations $\frac{d}{d \ln \mu^2} PDF(x, \mu^2) = \left[P(\alpha_s(\mu^2)) \otimes PDF(\mu^2)\right](x)$
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Methods

Optical theorem

- - three-loop calculation in DIS with help of loop technology



- Coefficient of $(2p \cdot q)^N$ gives *N*-th Mellin moment $A^N = \int_0^1 dx \ x^{N-1} A(x)$
- UV and mass singularities in dimensional regularization $D = 4 2\epsilon$
 - splitting functions: $\frac{1}{\epsilon}$ -poles
 - coefficient functions: ϵ^0 -part

Three-loop calculcation of DIS

- Splitting functions P_{gg} , P_{gq}
 - DIS with scalar ϕ coupling to $G^a_{\mu\nu}G^{\mu\nu}_a$ (cf. Higgs)
 - gluon polarization sum \longleftrightarrow diagrams with external ghost h

	tree	1-loop	2-loop	3-loop
$q\gamma$	1	3	25	359
${\sf g}\gamma$		2	17	345
$h\gamma$			2	56
qW	1	3	32	589
$q\phi$		1	23	696
$g\phi$	1	8	218	6378
$h\phi$		1	33	1184
sum	3	18	350	9607

- Highly optimised symbolic manipulation of formulae
 - computer algebra system Form Vermaseren '89-'05
 - capabilities substantially extended for this project

Results

NNLO non-singlet splitting functions

S.M., Vermaseren, Vogt '04

 $P_{\text{ms}}^{(2)+}(x) = 16C_A C_F n_f \left(\frac{1}{6}p_{\text{eq}}(x) \left[\frac{10}{3}\zeta_2 - \frac{209}{36} - 9\zeta_3 - \frac{167}{18}H_0 + 2H_0\zeta_2 - 7H_{0,0} - 2H_{0,0,0}\right]\right)$ $+ 3 H_{1,0,0} - H_3 \Big] + \frac{1}{3} p_{qq}(-x) \Big[\frac{3}{2} \zeta_3 - \frac{5}{3} \zeta_2 - H_{-2,0} - 2 H_{-1} \zeta_2 - \frac{10}{3} H_{-1,0} - H_{-1,0,0} \Big] \\$ $+2H_{-1,2}+\frac{1}{2}H_{0}\zeta_{2}+\frac{5}{3}H_{0,0}+H_{0,0,0}-H_{3}\Big]+(1-x)\Big[\frac{1}{6}\zeta_{2}-\frac{257}{54}-\frac{43}{18}H_{0}-\frac{1}{6}H_{0,0}-H_{1}\Big]$ $-(1+x)\Big[\frac{2}{3}H_{-1,0}+\frac{1}{2}H_2\Big]+\frac{1}{3}\zeta_2+H_0+\frac{1}{6}H_{0,0}+\delta(1-x)\Big[\frac{5}{4}-\frac{167}{54}\zeta_2+\frac{1}{20}\zeta_2^2+\frac{25}{18}\zeta_3\Big]\Big)$ $+16C_AC_F^2 \left(p_{qq}(x) \left[\frac{5}{5}\zeta_3 - \frac{69}{20}\zeta_2^2 - H_{-3,0} - 3H_{-2}\zeta_2 - 14H_{-2,-1,0} + 3H_{-2,0} + 5H_{-2,0,0}\right]\right)$ $-4H_{-2,2} - \frac{151}{48}H_0 + \frac{41}{12}H_0\zeta_2 - \frac{17}{2}H_0\zeta_3 - \frac{13}{4}H_{0,0} - 4H_{0,0}\zeta_2 - \frac{23}{12}H_{0,0,0} + 5H_{0,0,0,0} + \frac{2}{5}H_3$ $-24H_1\zeta_3 - 16H_{1,-2,0} + \frac{67}{6}H_{1,0} - 2H_{1,0}\zeta_2 + \frac{31}{2}H_{1,0,0} + 11H_{1,0,0,0} + 8H_{1,1,0,0} - 8H_{1,3} + H_4$ $+\frac{67}{9}H_2 - 2H_2\zeta_2 + \frac{11}{3}H_{2,0} + 5H_{2,0,0} + H_{3,0}\Big] + p_{99}(-x)\Big[\frac{1}{4}\zeta_2^2 - \frac{67}{9}\zeta_2 + \frac{31}{4}\zeta_3 + 5H_{-3,0}$ $-32 H_{-2} \zeta_2 -4 H_{-2,-1,0} -\frac{31}{6} H_{-2,0} +21 H_{-2,0,0} +30 H_{-2,2} -\frac{31}{3} H_{-1} \zeta_2 -42 H_{-1} \zeta_3 +\frac{9}{4} H_0$ $-4H_{-1,-2,0} + 56H_{-1,-1}\zeta_2 - 36H_{-1,-1,0,0} - 56H_{-1,-1,2} - \frac{134}{9}H_{-1,0} - 42H_{-1,0}\zeta_2 - H_{3,0}$ $+ 32 H_{-1,3} - \frac{31}{6} H_{-1,0,0} + 17 H_{-1,0,0,0} + \frac{31}{2} H_{-1,2} + 2 H_{-1,2,0} + \frac{13}{12} H_0 \zeta_2 + \frac{29}{2} H_0 \zeta_3 + \frac{67}{9} H_{0,0}$ $+13H_{0,0}\zeta_2 + \frac{89}{12}H_{0,0,0} - 5H_{0,0,0,0} - 7H_2\zeta_2 - \frac{31}{6}H_3 - 10H_4 + (1-x)\left[\frac{133}{26} + 4H_{0,0,0,0}\right]$ $-\frac{167}{4}\zeta_3 - 2H_0\zeta_3 - 2H_{-3,0} + H_{-2}\zeta_2 + 2H_{-2,-1,0} - 3H_{-2,0,0} + \frac{77}{4}H_{0,0,0} - \frac{209}{6}H_1 - 7H_1\zeta_2$ $+4H_{1,0,0}+\frac{14}{2}H_{1,0}]+(1+x)\left[\frac{43}{2}\zeta_{2}-3\zeta_{2}^{2}+\frac{25}{2}H_{-2,0}-31H_{-1}\zeta_{2}-14H_{-1,-1,0}-\frac{13}{2}H_{-1,0}\right]$ $+24 H_{-1,2}+23 H_{-1,0,0}+\frac{55}{2} H_0 \zeta_2+5 H_{0,0} \zeta_2+\frac{\bar{1} 457}{48} H_0-\frac{1025}{36} H_{0,0}-\frac{155}{6} H_2+H_2 \zeta_2-15 H_3$ $+2H_{2,0,0}-3H_4\Big]-5\zeta_2-\frac{1}{2}\zeta_2{}^2+50\zeta_3-2H_{-3,0}-7H_{-2,0}-H_0\zeta_3-\frac{37}{2}H_0\zeta_2-\frac{242}{9}H_0$ $-H_{0,0,0,0} + 9H_1\zeta_3 + 6H_{1,-2,0} - H_{1,0}\zeta_2 - \frac{11}{4}H_{1,0,0} - 3H_{1,0,0,0} - 4H_{1,1,0,0} + 4H_{1,3} + \frac{31}{12}H_{0,0,0}$ $+\frac{11}{17}H_3 + H_4 + p_{qq}(-x) \left[\frac{67}{18}\zeta_2 - \zeta_2^2 - \frac{11}{4}\zeta_3 - H_{-3,0} + 8H_{-2}\zeta_2 + \frac{11}{6}H_{-2,0} - 4H_{-2,0,0}\right]$ $-3H_{-1,0,0,0} + \frac{11}{2}H_{-1}\zeta_2 + 12H_{-1}\zeta_3 - 16H_{-1,-1}\zeta_2 + 8H_{-1,-1,0,0} + 16H_{-1,-1,2} + \frac{67}{9}H_{-1,0}$

