Massive 2-loop Bhabha Scattering

Tord Riemann, DESY, Zeuthen

RADCOR 2005

Shonan Village, Japan, October 2–7, 2005

A project in collaboration with

Michal Czakon Univ. Würzburg (and Katowice)

Janusz Gluza Katowice (and DESY)

See also:
- **NPB(PS) 135 (2004), hep-ph/0406203**
- **PRD 71 (2005), hep-ph/0412164**
- **http://www-zeuthen.desy.de/theory/research/bhabha/**

- **Introduction: What do we need?** $\rightarrow 10^{-4}$ **for** $d\sigma/d\cos\vartheta$ **at small** ϑ
- **Higher Order Corrections – Status** $\rightarrow$ **Boxes**
- **Summary**

The Physics Needs

For more details see e.g.:

K. Mönig, "Bhabha scattering at the ILC"

talk at Mini-WS on Bhabha scattering, Univ. Karlsruhe, April 2005

/afs/ifh.de/user/m/moenig/public/www/bhabha_ilc.pdf

ILC – Need Bhabha cross-sections with 3–4 significant digits.

Why?

- ILC: $e^+e^- \to W^+W^-, f\bar{f}$ with $O(10^6)$ events $\to 10^{-3}$
- GigaZ: relevant physics derived from $Z \to$ hadrons, l^+l^-, the latter with $O(10^8)$ events $\to 10^{-4}$, the systematic errors (luminosity!) influence this
- ILC: $e^+e^- \to e^+e^-$, a probe for New Physics with $O(10^5)$ events/year $\to 10^{-3}$

Conclude: will need $\Delta\mathcal{L}/\mathcal{L} \approx 2 \times 10^{-4}$

The luminosity comes from very forward Bhabha scattering.

Some Kinematics

Need a cross-section prediction with 5 significant digits.

Perturbative orders:

$$\left(\frac{\alpha}{\pi}\right) = 2 \times 10^{-3}$$

$$\left(\frac{\alpha}{\pi}\right)^2 = 0.6 \times 10^{-5}$$

Kinematics:

$$\begin{aligned}
\sqrt{s} &= 90 \ldots 1000 \text{ GeV} \\
\vartheta &= 26 \ldots 82 \text{ mrad} \\
\cos\vartheta &\sim 0.999\,66 \ldots 0.996\,64 \\
T &= \frac{s}{2}(1-\beta^2\cos\vartheta) > 1.36 \text{ GeV}|_{GigaZ},\ 42.2 \text{ GeV}|_{ILC500}
\end{aligned}$$

Conclude:

- t-**channel exchange of** γ **dominates everything else**
- $m_e^2/s \ < \ m_e^2/T \ \leq \ 10^{-5} \ldots 10^{-7}$
- **Calculate: 1-loop EWRC + 2-loop QED + corresp. bremsstrahlung**

.

The 1-loop electroweak corrections (plus some leading higher order terms) are well-known, with rising technical precision, since about 1988/91.
Böhm, Denner, Hollik; Bardin, TR 1991 → Fig. 2004 Lorca, TR

2-loop Bhabha scattering: What to be done?

- Calculate:

$$\sigma = (2 \to 2) + (2 \to 3) + (2 \to 4)$$

$$\begin{aligned}\sigma = & \left|\textbf{Born} + \textbf{1-loop} + \textbf{2-loop}\right|^2 \\ & + \left|(\textbf{Born} + \textbf{1-}\gamma) + (\textbf{1-loop} + \textbf{1-}\gamma)\right|^2 \\ & + \left|(\textbf{Born} + \textbf{2-}\gamma)\right|^2\end{aligned}$$

- Do not include: $|\textbf{2-loop}|^2$
  $|(\textbf{1-loop} + \textbf{1-}\gamma)|^2$

Results: Numerical comparison in all $f\bar{f}$

Bhabha $e^-e^+ \to e^-e^+\ (\gamma)$ at LC: $\sqrt{s} = 500$ GeV, $E_{\max}(\gamma_{\rm soft}) = \frac{\sqrt{s}}{10}$

$\cos\theta$	$\left[\frac{d\sigma}{d\cos\theta}\right]_{\rm Born}$ (pb)	$\left[\frac{d\sigma}{d\cos\theta}\right]_{\mathcal{O}(\alpha^3)=\text{Born+QED+weak+soft}}$	Group
−0.9999	0.21482 70434 05632 5	0.14889 12125 78083 7	aITALC
−0.9999	0.21482 70434 05632 6	0.14889 12189 28404 0	*FeynArts*
−0.9	0.21699 88288 10920 5	0.19344 50785 26863 6	aITALC
−0.9	0.21699 88288 10920 0	0.19344 50785 26862 2	*FeynArts*
−0.9	0.21699 88288 41513 1	0.19344 50785 62637 9	$m_e = 0$
+0.0	0.59814 23072 50330 3	0.54667 71794 69423 1	aITALC
+0.0	0.59814 23072 50329 4	0.54667 71794 69421 8	*FeynArts*
+0.0	0.59814 23072 88584 4	0.54667 71794 99961 4	$m_e = 0$
+0.9	$0.18916\,03223\,32270\,6 \cdot 10^3$	$0.17292\,83490\,66507\,2 \cdot 10^3$	aITALC
+0.9	$0.18916\,03223\,32270\,6 \cdot 10^3$	$0.17292\,83490\,66508\,0 \cdot 10^3$	*FeynArts*
+0.9	$0.18916\,03223\,31848\,5 \cdot 10^3$	$0.17292\,83490\,61347\,4 \cdot 10^3$	$m_e = 0$
+0.9999	$0.20842\,90676\,46142\,9 \cdot 10^9$	$0.19140\,17861\,11341\,6 \cdot 10^9$	aITALC
+0.9999	$0.20842\,90676\,46436\,4 \cdot 10^9$	$0.19140\,17861\,11979\,0 \cdot 10^9$	