 $-8H_{-2,2}+11H_{-1,0}\zeta_{2}+\frac{11}{6}H_{-1,0,0}-\frac{11}{3}H_{-1,2}-8H_{-1,3}-\frac{3}{4}H_{0}-\frac{1}{6}H_{0}\zeta_{2}-4H_{0}\zeta_{3}-\frac{67}{18}H_{0,$ $-3H_{0,0}\zeta_2 - \frac{31}{12}H_{0,0,0} + H_{0,0,0,0} + 2H_2\zeta_2 + \frac{11}{6}H_3 + 2H_4 + (1-x)\left[\frac{1883}{108} - \frac{1}{2}H_{0,0,0,0} + 11H_1\right]$ $-H_{-2,-1,0} + \frac{1}{2}H_{-3,0} - \frac{1}{2}H_{-2}\zeta_2 + \frac{1}{2}H_{-2,0,0} + \frac{523}{36}H_0 + H_0\zeta_3 - \frac{13}{3}H_{0,0} - \frac{5}{2}H_{0,0,0} + 2H_1\zeta_2$ $-2H_{1,0,0}$ + (1 + x) $\left[8H_{-1}\zeta_{2} + 4H_{-1,-1,0} + \frac{8}{2}H_{-1,0} - 5H_{-1,0,0} - 6H_{-1,2} - \frac{13}{2}\zeta_{2} + \frac{3}{2}\zeta_{2}^{2}\right]$ $-\frac{43}{4}\zeta_3 - \frac{5}{2}H_{-2,0} - \frac{11}{2}H_0\zeta_2 - \frac{1}{2}H_2\zeta_2 - \frac{5}{4}H_{0,0}\zeta_2 + 7H_2 - \frac{1}{4}H_{2,0,0} + 3H_3 + \frac{3}{4}H_4\Big] + \frac{1}{2}H_{0,0}\zeta_2$ $+\frac{4}{1}\zeta_{2}^{2}-\frac{8}{3}\zeta_{2}+\frac{17}{2}\zeta_{3}^{2}+H_{-2,0}-\frac{19}{2}H_{0}+\frac{4}{5}H_{0}\zeta_{2}-H_{0}\zeta_{3}+\frac{4}{13}H_{0,0}+\frac{5}{2}H_{0,0,0}+\frac{4}{1}H_{0,0,0,0}$ $-\delta(1-x)\left[\frac{1657}{576}-\frac{281}{27}\zeta_2+\frac{1}{8}\zeta_2^2+\frac{97}{9}\zeta_3-\frac{5}{2}\zeta_5\right]\right)+16C_F n_f^2\left(\frac{1}{18}p_{eq}(x)\left[H_{0,0}-\frac{1}{3}+\frac{5}{3}H_0\right]\right)$ $+ (1-x) \left[\frac{15}{54} + \frac{1}{9} H_0 \right] - \delta(1-x) \left[\frac{17}{144} - \frac{5}{27} \zeta_2 + \frac{1}{9} \zeta_3 \right] + 16 C_F^{-2} n_f \left(\frac{1}{3} \rho_{44}(x) \left[5 \zeta_3 - 4 H_{1,0,0} \right] \right) + 16 C_F^{-2} n_f \left(\frac{1}{3} \rho_{44}(x) \left[5 \zeta_3 - 4 H_{1,0,0} \right] \right) + 16 C_F^{-2} n_f \left(\frac{1}{3} \rho_{44}(x) \left[5 \zeta_3 - 4 H_{1,0,0} \right] \right) + 16 C_F^{-2} n_f \left(\frac{1}{3} \rho_{44}(x) \left[5 \zeta_3 - 4 H_{1,0,0} \right] \right) + 16 C_F^{-2} n_f \left(\frac{1}{3} \rho_{44}(x) \left[5 \zeta_3 - 4 H_{1,0,0} \right] \right) + 16 C_F^{-2} n_f \left(\frac{1}{3} \rho_{44}(x) \left[5 \zeta_3 - 4 H_{1,0,0} \right] \right) + 16 C_F^{-2} n_f \left(\frac{1}{3} \rho_{44}(x) \left[5 \zeta_3 - 4 H_{1,0,0} \right] \right) + 16 C_F^{-2} n_f \left(\frac{1}{3} \rho_{44}(x) \left[5 \zeta_3 - 4 H_{1,0,0} \right] \right) + 16 C_F^{-2} n_f \left(\frac{1}{3} \rho_{44}(x) \left[5 \zeta_3 - 4 H_{1,0,0} \right] \right) + 16 C_F^{-2} n_f \left(\frac{1}{3} \rho_{44}(x) \left[5 \zeta_3 - 4 H_{1,0,0} \right] \right) + 16 C_F^{-2} n_f \left(\frac{1}{3} \rho_{44}(x) \left[5 \zeta_3 - 4 H_{1,0,0} \right] \right) + 16 C_F^{-2} n_f \left(\frac{1}{3} \rho_{44}(x) \left[5 \zeta_3 - 4 H_{1,0,0} \right] \right) + 16 C_F^{-2} n_f \left(\frac{1}{3} \rho_{44}(x) \left[5 \zeta_3 - 4 H_{1,0,0} \right] \right) + 16 C_F^{-2} n_f \left(\frac{1}{3} \rho_{44}(x) \left[5 \zeta_3 - 4 H_{1,0,0} \right] \right) + 16 C_F^{-2} n_f \left(\frac{1}{3} \rho_{44}(x) \left[5 \zeta_3 - 4 H_{1,0,0} \right] \right) + 16 C_F^{-2} n_f \left(\frac{1}{3} \rho_{44}(x) \left[5 \zeta_3 - 4 H_{1,0,0} \right] \right) + 16 C_F^{-2} n_f \left(\frac{1}{3} \rho_{44}(x) \left[\frac{$ $\frac{55}{12} + \frac{5}{9}H_0 + H_0\zeta_2 + \frac{3}{2}H_{0,0} - H_{0,0,0} - \frac{10}{3}H_{1,0} - \frac{10}{3}H_2 - 2H_{2,0} - 2H_3 \Big] + \frac{2}{3}p_{qq}(-x) \Big[\frac{5}{3}\zeta_2 + \frac{10}{3}H_1 - \frac{10}{3}H_2 - \frac{10}$ $-\frac{3}{2}\zeta_3 + H_{-2,0} + 2H_{-1}\zeta_2 + \frac{10}{3}H_{-1,0} + H_{-1,0,0} - 2H_{-1,2} - \frac{1}{2}H_0\zeta_2 - \frac{5}{3}H_{0,0} - H_{0,0,0} + H_3$ $-(1-x)\left[\frac{10}{9}+\frac{19}{18}H_{0,0}-\frac{4}{3}H_{1}+\frac{2}{3}H_{1,0}+\frac{4}{3}H_{2}\right]+(1+x)\left[\frac{4}{3}H_{-1,0}-\frac{25}{24}H_{0}+\frac{1}{2}H_{0,0,0}\right]+\frac{2}{9}H_{0,0}$ $+\frac{7}{9}H_{0,0}+\frac{4}{3}H_2-\delta(1-x)\Big[\frac{23}{16}-\frac{5}{12}\zeta_2-\frac{29}{30}\zeta_2^2+\frac{17}{6}\zeta_3\Big]\Big)+16C_F^{-3}(p_{eq}(x)\Big[\frac{9}{10}\zeta_2^2-2H_{-3,0}(x)\Big]$ $+ 6 H_{-2} \zeta_2 + 12 H_{-2,-1,0} - 6 H_{-2,0,0} - \frac{3}{16} H_0 - \frac{3}{2} H_0 \zeta_2 + H_0 \zeta_3 + \frac{13}{8} H_{0,0} - 2 H_{0,0,0,0} + 8 H_{1,3}$ $+12H_{1}\zeta_{3}+8H_{1,-2,0}-6H_{1,0,0}-4H_{1,0,0,0}+4H_{1,2,0}-3H_{2,0}+2H_{2,0,0}+4H_{2,1,0}+4H_{2,2}$ $\begin{aligned} &+4\mathrm{H}_{3,0}+4\mathrm{H}_{3,1}+2\mathrm{H}_4\Big]+p_{qq}(-x)\Big[\frac{7}{2}\zeta_2^2-\frac{9}{2}\zeta_3-6\mathrm{H}_{-3,0}+32\mathrm{H}_{-2}\zeta_2+8\mathrm{H}_{-2,-1,0}+3\mathrm{H}_{-2,0}\\ &-26\mathrm{H}_{-2,0,0}-28\mathrm{H}_{-2,2}+6\mathrm{H}_{-1}\zeta_2+36\mathrm{H}_{-1}\zeta_3+8\mathrm{H}_{-1,-2,0}-48\mathrm{H}_{-1,-1}\zeta_2+40\mathrm{H}_{-1,-1,0,0} \end{aligned}$ $+\,48H_{-1,-1,2}+40H_{-1,0}\zeta_{2}+3H_{-1,0,0}-22H_{-1,0,0,0}-6H_{-1,2}-4H_{-1,2,0}-32H_{-1,3}-\frac{3}{2}H_{0}$ $-\frac{3}{2}H_0\zeta_2 - 13H_0\zeta_3 - 14H_{0,0}\zeta_2 - \frac{9}{2}H_{0,0,0} + 6H_{0,0,0,0} + 6H_2\zeta_2 + 3H_3 + 2H_{3,0} + 12H_4\Big]$ + (1-x) $\left[2H_{-3,0} - \frac{31}{9} + 4H_{-2,0,0} + H_{0,0}\zeta_2 - 3H_{0,0,0,0} + 35H_1 + 6H_1\zeta_2 - H_{1,0} + \frac{5}{2}H_{2,0}\right]$ $+(1+x)\left[\frac{37}{10}\zeta_{2}^{2}-\frac{93}{4}\zeta_{2}-\frac{81}{2}\zeta_{3}-15H_{-2,0}+30H_{-1}\zeta_{2}+12H_{-1,-1,0}-2H_{-1,0}-26H_{-1,0,0}\right]$ $-24H_{-1,2} - \frac{539}{16}H_0 - 28H_0\zeta_2 + \frac{191}{8}H_{0,0} + 20H_{0,0,0} + \frac{85}{4}H_2 - 3H_{2,0,0} - 2H_{3,0} + 13H_3$ $-H_4$ + 4 ζ_2 + 33 ζ_3 + 4 $H_{-3,0}$ + 10 $H_{-2,0}$ + $\frac{67}{2}H_0$ + 6 $H_0\zeta_3$ + 19 $H_0\zeta_2$ - 25 $H_{0,0}$ - 17 $H_{0,0,0}$ $-2H_2 - H_{2,0} - 4H_3 + \delta(1-x) \left[\frac{29}{27} - 2\zeta_2\zeta_3 + \frac{9}{6}\zeta_2 + \frac{18}{5}\zeta_2^2 + \frac{17}{17}\zeta_3 - 15\zeta_5\right]$

 $P_{ms}^{(2)-}(x) = P_{ms}^{(2)+}(x) + 16C_AC_F\left(C_F - \frac{C_A}{2}\right)\left(p_{qq}(-x)\left[\frac{134}{9}\zeta_2 - 4\zeta_2^2 - 11\zeta_3 - 4H_{-3,0}\right]\right)$ $+ 32 H_{-2} \zeta_2 + \frac{22}{3} H_{-2,0} - 16 H_{-2,0,0} - 32 H_{-2,2} + \frac{44}{3} H_{-1} \zeta_2 + 48 H_{-1} \zeta_3 - 64 H_{-1,-1} \zeta_2$ $+ 32 H_{-1,-1,0,0} + 64 H_{-1,-1,2} + \frac{268}{9} H_{-1,0} + 44 H_{-1,0} \zeta_2 + \frac{22}{2} H_{-1,0,0} - 12 H_{-1,0,0,0} - \frac{44}{2} H_{-1,2}$ $-32 H_{-1,3} - 3 H_0 - \frac{2}{2} H_0 \zeta_2 - 16 H_0 \zeta_3 - \frac{134}{0} H_{0,0} - 12 H_{0,0} \zeta_2 - \frac{31}{2} H_{0,0,0} + 4 H_{0,0,0,0} + 8 H_2 \zeta_2$ $+\frac{22}{2}H_{3}+8H_{4}]+(1-x)\Big[\frac{367}{10}+\frac{1}{2}\zeta_{2}^{2}+2H_{-3,0}-2H_{-2}\zeta_{2}-4H_{-2,-1,0}-10H_{-2,0}-2H_{0,0}$ $+2H_{-2,0,0} + 2H_0\zeta_3 + H_{0,0}\zeta_2 - H_{0,0,0,0} + 8H_1\zeta_2 + \frac{140}{2}H_1 + (1+x)[32H_{-1}\zeta_2 - 18\zeta_2]$ $-23\zeta_3 + \frac{26}{2}H_{-1,0} - 16H_{-1,0,0} - 32H_{-1,2} - \frac{481}{18}H_0 - 29H_0\zeta_2 + 5H_{0,0,0} + 24H_3 + \frac{70}{2}H_2$ $-2\zeta_2 - 2\zeta_3 + 32H_0 + 14H_0\zeta_2 + 2H_{0,0,0} - 16H_3 + 16C_F n_f \left(C_F - \frac{C_A}{2}\right) \left(p_{qq}(-x) \left[2\zeta_3\right]\right)$ $-\frac{20}{2}\zeta_2 - \frac{4}{2}H_{-2,0} - \frac{8}{2}H_{-1}\zeta_2 - \frac{40}{2}H_{-1,0} - \frac{4}{2}H_{-1,0,0} + \frac{8}{2}H_{-1,2} + \frac{2}{2}H_0\zeta_2 + \frac{20}{2}H_{0,0} + \frac{4}{2}H_{0,0,0}$ $-\frac{4}{3}H_3$ + $(1-x)\left[\frac{61}{9}-\frac{8}{3}H_1\right]$ + $(1+x)\left[2H_{0,0}-\frac{8}{3}H_{-1,0}+\frac{41}{9}H_0-\frac{4}{3}H_2\right]$ + $16C_F^2 (C_F - \frac{C_A}{2}) (p_{qq}(-x) [9\zeta_3 - 7\zeta_2^2 + 12H_{-3,0} - 64H_{-2}\zeta_2 - 16H_{-2,-1,0} - 6H_{-2,0}]$ $+52H_{-2,0,0} + 56H_{-2,2} - 12H_{-1}\zeta_2 - 72H_{-1}\zeta_3 - 16H_{-1,-2,0} + 96H_{-1,-1}\zeta_2 - 80H_{-1,-1,0,0}$ $-96H_{-1,-1,2} - 80H_{-1,0}\zeta_2 - 6H_{-1,0,0} + 44H_{-1,0,0,0} + 12H_{-1,2} + 8H_{-1,2,0} + 64H_{-1,3} + 3H_{-1,2,0} + 64H_{-1,3,0} + 64H_{ +3H_0\zeta_2 + 26H_0\zeta_3 + 28H_{0.0}\zeta_2 + 9H_{0.0.0} - 12H_{0.0.0.0} - 12H_2\zeta_2 - 6H_3 - 4H_{3.0} - 24H_4$ $-(1-x)\left[15+8H_{-3.0}+8H_{-2.0.0}+61H_{0}+6H_{0}\zeta_{3}+2H_{0.0}\zeta_{2}-6H_{0.0.0.0}+12H_{1}\zeta_{2}+60H_{-2.0.0}+6H_{0.0.0}+12H_{1}\zeta_{2}+60H_{-2.0.0}+6H_{0.0.0}+12H_{1}\zeta_{2}+60H_{-2.0.0}+6H_{0.0.0}+12H_{1}\zeta_{2}+6H_{0.0.0}+12H_{0.0}$ $+8H_{1,0}$ + (1 + x) 24 ζ_2 + 57 ζ_3 + 10H_{-2,0} - 48H₋₁ ζ_2 - 4H_{-1,0} + 40H_{-1,0,0} + 48H_{-1,2} $+59 H_0 \zeta_2 - 22 H_{0,0} - 35 H_{0,0,0} - 22 H_2 - 4 H_{2,0} - 44 H_3 \Big] + 8 \zeta_2 - 42 \zeta_3 - 4 H_{-2,0} + 42 H_0$ $-38H_0\zeta_2 + 14H_{0,0} - 16H_2 + 26H_{0,0,0} + 24H_3$

$$\begin{split} \mu_{2}^{(1)*}(x) &= 1.6n_{f}\frac{d^{20}}{d_{10}} \frac{d_{10}}{d_{10}} \left(\frac{1}{2}(1-x)\right) \left[\frac{69}{2} + \frac{41}{12} \zeta_{2} - \frac{5}{4} \zeta_{2}^{-2} - H_{-3,0} + H_{-2} \zeta_{2} - H_{-3,00} + \frac{9}{4} H_{1} \right. \\ &+ 2H_{-2,-1,0} - \frac{3}{2} H_{0} \zeta_{2} - \frac{1}{2} H_{1} \zeta_{2} - \frac{3}{4} H_{1,0,0} + \frac{91}{2} H_{1} + \frac{1}{2} (1+x) \left[H_{-1,-1,0} - \frac{3}{2} H_{-1} \zeta_{2} + \frac{3}{4} H_{0} \right. \\ &- \frac{13}{6} H_{-1,0} + \frac{1}{2} H_{-1,0,0} - \frac{2}{3} H_{-2,0} + \frac{9}{4} H_{0} \zeta_{2} - \frac{29}{2} H_{0,0} + \frac{41}{12} H_{2} - H_{2} \zeta_{2} - \frac{1}{2} H_{2,0,0} \\ &+ \frac{3}{4} H_{1} - \frac{1}{3} (\frac{1}{4} + x^{2}) \left[5H_{-1} \zeta_{2} + 2H_{-1,-1,0} - 2H_{-1,2,0} - 2H_{-1,2} + H_{1} \zeta_{2} \right] + \frac{1}{3} x^{2} \left[5\zeta_{3} - 2H_{3} \right] \\ &+ 2H_{-2,0} + 4H_{0} \zeta_{2} - 2H_{0,0,0} + 2H_{1} (\zeta_{2}) + \frac{3}{3} H_{0} + \zeta_{3} - \frac{9}{2} \zeta_{3} + \zeta_{3}^{2} - H_{0} \zeta_{3} - H_{3} - \frac{1}{2} \zeta_{3} + \zeta_{3}^{2} - 2H_{0,0,0} \\ &+ 2H_{-2,0} - 4H_{0} \zeta_{3} - 2H_{0,0,0} + 2H_{1} (\zeta_{3}) + \frac{3}{3} H_{0} + \zeta_{3} - \frac{9}{2} \zeta_{3} + \zeta_{3}^{2} - H_{0} \zeta_{3} - H_{0} \zeta_{3} - 2H_{0} \zeta_{3} \\ &+ 2H_{-2,0} - 4H_{0} \zeta_{3} - 2H_{0,0,0} + 2H_{1} (\zeta_{3}) + \frac{3}{2} H_{0} + \zeta_{3} - \frac{9}{2} \zeta_{3} + \zeta_{3}^{2} - H_{0} \zeta_{3} - H_{0} \zeta_{3} - 2H_{0} \zeta_{3} \\ &+ 2H_{0} \zeta_{3} - 2H_{0,0,0} + 2H_{0} \zeta_{3} - 2H_{0,0,0} + 2H_{0} \zeta_{3} - 2H_{0,0,0} \\ &+ 2H_{0} \zeta_{3} - 2H_{0,0,0} + 2H_{0} \zeta_{3} - 2H_{0,0,0} + 2H_{0} \zeta_{3} - 2H_{0,0,0} \\ &+ 2H_{0} \zeta_{3} - 2H_{0,0,0} + 2H_{0} \zeta_{3} - 2H_{0,0,0} + 2H_{0} \zeta_{3} - 2H_{0,0,0} \\ &+ 2H_{0} \zeta_{3} - 2H_{0,0,0} + 2H_{0} \zeta_{3} - 2H_{0,0,0} + 2H_{0} \zeta_{3} - 2H_{0,0,0} \\ &+ 2H_{0} \zeta_{3} - 2H_{0,0,0} + 2H_{0} \zeta_{3} - 2H_{0,0,0} + 2H_{0} \zeta_{3} - 2H_{0,0,0} \\ &+ 2H_{0} \zeta_{3} - 2H_{0,0,0} + 2H_{0} \zeta_{3} - 2H_{0,0,0} \\ &+ 2H_{0} \zeta_{3} - 2H_{0,0,0} + 2H_{0} \zeta_{3} - 2H_{0,0,0} \\ &+ 2H_{0} \zeta_{3} - 2H_{0,0,0} + 2H_{0} \zeta_{3} - 2H_{0,0,0} \\ &+ 2H_{0} \zeta_{3} - 2H_{0,0,0} \\ &+ 2H_{0} \zeta_{3} - 2H_{0,0,0} + 2H_{0} \zeta_{3} - 2H_{0,0,0} \\ &+ 2H_{0} \zeta_{3} - 2H_{0,0,0} \\ &+ 2H_{0} \zeta_{3} - 2H_{0,0,0} + 2H_{0} \zeta_{3} - 2H_{0,0,0} \\ &+ 2H_{0} \zeta_{3} - 2H_{0,0,0} \\ &+ 2H_{0} \zeta_{3} - 2H_{0,0,0} \\ &+ 2H_{0} \zeta_{3} - 2H_{0,0,0$$

 $+\frac{3}{8}H_{0,0}-\frac{1}{4}H_{0,0,0}+\frac{1}{2}H_{0,0,0}+H_{-2,0}-H_3$

NNLO singlet splitting functions S.M., Vermaseren, Vogt '04

 $P_{ps}^{(2)}(x) = 16C_A C_F n_f \left(\frac{4}{2}(\frac{1}{x}+x^2) \left[\frac{13}{2}H_{-1,0} - \frac{14}{9}H_0 + \frac{1}{2}H_{-1}\zeta_2 - H_{-1,-1,0} - 2H_{-1,0,0}\right]\right)$ $-H_{-1,2}\Big] + \frac{2}{3}(\frac{1}{x}-x^2)\Big[\frac{13}{16}\zeta_2 + H_{2,1} + 9\zeta_3 + \frac{9}{4}H_{1,0} - \frac{6761}{216} + \frac{571}{72}H_1 + \frac{10}{3}H_2 + H_1\zeta_2 - \frac{1}{6}H_{1,1}$ $-3H_{1,0,0} + 2H_{1,1,0} + 2H_{1,1,1} + (1-x) \left[\frac{182}{9}H_1 + \frac{158}{3} + \frac{397}{36}H_{0,0} - \frac{13}{2}H_{-2,0} + 3H_{0,0,0,0} \right]$ $+\frac{13}{2}H_{1,0} + 3xH_{1,0} + H_{-3,0} + H_{-2}\zeta_2 + 2H_{-2,-1,0} + 3H_{-2,0,0} + \frac{1}{2}H_{0,0}\zeta_2 + \frac{1}{2}H_1\zeta_2 - \frac{9}{2}H_{1,0,0}$ $\overset{0}{\overset{-3}{4}}H_{1,1}+H_{1,1,0}+H_{1,1,1}\Big]+(1+x)\Big[\frac{7}{12}H_0\zeta_2+\frac{31}{6}\zeta_3+\frac{91}{18}H_2+\frac{71}{12}H_3+\frac{113}{18}\zeta_2-\frac{826}{27}H_0$ $+\frac{5}{2}H_{2,0} + \frac{16}{3}H_{-1,0} + 6xH_{-1,0} + \frac{31}{6}H_{0,0,0} - \frac{17}{6}H_{2,1} + \frac{117}{20}\zeta_2^2 + 9H_0\zeta_3 + \frac{5}{2}H_{-1}\zeta_2 + 2H_{2,1,0}$ $\frac{1}{2}H_{-1,0,0} - 2H_{-1,2} + H_2\zeta_2 - \frac{7}{2}H_{2,0,0} + H_{-1,-1,0} + 2H_{2,1,1} + H_{3,1} - \frac{1}{2}H_4 + 5H_{-2,0} + H_{2,1}$ $+\frac{2^{10}}{12}H_{0,0,0,0} - \frac{1}{7}\zeta_{2}^{2} + 4H_{-3,0} + 4H_{0}\zeta_{3} - \frac{32}{9}H_{0,0} - \frac{29}{12}H_{0} - \frac{235}{12}\zeta_{2} - \frac{511}{12} - \frac{97}{12}H_{1} + \frac{33}{4}H_{2} - H_{3} - \frac{1}{12}H_{1} + \frac{1}{12}H_{1} - \frac{1}{12}$ $-\frac{11}{2}H_0\zeta_2 - \frac{11}{2}\zeta_3 - \frac{3}{2}H_{2,0} - 10H_{0,0,0} + \frac{2}{3}x^2 \left[\frac{83}{4}H_{0,0} - \frac{243}{4}H_0 + 10\zeta_2 + \frac{511}{8} + \frac{97}{8}H_1 - \frac{4}{3}H_2\right]$ $-4\zeta_3 - H_0\zeta_2 + H_3 + H_{2,0} - 6H_{-2,0}$ + $16C_F n_f^2 \left(\frac{2}{2\pi}H_0 - 2 - H_2 + \zeta_2 + \frac{2}{3}x^2 \left[H_2 - \zeta_2 + 3\right]\right)$ $-\frac{19}{6}H_0$ + $\frac{2}{9}(\frac{1}{2}-x^2)$ H_{1,1} + $\frac{5}{3}H_1$ + $\frac{2}{3}$ + (1-x) + $(\frac{1}{6}H_{1,1}-\frac{7}{6}H_1+xH_1+\frac{35}{27}H_0+\frac{185}{54})$ $+\frac{1}{2}(1+x)\left[\frac{4}{2}H_2-\frac{4}{2}\zeta_2+\zeta_3+H_{2,1}-2H_3+2H_0\zeta_2+\frac{29}{6}H_{0,0}+H_{0,0,0}\right]\right)+16C_F^{-2}n_f\left(\frac{85}{12}H_1-\frac{1}{2}H_1$ $\frac{3}{-25}H_{0,0} - H_{0,0,0} + \frac{583}{12}H_0 - \frac{101}{54} + \frac{73}{4}\zeta_2 - \frac{73}{4}H_2 + H_3 - 5H_{2,0} - H_{2,1} - H_0\zeta_2 + x^2 \left[\frac{55}{12}\right]$ $\frac{4}{12} + \frac{12}{3} + \frac{54}{12} + \frac{4}{9} + \frac{4}{9} + \frac{4}{9} + \frac{4}{9} + \frac{16}{3} + \frac{112}{9} + \frac{1$ $\frac{22}{3}H_{0,0,0} + \frac{4}{3}(\frac{1}{x}-x^2)\left[\frac{23}{12}H_{1,0} - \frac{523}{144}H_1 - 3\zeta_3 + \frac{55}{16} + \frac{1}{2}H_{1,0,0} + H_{1,1} - H_{1,1,0} - H_{1,1,1}\right]$ $+3H_0\zeta_2 - 3H_3 - H_{1,1,0} - H_{1,1,1} + (1+x) \left[\frac{1669}{216} + \frac{5}{2}H_{0,0,0} + 4H_{2,1} + 7H_{2,0} + 10x\zeta_3 - \frac{37}{10}\zeta_2^2 \right]$ $-7H_0\zeta_3 + 6H_{0.0}\zeta_2 - 4H_{0.0,0,0} + H_{2.0,0} - 2H_{2.1,0} - 2H_{2.1,1} - 4H_{3.0} - H_{3.1} - 6H_4$

$$\begin{split} \mu^{(2)}_{01}(s) &= 16C_{4}C_{5}\sigma_{1}/\left(p_{01}(s)\frac{129}{2}H_{5}\zeta_{1}-4H_{1,11}+3H_{2,0,0}-\frac{15}{2}H_{1,2}+\frac{9}{2}H_{1,1,0}+3H_{2,1,0}\right.\\ &+H_{6}\zeta_{3}-2H_{2,1,1}+4H_{5}\zeta_{2}-\frac{172}{2}H_{0}\zeta_{3}-\frac{52}{72}H_{0,0}+\frac{64}{5}\zeta_{3}-\zeta_{3}^{2}-\frac{49}{2}H_{2}-\frac{3}{2}H_{1,0,0,0}-\frac{1}{2}H_{1,0,0,0}\right.\\ &-\frac{385}{72}H_{1,0}-\frac{31}{2}H_{1,1}-\frac{113}{2}H_{1}+\frac{49}{4}H_{2,1,0}+\frac{5}{2}H_{1,2}+\frac{69}{6}H_{0,0,0}+\frac{173}{2}H_{1,1}-\frac{139}{2}+\frac{233}{216}H_{0}\\ &+H_{0,1}+3H_{1,2}-9+H_{10}\zeta_{3}+4H_{1,1}\zeta_{3}+H_{1,1,0,0}-3H_{1,1,1}-3H_{1,1,2}-3H_{1,2,1}-3H_{1,2,$$

 $-2H_{3,0} - \frac{13}{2}H_0\zeta_2 - 13H_{-3,0} - \frac{13}{2}H_{3,1} + \frac{15}{2}H_3 - \frac{2005}{64} + \frac{157}{4}\zeta_2 + 8\zeta_3 + \frac{1291}{432}H_1 + \frac{55}{12}H_{1,1}$ $+\frac{3}{2}H_{2}+\frac{1}{2}H_{2,1}+\frac{27}{4}H_{-1,0}-\frac{11}{2}H_{1,0,0}-8H_{2,0,0}-4\zeta_{2}{}^{2}+\frac{3}{2}H_{1,2}-H_{2,2}+\frac{5}{2}H_{1}\zeta_{2}+8H_{-1,-1,0}$ $+4H_{2,0}+\frac{3}{2}H_{2,1,1}-H_{-1}\zeta_2+7H_2\zeta_2+6H_{-2}\zeta_2+12H_{-2,-1,0}-6H_{-2,0,0}+x\Big[3H_{1,1,1}-H_{0,0}\zeta_2+2H_{-2,0}+2H_{-2,$ $+\frac{9}{2}H_{-1,0,0}-\frac{35}{8}H_{1,0}+2H_4+3H_{1,1,0}+H_{-1,2}\right]+16C_A^{-2}C_F\left(x^2\left[\frac{2}{3}H_1\zeta_2-\frac{2105}{81}-\frac{77}{18}H_{0,0}-\frac{1}{18}H_{0,0}$ $-6H_3 + \frac{16}{3}\zeta_3 - 10H_{-1,0} - \frac{14}{3}H_{2,0} - \frac{2}{3}H_{-1}\zeta_2 - \frac{14}{3}H_{0,0,0} + \frac{104}{9}H_2 - \frac{4}{3}H_{1,0,0} + \frac{37}{9}H_{1,1}$ $+\frac{4}{3}H_{-1,-1,0} - \frac{104}{9}\zeta_2 - \frac{8}{3}H_{2,1} + \frac{145}{18}H_{1,0} + \frac{4}{3}H_{-1,2} + \frac{2}{3}H_{1,1,1} - \frac{109}{27}H_1 + \frac{8}{3}H_{-1,0,0} + 6H_0\zeta_2 + \frac{1}{3}H_{-1,0,0} + \frac{1}{3}$ $\begin{array}{c} 3 \\ +4H_{-2,0} + \frac{584}{27}H_0 \end{array} + p_{\mathbb{P}^q}(x) \left[\frac{7}{2}H_1\zeta_3 + \frac{138305}{2592} - \frac{1}{3}H_{2,0} + \frac{13}{4}H_{-1}\zeta_2 + 2H_{2,1,1} + \frac{11}{2}H_{1,0,0} \right] \end{array}$ $+4H_{3,1}-\frac{43}{6}H_{1,1,1}-\frac{109}{12}\zeta_2-\frac{17}{3}H_{2,1}-\frac{71}{24}H_{1,0}-\frac{11}{6}H_{-2,0}-\frac{21}{2}\zeta_3+\frac{3}{2}H_{1,0,0,0}-H_{1,-2,0}$ $+\frac{395}{54}H_0 - 2H_{1,0}\zeta_2 - H_{1,1}\zeta_2 - \frac{55}{12}H_{1,1,0} + 2H_{1,1,0,0} + 4H_{1,1,1,0} + 2H_{1,1,1,1} + 4H_{1,1,2} - \frac{55}{12}H_{1,2}$ $+6H_{1,2,0} + 4H_{1,2,1} + 4H_{1,3} + 3H_{2,1,0} + 3H_{2,2} + p_{99}(-x) \left[\frac{23}{2}H_{-1}\zeta_3 + 5H_{-2}\zeta_2 + 2H_{-2,-1,0}\right]$ $+\frac{109}{12}H_{-1,0}+H_0\zeta_3+\frac{17}{5}\zeta_2{}^2+\frac{1}{6}H_1\zeta_2+2H_2\zeta_2-\frac{65}{24}H_{1,1}-\frac{19}{2}H_{-1,-1,0}-4H_{3,0}-3H_{2,0,0}$ $\begin{array}{c} 12 \\ -7H_{-2,0,0} - \frac{3}{2}H_{-1,2} + \frac{3379}{216}H_1 - 4H_{-2,2} - \frac{49}{6}H_{-1,0,0} - \frac{11}{2}H_{-1,0,0,0} - 13H_{-1,-1}\zeta_2 - 8H_{-1,3} \end{array}$ $-6H_{-1,-1,-1,0} + 12H_{-1,-1,0,0} + 10H_{-1,-1,2} + 10H_{-1,0}\zeta_2 + 5H_{-1,-2,0} - 2H_{-1,2,0} - 2H_{-1,2,1} -$ $\begin{array}{l} \underset{i=1,\ldots,-1}{\underset{i=1}{10}} 0 & \underset{i=1,\ldots,0}{\underset{i=1,\ldots,0}{10}} 0 & \underset{i=1,\ldots,0}{\underset{i=1,\ldots,0}{10}} & \underset{i=1,\ldots,$ $+\frac{17}{6}H_{-2,0}-12H_{2,0}-\frac{31}{12}H_{2,1}+\frac{1}{2}H_{2,0,0}-H_2\zeta_2+\frac{61}{36}H_{1,0}-4H_0\zeta_3-\frac{13}{3}H_{-1}\zeta_2-\frac{46}{3}H_{-1,-1,0}$ $+\frac{9}{2^{5}}\frac{4}{4}H_{4}+\frac{93}{4}H_{0}\zeta_{2}-\frac{5^{2}}{9}H_{1,1}-\frac{71}{18}H_{2}+\frac{49}{18}H_{0,0}-\frac{13}{2}H_{0,0}\zeta_{2}-\frac{47}{40}\zeta_{2}^{-2}\right]+\frac{6131}{2592}-\frac{31}{2}H_{-2}\zeta_{2}$ $-15H_{-2,-1,0} + \frac{9}{2}H_{-1,0,0} - 3H_{2,1,1} - \frac{9}{4}H_{2,1} + \frac{53}{2}H_{-2,0} - \frac{1}{5}H_{-2,0,0} - 5H_{2,0} - \frac{7}{6}H_{1,1,1} - 8H_0\zeta_3$ $-\frac{67}{40}\xi_2^2 + \frac{29}{6}H_{-1,2} - H_{-1,0} + 8H_{-2,2} + 25H_0\xi_2 + \frac{412}{9}H_1 + \frac{928}{9}H_0 + \frac{1}{4}H_4 - 65H_3 - 38H_{0,0}$ $-9H_{-3,0} - \frac{17}{3}H_{0,0,0} + x \left[\frac{27}{2}H_{-1,0} - \frac{1}{2}H_{0,0,0,0} + \frac{3}{4}H_{0,0}\zeta_2 + \frac{1}{2}H_{-3,0} - 14H_{0,0,0} + \frac{1}{12}H_{1,1,1}\right]$ $\begin{array}{c} -\frac{43}{36}\zeta_2 - \frac{1}{2}H_5\zeta_2 + \frac{7}{72}H_0 + \frac{749}{54}H_1 + \frac{135}{4}\zeta_3 + \frac{97}{24}H_{1,0} + \frac{43}{12}H_1\zeta_2 - \frac{85}{12}H_{-1}\zeta_2 - \frac{13}{3}H_{1,0,0} \\ +\frac{53}{12}H_2 + \frac{39}{4}H_{1,1} - 2H_{3,1} + \frac{13}{6}H_{-1,-1,0} + \frac{7}{4}H_{2,0,0} - 4H_{1,1,0} - 4H_{1,2} \Big) + 16C_F n_\ell^2 \Big(\frac{1}{9} - \frac{11}{9x}H_{1,0} + \frac{13}{9x}H_{1,0} + \frac{13}{9x}H_{1,0}$ $+\frac{2}{6}x - \frac{1}{c}xH_{1} + \frac{1}{c}p_{89}(x)\left[H_{1,1} - \frac{5}{3}H_{1}\right] + 16C_{F}^{2}n_{f}\left(\frac{4}{6}x^{2}\left[H_{0,0} - \frac{11}{c}H_{0} - \frac{7}{3} + H_{-1,0}\right]\right]$

 $-6H_{1,3} + \frac{49}{4}\zeta_2 + p_{qg}(-x) \left[\frac{17}{2}H_{-1}\zeta_3 - \frac{5}{2}H_{-1,-1,0} - \frac{5}{2}H_{-1,2} - \frac{9}{2}H_{-1,0} + \frac{5}{2}H_{-2,0} + \frac{3}{2}H_{-1,0,0}\right]$ $-2H_{3,1} - 2H_4 - 6H_{-2,2} + 6H_{-2,-1,0} - 6H_{-2,0,0} + 2H_{0,0}\zeta_2 + 9H_{-2}\zeta_2 + 3H_{-1,-2,0} - 2H_{-1,2,1}$ $-6H_{-1,-1,-1,0} + 6H_{-1,-1,0,0} + 6H_{-1,-1,2} + 9H_{-1,0}\zeta_2 - 9H_{-1,-1}\zeta_2 - 2H_{-1,2,0} - \frac{11}{2}H_{-1,0,0,0}$ $-6H_{-1,3}$ + $(\frac{1}{x}-x^2)\left[\frac{55}{12}-4\zeta_3+\frac{23}{9}H_{1,0}-\frac{4}{3}H_{1,1,0}\right]$ + $(\frac{1}{x}+x^2)\left[\frac{2}{3}H_{1,0,0}-\frac{371}{108}H_1+\frac{23}{9}H_{1,1}\right]$ $-\frac{2}{3}H_{1,1,1}\Big] + (1-x)\Big[6H_{2,1,0} + 3H_{2,1,1} - \frac{5}{6}H_{1,1,1} - 7H_{2,0,0} - 2H_{1,2} + 39H_0\zeta_3 - 4H_2\zeta_2 - \frac{16}{3}\zeta_3 - 4H_2\zeta_3 - \frac{16}{3}\zeta_3 - \frac$ $+H_{1,1,0} + \frac{154}{3}H_0\zeta_2 + \frac{899}{24}H_{0,0} + \frac{121}{10}\zeta_2^2 + \frac{607}{36}H_2 - \frac{5}{2}H_1\zeta_2 + \frac{65}{6}H_{1,0,0} - \frac{29}{12}H_{1,0} - \frac{13}{18}H_{1,1}$ $-\frac{1189}{108}H_1 - \frac{67}{3}H_{2,1} - 29H_{2,0} - \frac{949}{36}\zeta_2 - \frac{67}{2}H_{0,0,0} - \frac{142}{3}H_3 + \frac{215}{32} - \frac{3989}{48}H_0 + 2H_{-3,0}]$ +(1+x) $H_{-1,0,0} - 10H_{-2}\zeta_2 + 6H_{-2,0,0} + 2H_{0,0}\zeta_2 - 9H_{-1,-1,0} - 7H_{-1,2} - 9H_{-2,0} - 2H_{3,1}$ $-4H_{-2,-1,0} - 4H_4 - 4H_{3,0} - 4H_{0,0,0,0} + \frac{37}{2}H_{-1,0} + \frac{5}{2}(1+x)H_{-1}\zeta_2 - 4H_{-2,0,0} + 2H_{0,0}\zeta_2$ $+H_2\zeta_2 - 3H_{1,1,0} + 2H_{0,0,0,0} + H_{-3,0} - 9H_{2,1,0} - \frac{9}{2}H_{2,1,1} + \frac{11}{2}H_{1,1,1} + \frac{19}{2}H_{2,0,0} + \frac{9}{2}H_{1,2}$ $-\frac{91}{2}H_0\zeta_3 + 8H_{-2}\zeta_2 + \frac{5}{2}H_{-1,-1,0} + \frac{5}{2}H_{-1,2} + \frac{9}{2}H_{-1,0} + \frac{39}{2}H_{-2,0} - \frac{473}{12}H_0\zeta_2 - \frac{1853}{48}H_{0,0}$ $\frac{217}{2}\zeta_3 - \frac{59}{2}\zeta_2^2 - \frac{169}{18}H_2 - \frac{13}{4}H_1\zeta_2 - \frac{2}{3}H_{1,0,0} + \frac{167}{24}H_{1,0} + \frac{191}{18}H_{1,1} + \frac{1083}{128}H_1 + \frac{185}{12}H_{2,1}$ $\frac{12}{75} \frac{4}{H_{2,0}} + \frac{170}{9} \zeta_2 + \frac{85}{4} \frac{4}{H_{0,0,0}} + \frac{425}{12} H_3 + \frac{7693}{192} + \frac{3659}{48} H_0 - 2x \Big[xH_{2,2} + 4H_{3,0} - 4H_{-2,2} \Big] \Big)$ $+16C_A n_f^2 \left(\frac{1}{z} p_{qg}(x) \left[H_{1,2} - H_1 \zeta_2 - H_{1,0,0} - H_{1,1,0} - H_{1,1,1} - \frac{229}{19}H_0 + \frac{4}{2}H_{0,0} + \frac{11}{2}\right] + x \left[\frac{1}{z}H_2 - H_1 \zeta_2 -$ $-\frac{53}{18}H_0 + \frac{17}{6}H_{0,0} - \zeta_3 + \frac{11}{18}\zeta_2 - \frac{139}{108}] + \frac{1}{3}p_{48}(-x)H_{-1,0,0} - \frac{53}{162}(\frac{1}{x} - x^2) - \frac{2}{9}(1-x)\Big[6H_{0,0,0}(1-x)H_{0,0,0}(1-x)] + \frac{11}{3}H_{0,0,0}(1-x)H_{0,0}(1-x)H_{0,$ $\frac{1}{6} xH_1 - H_{0,0} + \frac{7}{2} xH_{1,1} + \frac{7}{9} x(1+x)H_{-1,0} + \frac{7}{4}H_0 - \frac{19}{54}H_1 + H_{0,0,0} + \frac{5}{9}H_{1,1} + \frac{5}{9}H_{-1,0}$ $\frac{6}{5} - \frac{85}{216} + 16C_A^{-2}n_f \left(p_{qg}(x)\right) \left[3H_{1,3} + \frac{3}{16}H_{1,0,0} - \frac{17}{2}H_{2,1} + \frac{7}{5}\zeta_2^2 - \frac{55}{12}H_{1,1,0} + \frac{3}{12}H_3 - \frac{31}{2}H_1\zeta_3$ $\frac{210}{52}H_{2,0} - \frac{63}{8}H_{1,0} - \frac{23}{12}H_{1,2} - \frac{155}{6}\zeta_3 + \frac{25}{24}H_2 - \frac{2537}{27}H_0 + \frac{867}{7} - \frac{23}{24}H_{-1,0,0} + 3H_4 - H_{1,1,1}$ $\frac{12}{72}H_{1,1} - \frac{25}{2}H_{-2,0} - \frac{3}{8}\zeta_2 - \frac{7}{4}H_1\zeta_2 - 3H_{0,0}\zeta_2 - \frac{3}{12}H_0\zeta_2 + \frac{103}{216}H_1 + \frac{5}{2}H_{1,0,0,0} + \frac{2561}{72}H_{0,0,0}$ 12 2 2 8 4 12 210 2 12 $-2H_{1,1,2} - 2H_{1,2,0}$ + $p_{qg}(-x)$ $\left[H_{-1,-1}\zeta_2 - 2H_{-1,2} - 6H_{-1,-1,0} + H_{1,1,1} + 2H_{-2}\zeta_2 - H_{-2,0,0}\right]$ $+\frac{727}{36}H_{-1,0}-H_{-1}\zeta_2-2H_{-2,2}-\frac{5}{2}H_{-1}\zeta_3-H_{-1,-2,0}+2H_{-1,-1,0,0}+2H_{-1,-1,2}-\frac{3}{2}H_{-1,0,0,0}$ $+6H_{-1,-1,-1,0}-2H_{-1,3}+2H_{-1,2,1}] + (\frac{1}{2}-x^2)\left[\frac{2}{2}H_{2,1}+\frac{32}{9}\zeta_2-2H_{1,0,0}+\frac{4}{2}H_{1,1,0}-\frac{10}{9}H_{1,1}\right]$ $-\frac{8}{9}H_{-1,0,0} + \frac{3}{9}H_{1,0} + 6\zeta_3 + \frac{161}{2c}H_1 - \frac{2351}{2c} + \frac{2}{2}(\frac{1}{2}+x^2)\left[\frac{26}{2}H_{-1,0} - \frac{26}{2}H_{0,0} - \frac{2}{3}H_{0,0} - \frac{2}{2}H_{-1,0} - \frac{26}{2}H_{-1,0} - \frac{26}{2}H_{-1$

 $+\frac{1}{3}p_{03}(x)\Big[H_{1,2}-H_{1,0}-H_{1}\zeta_{2}+9\zeta_{3}+\frac{83}{12}H_{1,1}+2H_{-2,0}-\frac{7}{36}H_{1}+2H_{0}\zeta_{2}-\frac{1625}{48}+\frac{3}{2}H_{1,0,0}$ $+2H_{1,1,0} - \frac{5}{2}H_{1,1,1} + \frac{31}{18}p_{gq}(-x) \left[\frac{95}{92}H_0 - \zeta_2 - H_{-1,0}\right] + \frac{1}{3}(2-x) \left[6H_{0,0,0,0} - H_3 - \frac{13051}{288}H_0 - \zeta_2 - H_{-1,0}\right]$ $-\frac{13}{2}\zeta_3 - 4H_{-2,0} - H_{2,0} - \frac{1}{2}H_{1,0} - \frac{1}{2}H_{2,1} + 2H_{0,0,0} - \frac{653}{24}H_{0,0} + (1+x)\left[H_0\zeta_2 - \frac{1187}{216}H_0\right]$ $\frac{2}{8}H_{2} - \frac{85}{18}H_{-1,0} - \frac{101}{18}\zeta_{2} - \frac{80}{27}H_{0} + \frac{23}{18}\zeta_{2} - \frac{1}{3}H_{1,1} + \frac{5}{4}xH_{1,1} - \frac{1}{9}H_{1} - \frac{37}{17}xH_{1} + \frac{210}{18}H_{-1,0}$ $\frac{9}{164} + \frac{10}{164} + \frac{10}{10} + \frac{10}{3} + \frac{10}{10} + \frac{10}{3} + \frac{1}{100} + \frac{10}{3} + \frac{10}{100} + \frac{1$ $+\frac{1}{9}\frac{1}{9}\frac{1}{9}\frac{1}{10}\frac{1}{9}\frac{1}{9}\frac{1}{10}\frac{1}{9}\frac{1}{10}\frac{1}{9}\frac{1}{9}\frac{1}{10}\frac{1}{9}\frac{1}{9}\frac{1}{10}\frac{1}{9}\frac{1}{10}\frac{1}{9}\frac{1}{10}\frac{1}{9}\frac{1}{10}$ $\frac{8}{8} + p_{1,2,1} - \frac{9}{2}H_{1,2,1} - \frac{9}{2}H_{1,1,1} - \frac{3}{2}H_{1,0,0} - \frac{47}{16} - \frac{47}{16}H_1 - \frac{15}{2}\zeta_3 + p_{gg}(-x) \left[2H_{-1,-2,0} - \frac{16}{2}H_{-1,-2,0} - \frac{16$ $+6H_{-1,-1,0} + 3H_{-1}\zeta_2 + \frac{7}{4}H_{1,0} - \frac{16}{5}\zeta_2^2 - 6H_{-1,0,0} - \frac{7}{2}H_{-1,0} + 4H_{-1,-1,0,0} - 2H_{-1,0}\zeta_2$ $-H_{-1,0,0,0}$ + (1 - x) [9H_{1,0,0} + H_{1,1,1} - 10H₁ζ₂ + 3H₀ζ₃ + H_{2,2} - H₂ζ₂ + H_{0,0,0} + 5H_{2,0,0} $-4H_3 + H_{2,1,1} + 3H_{0,0}\zeta_2 + 3H_{3,1} - 3H_4 + \frac{211}{16}H_1 + \frac{49}{20}\zeta_2^2 + (1+x) \left[11\zeta_3 + \frac{1}{4}H_{1,1} + \frac{1}{4}H_{1,0} + \frac{1}{4}H_{1,1} + \frac{1}{$ $+\frac{91}{16}H_{0}+36H_{-1,0}+8H_{-1,0,0}-14H_{-1,-1,0}-7H_{-1}\zeta_{2}+2H_{1,2}+4H_{0}\zeta_{2}-H_{2,1}+2H_{-2,0,0}$ $+5H_{-2,0} + \frac{11}{2}H_2 - 2H_{0,0,0,0} - 2H_{-1,-1,0} - H_{-1}\zeta_2 - \frac{13}{4}\zeta_2 + \frac{9}{4}H_{1,0} + \frac{9}{20}\zeta_2^2 + \frac{287}{422} + \frac{11}{12}H_1$ $+4H_{-1,0,0}+16H_{-3,0}-4H_{-2}\zeta_2-8H_{-2,-1,0}-5H_2\zeta_2+\frac{19}{4}H_2+H_{2,2}-\frac{35}{6}H_{0,0}+9H_0\zeta_3$ $+25H_{-2,0}+6H_{-2,0,0}+\frac{3}{2}x\Big[\frac{58}{3}\zeta_2-\frac{7}{2}H_1\zeta_2+4H_{1,1}-\frac{3}{2}H_{1,1,1}+\frac{5}{2}H_{1,0,0}-\frac{175}{96}+H_{3,1}+\frac{19}{3}\zeta_3+\frac{$ $+2H_{2,0}-14H_0+H_{0,0}\zeta_2-H_{-1,0}-H_4-\frac{3}{2}H_{2,1}+\frac{1}{3}H_{2,1,1}+3H_{2,0,0}-\frac{5}{6}H_3-H_{1,2}-\frac{7}{6}H_0\zeta_2$ $+\frac{2}{2}H_{1,1,0}-\frac{29}{6}H_{0,0,0}-\frac{185}{9}H_{0,0}$]).

$$\begin{split} \mu_{11}^{(2)}(z) &= 16C_{4}C_{5}\phi_{1}\left(z^{2} \begin{bmatrix} \frac{1}{2}H_{1}+3H_{1,0}-\frac{97}{12}H_{1}+\frac{8}{3}H_{-2,0}-\frac{2}{3}H_{5}L_{5}+\frac{100}{27}H_{0}-\frac{16}{5}c_{2}+2H_{1} \\ -6H_{-1,0}+2H_{2,0}+\frac{217}{12}H_{0,0}-\frac{11}{12}\right)+\mu_{01}(z)\left[2\zeta_{5}-\frac{51}{25}\right]+\frac{4}{3}(\frac{1}{a}-z^{2})\left[\frac{27}{2}H_{1,0}-\frac{48}{3}H_{0} \\ -\frac{141}{4}H_{-1,0}+8H_{5}\phi_{1}-6H_{-2,0}-\frac{85}{3}H_{5}\phi_{2}-2H_{1,0,0}+\frac{112}{12}H_{-1}-H_{1,1,0}-H_{1,1,1}-H_{1,1,1}\right]-\frac{173}{12}H_{1} \\ +6H_{-1,0}+8H_{5}\phi_{0}-6H_{-2,0}-\frac{85}{3}H_{5}\phi_{2}-\frac{29}{2}H_{0}+\frac{185}{4}c_{2}-\frac{112}{2}H_{1,0}-3H_{1,0}-4H_{0,00} \\ -\frac{7}{12}H_{0,0}+\frac{4}{2}c_{3}-H_{2}+\frac{17}{2}H_{1}-4C_{4}c_{2}-\frac{2}{2}H_{3}-8H_{-3}H_{-3}-\frac{3}{2}H_{0,0}+\frac{4}{3}(\frac{1}{a}+z^{2})\left[\frac{1}{2}H_{-2}-H_{2,0}\right] \\ +\frac{1}{4}H_{-1}H_{-1}H_{-1}H_{-1}H_{-2}H_{-2}+\frac{2}{2}H_{-3}-8H_{-3}H_{-3}-\frac{3}{4}H_{-4}H_{-4}+\frac{1}{4}H_{-1}+\frac{1}{4}H_{-1}-4H_{0,00} \\ +\frac{1}{4}H_{-1}H_{-1}H_{-1}H_{-1}H_{-2}H_{-2}-\frac{2}{2}H_{-3}-8H_{-3}H_{-3}-\frac{3}{4}H_{0,0}+\frac{4}{3}(\frac{1}{a}+z^{2})\left[\frac{1}{2}H_{-}-H_{2,0}\right] \\ +\frac{1}{4}H_{-1}H_{-1}H_{-1}H_{-1}H_{-2}H_{-2}H_{-2}-4H_{-1,00}-4H_{-1,10}-4H_{-1$$

 $-2H_{-1,2} + H_1\zeta_2 + H_{-1}\zeta_2 + \frac{10}{2}H_2 + H_{1,1,1} + (1-x)\left[15H_{0,0,0,0} - 5H_2\zeta_2 - \frac{65}{6}\zeta_3 + \frac{11}{6}H_{1,1,1}\right]$ $-\frac{3}{2}H_4 + \frac{5}{2}H_{0,0}\zeta_2 + H_{1,1,0} - \frac{31}{6}H_{2,0} + \frac{17}{12}H_{1,0} - \frac{551}{20}\zeta_2^2 - \frac{29}{4}H_{1,0,0} - \frac{113}{4}H_2 + \frac{18691}{72}H_0$ $+\frac{2243}{108}+\frac{265}{6}H_{-1,0,0}+\frac{33}{2}H_{2,0,0}+19H_{2,1}+\frac{31}{12}H_{1,1}+\frac{23}{2}H_{-2,0}-\frac{497}{36}\zeta_2+\frac{29}{6}H_1\zeta_2-\frac{143}{12}H_{2,1}$ $\frac{108}{14} \frac{10}{14} \frac{10}{14} \frac{19}{12} \frac{1223}{14} \frac{1223}{14} \frac{143}{14} \frac{12}{14} \frac{12}{14$ $+7H_{-1,-1,0} - \frac{35}{6}H_{1,1,1} - 5H_{-2}\zeta_2 - 11H_{-2,0,0} + \frac{1}{3}H_{-1,0} + \frac{15}{2}H_{-1}\zeta_2 + 8H_{3,1} - 10H_{-2,-1,0}$ $+5H_2\zeta_2 + 4H_{2,1,1} - H_{-3,0} + 36H_0\zeta_3 - 5H_2\zeta_2 + 2H_{-1,2} + 6H_{-1,-1,0} - 6H_{2,1,0} - 3H_{2,1,1}$ $-11H_{0,0,0,0}-5H_{3,1}+\frac{25}{4}H_{1,1,1}+\frac{13}{2}H_{-2}\zeta_2+\frac{27}{2}H_{-2,0,0}+\frac{11}{2}H_{-3,0}+\frac{13}{2}H_{2}\zeta_2-\frac{17}{4}H_{1,0,0}$ $+13H_{-2,-1,0} - \frac{17}{12}H_{1,1,1} - \frac{3}{4}H_4 - \frac{1}{4}H_{0,0}\zeta_2 + H_{1,2} + \frac{12}{2}H_{1,1,0} + \frac{79}{12}H_{2,0} + \frac{67}{8}H_{1,0} + \frac{263}{8}\zeta_2^2$ $+\frac{119}{3}\zeta_{3}+\frac{967}{24}H_{2}-\frac{305}{12}H_{-1,0}-24H_{0}\zeta_{3}+H_{-1}\zeta_{2}-\frac{13375}{72}H_{0}-\frac{1889}{18}-38H_{-1,0,0}-\frac{21}{2}H_{2,1}$ $\frac{4}{12} + \frac{145}{12} + \frac{153}{12} + \frac{153}{12} + \frac{1}{12} + \frac{1$ $\begin{array}{c} 12 & 24 & 10 \\ -\frac{5}{9}H_{1,1} - \frac{5}{9}H_2 - \frac{5}{18}H_{1,0} + \frac{5}{9}\zeta_2 + \frac{1}{6}\rho_{SB}(x) \Big[H_{2,1} + \frac{41}{2} - \frac{35}{3}H_0 - \frac{24}{2} - \frac{24}{3}H_{0,0} + H_{1,1,1} + 6H_{0,0,0} \\ -\zeta_3 - 2H_{1,0,0} + \frac{7}{9}H_1 \Big] + \frac{77}{81} \Big[\frac{1}{x} - x^2 \Big) + (1 - x) \Big[\frac{1}{12}H_1 - \frac{643}{432} - 4H_{0,0,0,0} - \frac{16}{3}H_{0,0,0} + \frac{7}{9}\pi H_{1,1} \Big] \\ \end{array}$ $+\frac{7}{9}xH_2 + \frac{8}{9}xH_{1,0} - \frac{7}{9}x\zeta_2 \left[-(1+x) \left[\frac{3475}{216}H_0 + \frac{103}{12}H_{0,0} \right] \right] + 16C_F^{-2}n_f \left(p_{qg}(x) \left[7H_{1,3} + 7H_4 + \frac{103}{12}H_0 \right] \right) + 16C_F^{-2}n_f \left(p_{qg}(x) \left[7H_{1,3} + 7H_4 + \frac{103}{12}H_0 \right] \right) + 16C_F^{-2}n_f \left(p_{qg}(x) \left[7H_{1,3} + 7H_4 + \frac{103}{12}H_0 \right] \right) + 16C_F^{-2}n_f \left(p_{qg}(x) \left[7H_{1,3} + 7H_4 + \frac{103}{12}H_0 \right] \right) + 16C_F^{-2}n_f \left(p_{qg}(x) \left[7H_{1,3} + 7H_4 + \frac{103}{12}H_0 \right] \right) + 16C_F^{-2}n_f \left(p_{qg}(x) \left[7H_{1,3} + 7H_4 + \frac{103}{12}H_0 \right] \right) + 16C_F^{-2}n_f \left(p_{qg}(x) \left[7H_{1,3} + 7H_4 + \frac{103}{12}H_0 \right] \right) + 16C_F^{-2}n_f \left(p_{qg}(x) \left[7H_{1,3} + 7H_4 + \frac{103}{12}H_0 \right] \right) + 16C_F^{-2}n_f \left(p_{qg}(x) \left[7H_{1,3} + 7H_4 + \frac{103}{12}H_0 \right] \right) + 16C_F^{-2}n_f \left(p_{qg}(x) \left[7H_{1,3} + 7H_4 + \frac{103}{12}H_0 \right] \right) + 16C_F^{-2}n_f \left(p_{qg}(x) \left[7H_{1,3} + 7H_4 + \frac{103}{12}H_0 \right] \right) + 16C_F^{-2}n_f \left(p_{qg}(x) \left[7H_{1,3} + 7H_4 + \frac{103}{12}H_0 \right] \right) + 16C_F^{-2}n_f \left(p_{qg}(x) \left[7H_{1,3} + 7H_4 + \frac{103}{12}H_0 \right] \right) + 16C_F^{-2}n_f \left(p_{qg}(x) \left[7H_{1,3} + 7H_4 + \frac{103}{12}H_0 \right] \right) + 16C_F^{-2}n_f \left(p_{qg}(x) \left[7H_{1,3} + 7H_4 + \frac{103}{12}H_0 \right] \right) + 16C_F^{-2}n_f \left(p_{qg}(x) \left[7H_{1,3} + 7H_4 + \frac{103}{12}H_0 \right] \right) + 16C_F^{-2}n_f \left(p_{qg}(x) \left[7H_{1,3} + 7H_4 + \frac{103}{12}H_0 \right] \right) + 16C_F^{-2}n_f \left(p_{qg}(x) \left[7H_{1,3} + 2H_4 + \frac{103}{12}H_0 \right] \right) + 16C_F^{-2}n_f \left(p_{qg}(x) \left[7H_{1,3} + 2H_4 + \frac{103}{12}H_0 \right] \right) + 16C_F^{-2}n_f \left(p_{qg}(x) \left[7H_{1,3} + 2H_4 + \frac{103}{12}H_0 \right] \right) + 16C_F^{-2}n_f \left(p_{qg}(x) \left[7H_{1,3} + 2H_4 + \frac{10}{12}H_0 \right] \right) + 16C_F^{-2}n_f \left(p_{qg}(x) \left[7H_{1,3} + 2H_4 + \frac{10}{12}H_0 \right] \right) + 16C_F^{-2}n_f \left(p_{qg}(x) \left[7H_{1,3} + 2H_4 + \frac{10}{12}H_0 \right] \right) + 16C_F^{-2}n_f \left(p_{qg}(x) \left[7H_{1,3} + 2H_4 + \frac{10}{12}H_0 \right] \right) + 16C_F^{-2}n_f \left(p_{qg}(x) \left[7H_{1,3} + 2H_4 + \frac{10}{12}H_0 \right] \right) + 16C_F^{-2}n_f \left(p_{qg}(x) \left[7H_{1,3} + 2H_4 + \frac{10}{12}H_0 \right] \right) + 16C_F^{-2}n_f \left(p_{qg}(x) \left[7H_{1,3} + 2H_4 + \frac{10}{12}H_0 \right] \right) + 16C_F^{-2}n_f \left(p_{qg}(x) \left[7H_{1,3} + 2H_4 + \frac{10}{12}H_$ $-2H_{-3,0} - 7H_1\zeta_3 + 5H_{2,2} + 6H_{3,0} + 6H_{3,1} + H_{2,1,0} + 4H_{2,0,0} + 3H_{2,1} + 2H_{2,1,1} + \frac{5}{4}H_{2,0}$ $+\frac{61}{8}H_2-\frac{61}{8}\zeta_2+\frac{87}{8}H_1+\frac{11}{2}H_{1,2}+\frac{61}{8}H_{1,1}+\frac{17}{2}H_{1,0}-7H_{0,0}\zeta_2+\frac{5}{2}H_{1,0,0}+\frac{5}{2}H_{1,1,0}-\frac{19}{2}\zeta_3$ $+ \frac{81}{32} + \frac{11}{2}H_3 - \frac{11}{2}H_0\zeta_2 - \frac{7}{2}H_1\zeta_2 + \frac{15}{2}H_{0,0,0} + \frac{87}{8}H_0 + \frac{11}{\epsilon}\zeta_2^2 + 3H_{1,1,1} - 5H_2\zeta_2 - 7H_0\zeta_3$ ${}^{32}_{+11} {}^{2}_{H_{0,0}-2} {}^{2}_{H_{1,-2,0}-7} {}^{2}_{H_{1,0}\zeta_2+3} {}^{2}_{H_{1,0,0,0}-5} {}^{8}_{H_{1,1}\zeta_2+4} {}^{5}_{H_{1,1,0,0}+H_{1,1,1,0}+2}_{H_{1,1,1,1}+5} {}^{4}_{H_{1,1,2}}$ $+6H_{1,2,0}+6H_{1,2,1}\Big]+4p_{qg}(-x)\Big[H_{0,0,0,0}-H_{-2,0}+H_{-1,-1,0}-H_{-2,0,0}+\frac{1}{2}H_{-1,-2,0}-\frac{5}{8}H_{-1,0}$ $-\frac{5}{4}H_{-1,0,0} - \frac{1}{2}H_{-3,0} + \frac{1}{2}H_{-1}\zeta_2 + H_{-1,-1,0,0} - \frac{1}{4}H_{-1,0,0,0} + 2(1-x)\left[H_{2,1,0} - H_{2,0,0} - H_{2,2}\right]$ $-H_{3,1} - 2H_{3,0} - 2H_{-1}\zeta_2 + H_{1,2} - H_{1,0,0} - H_{1,1,0} + H_2\zeta_2 - \zeta_2^2 + \frac{43}{9}H_2 + \frac{49}{9}\zeta_2 + \frac{13}{9}H_{1,1}$ $-\frac{33}{16}H_1+\frac{5}{2}H_{1,0}+\frac{7}{2}H_{0,0}\zeta_2+\frac{21}{4}\zeta_3+\frac{479}{64}-\frac{1}{2}H_{1,1,1}-\frac{1}{2}H_3+\frac{1}{4}H_{2,1}+\frac{1}{2}H_{2,1,1}+\frac{3}{2}H_0\zeta_2$ $+\frac{1}{2}H_{0}\zeta_{3}-\frac{7}{2}H_{4}+H_{1}\zeta_{2}-\frac{19}{2}H_{0,0,0}-\frac{239}{16}H_{0,0}-\frac{405}{32}H_{0}\Big]+8(1+x)\Big[H_{-1,-1,0}-H_{-1,0,0}-H_$

 $+12H_{0,0,0,0}-\frac{293}{108}+\frac{61}{6}H_0\zeta_2-\frac{7}{3}H_{1,0}-\frac{857}{36}H_1-9H_0\zeta_3+16H_{-2,-1,0}-4H_{-2,0,0}+8H_{-2}\zeta_2$ $-\frac{13}{2}H_{1,0,0} + \frac{3}{4}H_{1,1} - H_{1,1,0} - H_{1,1,1} + (1+x) \left[\frac{1}{6}H_{2,0} - \frac{95}{3}H_{-1,0} - \frac{149}{36}H_2 + \frac{3451}{108}H_0\right]$ $\begin{array}{c} 2\\ -7H_{-2,0}+\frac{43}{9}H_{0,0}+\frac{19}{6}H_3-\frac{991}{36}\zeta_2-\frac{163}{6}\zeta_3-\frac{35}{3}H_{0,0,0}+\frac{17}{6}H_{2,1}-\frac{43}{10}\zeta_2^2+13H_{-1}\zeta_2\\ +18H_{-1,-1,0}-H_{3,1}-6H_4-4H_{-1,2}+6H_{0,0}\zeta_2+8H_2\zeta_2-7H_{2,0,0}-2H_{2,1,0}-2H_{2,1,1}-4H_{3,0}\end{array}$ $-9H_{-1,0,0}\left]-\frac{241}{288}\delta(1-x)\right)+16C_An_f^2\left(\frac{19}{54}H_0-\frac{1}{24}xH_0-\frac{1}{27}p_{gg}(x)+\frac{13}{54}(\frac{1}{x}-x^2)\left[\frac{5}{3}-H_1\right]\right]$ $+(1-x)\left[\frac{11}{12}H_1-\frac{71}{216}\right]+\frac{2}{9}(1+x)\left[\zeta_2+\frac{13}{12}xH_0-\frac{1}{2}H_{0,0}-H_2\right]+\frac{29}{288}\delta(1-x)\right)$ $+16C_{A}^{2}n_{f}\left(x^{2}\left[\zeta_{3}+\frac{11}{9}\zeta_{2}+\frac{11}{9}H_{0,0}-\frac{2}{3}H_{3}+\frac{2}{3}H_{0}\zeta_{2}+\frac{1639}{108}H_{0}-2H_{-2,0}\right]+\frac{1}{3}p_{gg}(x)\left[\frac{10}{3}\zeta_{2}+\frac{1}{3}H_{0}^{2}+\frac{$ $-\frac{209}{36} - 8\zeta_3 - 2H_{-2,0} - \frac{1}{2}H_0 - \frac{10}{3}H_{0,0} - \frac{20}{3}H_{1,0} - H_{1,0,0} - \frac{20}{3}H_2 - H_3 \Big] + \frac{10}{9}p_{gg}(-x)\Big[\zeta_2$ $\begin{array}{c} 36 \\ +2H_{-1,0}+\frac{3}{10}H_0\zeta_2-H_{0,0} \right] + \frac{1}{3}(\frac{1}{x}-x^2) \left[H_3-H_0\zeta_2-\frac{13}{3}H_2+\frac{5443}{108}-3H_1\zeta_2+\frac{23}{36}H_1 \right] \\ -\frac{13}{3}H_{1,0}+H_{1,0,0} \right] + (\frac{1}{x}+x^2) \left[\frac{15}{54}H_0-\frac{3}{5}\zeta_2+\frac{1}{3}H_{-1}\zeta_2-\zeta_3+2H_{-1,-1,0}-\frac{2}{3}H_{-1,0,0} \right] \\ \end{array}$ $-\frac{37}{9}H_{-1,0} + \frac{2}{3}H_{-1,2}\Big] + (1-x)\Big[\frac{5}{6}H_{-2,0} + H_{-3,0} + 2H_{0,0,0} - \frac{269}{36}\zeta_2 - \frac{4097}{216} - 3H_{-2}\zeta_2 - \frac{4097}{216} - 3H_{-2}\zeta_2 - \frac{4097}{216} - 3H_{-2}\zeta_2 - \frac{10}{36}\zeta_2 - \frac{10}{36}\zeta$ 9 3 3 1 (6 36 216 -6H_{-2,-1,0} + 3H_{-2,0,0} $-\frac{7}{2}$ H₁ ζ_2 + $\frac{677}{22}$ H₁ + H_{1,0} + $\frac{7}{4}$ H_{1,0,0} + (1 + x) $\left[\frac{193}{32}$ H₂ - $\frac{11}{2}$ H₋₁ ζ_2 $+\frac{39}{20}\zeta_{2}^{-2}-\frac{7}{12}H_{3}-\frac{53}{0}H_{0,0}+\frac{7}{12}H_{0}\zeta_{2}-\frac{5}{2}H_{0,0}\zeta_{2}+5\zeta_{3}-7H_{-1,-1,0}+\frac{77}{2}H_{-1,0}+\frac{9}{2}H_{-1,0,0}$ $\begin{array}{c} 20 & 12 & 9 \\ +2H_{-1,2} - 3H_2\zeta_2 - \frac{2}{2}H_{2,0} + \frac{3}{2}H_{2,0,0} + \frac{3}{2}H_4 \\ \end{array} + \frac{1}{0}\zeta_2 + 7H_{-2,0} + 2H_2 + \frac{458}{27}H_0 + H_{0,0}\zeta_2 \end{array}$ $+\frac{3}{2}\zeta_{2}^{2}+4H_{-3,0}-x\left[\frac{131}{12}H_{0,0}-\frac{8}{3}H_{0}\zeta_{2}+\frac{7}{2}H_{3}-H_{0,0,0,0}+\frac{7}{6}H_{0,0,0}+\frac{1943}{216}H_{0}+6H_{0}\zeta_{3}\right]$ $-\delta(1-x)\left[\frac{233}{288}+\frac{1}{6}\zeta_{2}+\frac{1}{17}\zeta_{2}^{2}+\frac{5}{3}\zeta_{3}\right]\right)+16C_{A}^{-3}\left(x^{2}\left[33H_{-2,0}+33H_{0}\zeta_{2}-\frac{1249}{18}H_{0,0}\right]$ $-44H_{0,0,0} - \frac{110}{3}H_3 - \frac{43}{3}H_{2,0} + \frac{85}{85}\xi_2 + \frac{6409}{108}H_0 \Big] + p_{gg}(x) \Big[\frac{245}{24} - \frac{67}{9}\xi_2 - \frac{3}{10}\xi_2^2 + \frac{11}{3}\xi_3 + \frac$ $+\frac{134}{2}H_{1,0} + \frac{11}{2}H_{1,0,0} + 8H_{1,2,0} + 8H_{1,3} + \frac{134}{2}H_2 - 4H_2\zeta_2 + 8H_{3,1} + 8H_{2,2} + \frac{11}{2}H_3 + 10H_{3,0}$ $\begin{array}{c} 9 \\ +8H_{2,1,0} \\ + Rg_{2,1,0} \\ + P_{gg}(-x) \Big[\frac{11}{2}\zeta_2^2 - \frac{11}{6}H_0\zeta_2 - 4H_{-3,0} + 16H_{-2}\zeta_2 - 12H_{-2,2} - \frac{134}{9}H_{-1,0} + 2H_2\zeta_2 \\ +8H_{-2,-1,0} + 12H_{-1}\zeta_3 - 18H_{-2,0,0} + 8H_{-1,-2,0} - 16H_{-1,-1}\zeta_2 + 24H_{-1,-1,0} + 16H_{-1,-1,2} \\ \end{array}$
$$\begin{split} +18H_{-1,0}\zeta_2 - 16H_{-1,0,0,0} - 4H_{-1,2,0} - 16H_{-1,3} - 5H_0\zeta_3 - 8H_{0,0}\zeta_2 + 4H_{0,0,0,0} + 2H_{3,0} \\ -\frac{67}{9}\zeta_2 + \frac{67}{9}H_{0,0} + 8H_4 \Big] + \Big(\frac{1}{x} - x^2\Big) \Big[\frac{16619}{162} + \frac{22}{3}H_{2,0} - \frac{52}{5}\zeta_3 - \frac{11}{2}H_0\zeta_2 - \frac{67}{9}H_2 - \frac{67}{9}H_2 \\ -\frac{67}{9}H_2 - \frac{67}{9}H_1 - \frac{67}{9}H_1 - \frac{67}{9}H_1 - \frac{67}{9}H_2 - \frac{67}{9}H_2 \\ -\frac{67}{9}H_2 - \frac{67}{9}H_1 - \frac{67}{9}H_1 - \frac{67}{9}H_1 \\ -\frac{67}{9}H_2 - \frac{67}{9}H_1 - \frac{67}{9}H_1 - \frac{67}{9}H_1 \\ -\frac{67}{9}H_2 - \frac{67}{9}H_2 - \frac{67}{9}H_2 - \frac{67}{9}H_2 - \frac{67}{9}H_1 \\ -\frac{67}{9}H_2 - \frac{67}{9}H_2 - \frac{67}{9}H_2 - \frac{67}{9}H_2 \\ -\frac{67}{9}H_2 \\ -\frac{$$

$$\begin{split} &-H_{00,0,0}+\frac{3}{4}H_{-2,0}-\frac{9}{4}H_{-1,0}\Big]-4H_{-1,-1,0}+5H_{00,0,0}+5H_{-1,0,0}-13H_{-2,0}+\frac{1}{2}H_{-1,0}\\ &+44H_{-20,0}-\frac{11}{3}H_{0}-\frac{7}{1}\xi_{0}-\frac{3}{2}H_{0}-\frac{1}{2}H_{1,0}-\frac{3}{8}H_{1}-\frac{7}{2}H_{0}-\frac{7}{2}H_{0}\xi_{0}-\frac{5}{2}H_{0,00}\\ &-\frac{5}{2}H_{-1,0}-\frac{7}{2}\xi_{0}-\frac{5}{2}\xi_{0}-\frac{7}{2}\xi_{0}-\frac{7}{2}\xi_{0}-\frac{1}{2}H_{0}-\frac{7}{2}H_{0}-\frac{7}{2}H_{0}-\frac{7}{2}H_{0}+\frac{7}{2}H_{0}-\frac$$

 $P_{89}^{(2)}(x) = 16C_A C_F n_f \left(\frac{2}{9}x^2 \left[\frac{25}{6}H_1 - \frac{131}{4} + 3\zeta_2 - H_{-1,0} - 3H_2 + H_{1,1} + \frac{125}{6}H_0 - H_{0,0}\right]\right)$ $+\frac{5}{6}p_{89}(x)\Big[H_{1,2}+H_{2,1}+\frac{967}{120}+\frac{251}{90}H_1-\frac{39}{10}H_{1,1}-3\zeta_3-\frac{2}{5}H_0\zeta_2-\frac{1}{5}H_1\zeta_2-\frac{4}{3}H_{1,0}+H_{1,1,0}$ $\frac{2}{8}H_{1,0,0} + H_{1,1,1} + \frac{2}{8}H_{2,0} + \frac{2}{3}p_{88}(-x) \left[2H_{-1}\zeta_2 + \frac{7}{4}\zeta_2 + \frac{41}{12}H_{-1,0} - \frac{151}{72}H_0 + \frac{1}{2}H_{-2,0}\right]$ $+\frac{5}{2}H_2 + 2H_{-1,-1,0} - H_{-1,0,0} - H_{-1,2} + \frac{2}{3}(1-x)[H_{-2,0} + 2\zeta_3 - H_3] + (1+x)[\frac{179}{100}H_1]$ $\frac{5}{5}\zeta_2 + \frac{25}{9}H_{-1,0} - \frac{5}{26}H_{1,1} - \frac{167}{26}H_{0,0} - \frac{1}{2}H_{2,1} - \frac{4}{2}H_0\zeta_2 - \frac{193}{23} + \frac{1}{4}H_1 + \frac{1}{9}H_{-1,0} + 4H_2$ $-\frac{1}{4}H_{1,1} + \frac{227}{18}H_0 - \frac{35}{12}H_{0,0} - H_{2,1} - \frac{2}{2}H_0\zeta_2 + \frac{10}{2}H_{-2,0} + 3\zeta_3 + 2H_3 + \frac{2}{2}H_{0,0,0} + x\left[\frac{11}{4}\zeta_2 + \frac{10}{2}H_{-2,0} + \frac{10}{2}G_3 + \frac{10}{2}H_3 + \frac{10}{2}G_3 + \frac$ $\frac{523}{144} - \frac{19}{36}H_2 + \frac{271}{108}H_0 - \frac{5}{6}H_{1,0} \Big] + 16C_A C_F^{-2} \Big(x^2 \Big[\frac{7}{2} + \frac{173}{54}H_1 - 2\zeta_3 - \frac{2}{3}H_{1,1,1} - \frac{26}{9}H_{1,1} \Big] + 16C_A C_F^{-2} \Big(x^2 \Big[\frac{7}{2} + \frac{173}{54}H_1 - 2\zeta_3 - \frac{2}{3}H_{1,1,1} - \frac{26}{9}H_{1,1} \Big] + 16C_A C_F^{-2} \Big(x^2 \Big[\frac{7}{2} + \frac{173}{54}H_1 - 2\zeta_3 - \frac{2}{3}H_{1,1,1} - \frac{26}{9}H_{1,1} \Big] + 16C_A C_F^{-2} \Big(x^2 \Big[\frac{7}{2} + \frac{173}{54}H_1 - 2\zeta_3 - \frac{2}{3}H_{1,1,1} - \frac{26}{9}H_{1,1} \Big] + 16C_A C_F^{-2} \Big(x^2 \Big[\frac{7}{2} + \frac{173}{54}H_1 - 2\zeta_3 - \frac{2}{3}H_{1,1,1} - \frac{26}{9}H_{1,1} \Big] + 16C_A C_F^{-2} \Big(x^2 \Big[\frac{7}{2} + \frac{173}{54}H_1 - 2\zeta_3 - \frac{2}{3}H_{1,1,1} - \frac{26}{9}H_{1,1} \Big] + 16C_A C_F^{-2} \Big(x^2 \Big[\frac{7}{2} + \frac{173}{54}H_1 - 2\zeta_3 - \frac{2}{3}H_{1,1,1} - \frac{26}{9}H_{1,1} \Big] + 16C_A C_F^{-2} \Big(x^2 \Big[\frac{7}{2} + \frac{173}{54}H_1 - 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\frac{28}{9}H_{0,0} - \frac{8}{3}H_{0,0,0} \Big] + p_{B9}(x) \Big[\frac{3}{2}H_1\zeta_3 + \frac{163}{32} - 5\zeta_2 + \frac{27}{4}\zeta_3$ $+\frac{6503}{422}H_1 + \frac{2}{6}H_{1,1} + \frac{35}{2}H_{1,1,1} + 4H_2 + \frac{9}{2}H_{2,1} + 4H_{1,0,0} + 2H_{2,0,0} - H_2\zeta_2 + \frac{41}{12}H_{1,2} + H_{2,2}$ $+\frac{191}{24}H_{1,0} + 3H_{2,0} - 2H_{2,1,1} - \frac{3}{2}H_{-1}\zeta_2 - \frac{59}{12}H_1\zeta_2 + 5H_{1,-2,0} + H_{1,0}\zeta_2 + \frac{5}{2}H_{1,0,0,0} - 2H_{1,1}\zeta_2$ $+\frac{1}{12}H_{1,1,0}+5H_{1,1,0,0}-3H_{1,1,1,0}-4H_{1,1,1,1}-H_{1,1,2}-2H_{1,2,1}+H_{2,1,0}]+p_{00}(-x)[H_{-1,0}]$ $+H_{-1,0}\zeta_2 + \frac{3}{2}H_{-1,0,0} + \frac{27}{10}\zeta_2^2 - 3H_{-1,-1,0} - \frac{11}{2}H_{-1}\zeta_3 - 3H_{-1,-2,0} - \frac{3}{2}H_{-1,0,0,0} - 3H_{-1,2}$ $+5H_{-1,-1}\zeta_2 - 4H_{-1,-1,0,0} - 2H_{-1,-1,2} + 6H_{-1,-1,-1,0} + 2H_{-1,2,1} + (1-x) \left[H_2\zeta_2 - H_{2,2}\right]$ $+ \frac{23}{12}H_{1,0} - \frac{7061}{432}H_0 - \frac{4631}{144}H_{0,0} - \frac{38}{3}H_{0,0,0} - H_{-3,0} - 2H_{3,0} - \frac{443}{432}H_1 - 2H_{2,0,0} - \frac{2}{2}H_{1,0} - \frac{2}{14}H_{0,0} - \frac{2}{2}H_{0,0} - H_{0,0} - H_{0,0} - \frac{4}{12}H_{0,0} - H_{0,0} - \frac{2}{2}H_{0,0} - \frac{1}{2}H_{0,0} - \frac{1}{2}H$ $+\frac{555}{576}-\frac{151}{54}\zeta_3-\frac{185}{18}H_{1,1}+\frac{1}{6}H_{1,1,1}-\frac{95}{9}H_2+\frac{29}{6}H_{2,1}-\frac{171}{4}H_{-1,0}-12H_{-1,0,0}+7H_{-1}\zeta_2$ $+16H_{-1,-1,0} + \frac{5}{3}H_{2,0} + \frac{3}{2}H_{2,1,1} + 4H_{0,0,0,0} - 35H_{-2,0} - \frac{179}{27}H_0 + \frac{2041}{144}H_{0,0} - \frac{19}{6}H_{0,0,0}$

$$\begin{split} & -\frac{413}{100} H_1 - \frac{11}{21} H_1 \lesssim_2 + \frac{32}{3} H_{1,00} + 11 (\frac{1}{2} + z^2) [\frac{71}{31} H_0 - \frac{1}{6} H_0 - \frac{39}{100} \varsigma_0 - \frac{2}{2} H_{-2,00} - \frac{1}{2} H_{-1,00} + \frac{2}{30} H_{-1,00} + \frac{3}{3} H_{-1,00} + H_{-1,2} + (1-z) [\frac{31}{30} H_1 + \frac{7}{2} H_1 H_0 - \frac{2}{2} H_{-1,00} - \frac{1}{2} H_{-1,00} + \frac{1}{30} H_{-1,00} - \frac{3}{2} H_{0,0} - \frac{1}{3} H_{-1,00} + \frac{1}{30} H_{-1,00} - \frac{3}{2} H_{0,0} - \frac{1}{3} H_{-1,00} - \frac{1}{30} H_{-1,00} - \frac{3}{3} H_{0,00} - \frac{1}{30} H_{0,00} - \frac{1$$

Coefficient functions in DIS at three loops

S.M., Vermaseren, Vogt '05

• Exact (analytical) results for coefficient functions of F_2 and F_L fill $\mathcal{O}(100)$ pages (normalsize fonts)

Parton evolution at large N/large x

• Recall:
$$A^N = \int_0^1 dx \, x^{N-1} A(x)$$
, non-singlet $u + \bar{u} - (d + \bar{d})$ etc.



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Perturbative expansion very benign: expect < 1% beyond NNLO</p>

The large *x*-limit: $x \rightarrow 1$

• Large *x*-limit for diagonal splitting functions $P_{aa}^{(2)}$, a = q, g $P_{aa, \rightarrow 1}^{(2)}(x) = \frac{A_3^a}{(1-x)_+} + B_3^a \,\delta(1-x) + C_3^a \,\ln(1-x) + \mathcal{O}(1)$ one-loop $A_1^q = 4C_F$ two-loop $A_2^q = 8C_F C_A \left(\frac{67}{18} - \zeta_2\right) - \frac{5}{9}C_F n_f$ Kodaira, Trentadue '80

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• A_3^a important for threshold resummation in soft/collinear limit $A_3^q = 16C_F C_A^2 \left(\frac{245}{24} - \frac{67}{9}\zeta_2 + \frac{11}{6}\zeta_3 + \frac{11}{5}\zeta_2^2\right) + 16C_F^2 n_f \left(-\frac{55}{24} + 2\zeta_3\right)$

$$+ 16 C_F C_A n_f \left(-\frac{209}{108} + \frac{10}{9} \zeta_2 - \frac{7}{3} \zeta_3 \right) + 16 C_F n_f^2 \left(-\frac{1}{27} \right)$$

• Maximally non-Abelian colour structure, $A_3^{g} = \frac{C_A}{C_F} A_3^{q}$ Korchemsky '89

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- Maximally non-Abelian colour structure, $A_3^{g} = \frac{C_A}{C_F} A_3^{q}$ Korchemsky '89
- Subleading logarithms (unexplored structure) $C_1^{\rm a} = 0$, $C_2^{\rm a} = (A_1^{\rm a})^2$, $C_3^{\rm a} = 2A_1^{\rm a}A_2^{\rm a}$

Three-loop splitting functions at small *x*

Small momentum fractions $x: g \rightarrow i$ splitting P_{ig} most important



Three-loop splitting functions at small *x*

Small momentum fractions $x: g \rightarrow i$ splitting P_{ig} most important



• Leading $x \rightarrow 0$ -terms (BFKL) confirmed but insufficient at colliders

The small *x*-limit: $x \rightarrow 0$

Singlet

• Structure of singlet splitting functions at small x

$$P_{ab,\to0}^{(2)}(x) = E_1^{ab} \frac{\ln x}{x} + E_2^{ab} \frac{1}{x} + \mathcal{O}(\ln^4 x)$$
$$E_1^{gg} = \left(\frac{6320}{27} - \frac{176}{3}\zeta_2 - 32\zeta_3\right)C_A^3 + \left(\frac{1136}{27} - \frac{32}{3}\zeta_2\right)C_A^2 n_f - \left(\frac{1376}{27} - \frac{64}{3}\zeta_2\right)C_A C_F n_f$$
$$\cong 2675.85 + 157.269 n_f$$

 $E_2^{\text{gg}} \cong 14214.2 + 182.958 n_f - 2.79835 n_f^2$

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$$E_1^{gg} \approx 14214.2 + 182.058 m = 2.70825 m^2$$

$$E_2^{\text{gg}} \cong 14214.2 + 182.958 n_f - 2.79835 n_f^2$$

- No logarithm $\ln^2 x/x$ in $P_{gg}^{(2)}$
 - predicted by leading logarithmic BFKL equation
 Kuraev, Lipatov, Fadin '77; Balitsky, L.N. Lipatov '78; Jaroszewicz '82

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- Coefficients for quark case Catani, Hautmann '94

Evolution of parton distributions at small x

• Scale derivatives of quark and gluon distributions at $Q^2 \approx 30 \text{ GeV}^2$



Evolution of parton distributions at small x

Scale derivatives of quark and gluon distributions at $Q^2 \approx 30 \text{ GeV}^2$



Expansion very stable except for very small momenta $x \leq 10^{-4}$

Three-loop coefficient functions for F_2 (non-singlet)



• Large *x*-limit at nth-order
$$\alpha_s^n \frac{\ln^{2n-1}(1-x)}{(1-x)_+}$$

Small-x limit insufficient for accurate description

The structure function F_2 (non-singlet)

- Large-*x* convergence of perturbative series
 - approx. N³LO structure functions



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• At very large x: soft-gluon resummation S.M., Vermaseren, Vogt '05

Three-loop structure function F_2 (singlet)



Three-loop structure function F_2 (singlet)



- Perturbative expansion to N³LO of the quark and gluon contribution
- Perturbative stability of F_2

Three-loop coefficient functions for F_L



- Perturbative expansion of singlet-quark and gluon coefficient functions $c_{L,q}$ and $c_{L,g}$ for F_L with $n_f = 4$ and $\alpha_s(\mu^2) = 0.2$ (results divided by $a_s = \alpha_s/(4\pi)$)
- LO and NLO contributions remarkably small

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- Leading small-*x* term at NNLO results $xc_{L,a}^{(3)} \sim \ln x$

The longitudinal structure function



- Perturbative expansion of singlet-quark and gluon contributions to F_L for $n_f = 4$ and $\alpha_s(\mu^2) = 0.2$ (results divided by $\langle e^2 \rangle$)
- Parametrization of singlet distributions (order independent)

 $\begin{aligned} xq_{\rm s}(x,\mu_0^2) &= \\ 0.6 x^{-0.3} (1-x)^{3.5} (1+5.0 x^{0.8}) \\ xg(x,\mu_0^2) &= \\ 1.6 x^{-0.3} (1-x)^{4.5} (1-0.6 x^{0.3}) \end{aligned}$

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NNLO evolution and data analyses



NNLO evolution and data analyses



- Statistical error bands for xu_v and xd_v at 1σ at $Q_0^2 = 4.0 \text{ GeV}^2$ Alekhin '02; Martin, Roberts, Stirling, Thorne '03; Blümlein, Böttcher, Guffanti '04
- Evolution code at NNLO (in Mellin N-space): QCD-Pegasus Vogt '04

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- Further topics

CERN/DESY Workshop HERA and the LHC '04-'05; www.desy.de/heralhc

- precision parton distributions
- impact on LHC parton luminosity
- **_** ...

N = 4 Super Yang-Mills theory

- Maximally supersymmetric Yang-Mills theory in four dimensions (MYSM)
 - renewed interest from AdS/CFT and from twistor space methods
 - simple planar limit for large n_c
 - vanishing β -function (conformal invariance)

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Correspondence with QCD

QCD results carry over to MYSM (no formal proof exists)

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 - renewed interest from AdS/CFT and from twistor space methods
 - simple planar limit for large n_c
 - vanishing β -function (conformal invariance)

Correspondence with QCD

- QCD results carry over to MYSM (no formal proof exists)
- Conjecture
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- MSYM identification for colour coefficients $C_A = C_F = n_c$
 - n_f -terms do not contribute at highest transcendentality

Anomalous dimensions of MYSM

- Universal anomalous dimension in MYSM to three loops Kotikov, Lipatov, Onishchenko, Velizhanin '04
 - at loops *l*-loops harmonic sums of weight w = 2l 1

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is

$$\gamma(j) \equiv \gamma_{uni}(j) = \hat{a}\gamma_{uni}^{(0)}(j) + \hat{a}^2\gamma_{uni}^{(1)}(j) + \hat{a}^3\gamma_{uni}^{(2)}(j) + \dots, \qquad \hat{a} = \frac{\alpha N_c}{4\pi}, \qquad (9)$$

where

$$\frac{1}{4}\gamma_{uni}^{(0)}(j+2) = -S_1, \tag{10}$$

$$\frac{1}{8}\gamma_{uni}^{(1)}(j+2) = \left(S_3 + \overline{S}_{-3}\right) - 2\,\overline{S}_{-2,1} + 2\,S_1\left(S_2 + \overline{S}_{-2}\right),\tag{11}$$

$$\frac{1}{32} \gamma_{uni}^{(2)}(j+2) = 2\overline{S}_{-3}S_2 - S_5 - 2\overline{S}_{-2}S_3 - 3\overline{S}_{-5} + 24\overline{S}_{-2,1,1,1} \\
+ 6\left(\overline{S}_{-4,1} + \overline{S}_{-3,2} + \overline{S}_{-2,3}\right) - 12\left(\overline{S}_{-3,1,1} + \overline{S}_{-2,1,2} + \overline{S}_{-2,2,1}\right) \\
- \left(S_2 + 2S_1^2\right) \left(3\overline{S}_{-3} + S_3 - 2\overline{S}_{-2,1}\right) - S_1\left(8\overline{S}_{-4} + \overline{S}_{-2}^2 \\
+ 4S_2\overline{S}_{-2} + 2S_2^2 + 3S_4 - 12\overline{S}_{-3,1} - 10\overline{S}_{-2,2} + 16\overline{S}_{-2,1,1}\right)$$
(12)

and $S_a \equiv S_a(j)$, $S_{a,b} \equiv S_{a,b}(j)$, $S_{a,b,c} \equiv S_{a,b,c}(j)$ are harmonic sums

Precision Predictions for Deep-Inelastic Scattering – p.23

$$j$$
 1 j 1

Sven-Olaf Moch

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 - at loops *l*-loops harmonic sums of weight w = 2l 1
- Result agrees with predictions based on integrability for planar three-loop contribution to dilatation operator
 Beisert, Kristjansen, Staudacher '03
- Additional check up to spin N = 70 from a Bethe ansatz for S-matrix of spin chain Staudacher '04

Order $\alpha \alpha_s^2$ **photon-parton splitting functions**

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• Large x: approximation based on fixed-N results sufficiently reliable

Photon-quark splitting functions up to NNLO

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• Large x: corrections small in both \overline{MS} and DIS_{γ} factorization

Higgs production in soft limit

N³LO results for Higgs production in gluon-gluon fusion with $x = M^2/s$ S.M., Vogt '05

$$c_{3}^{\text{Higgs}} = \frac{\ln(1-x)^{5}}{(1-x)_{+}} \left\{ 512C_{A}^{3} \right\} + \dots + \frac{1}{(1-x)_{+}} \left\{ C_{A}^{3} \left[-\frac{594058}{729} + \frac{137008}{81} \zeta_{2} + \frac{143056}{27} \zeta_{3} + \frac{4048}{15} \zeta_{2}^{2} - \frac{23200}{3} \zeta_{2}\zeta_{3} + 11904 \zeta_{5} \right] + C_{A}^{2} n_{f} \left[\frac{125252}{729} - \frac{34768}{81} \zeta_{2} - \frac{7600}{9} \zeta_{3} - \frac{544}{15} \zeta_{2}^{2} \right] + C_{A} C_{F} n_{f} \left[\frac{3422}{27} - 32 \zeta_{2} - \frac{608}{9} \zeta_{3} - \frac{64}{5} \zeta_{2}^{2} \right] - C_{A} n_{f}^{2} \left[\frac{3712}{729} - \frac{640}{27} \zeta_{2} - \frac{320}{27} \zeta_{3} \right] \right\}$$

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• N³LO results for Drell-Yan process in $q\bar{q}$ -annihilation with $x = M^2/s$ and independent cross check on $\frac{1}{(1-x)_+}$ -term Laenen, Magnea '05



• Variation of cross section at LHC with renormalization scale for different Higgs masses: $M_H = 120$ GeV (left) and $M_H = 240$ GeV (right)

Cross section Higgs production (cont'd)



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Harlander, Kilgore '02; Anastasiou, Melnikov '02; Ravindran, Smith, van Neerven '03

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 - complete soft N³LO corrections S.M., Vogt '05

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Summary

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Outlook

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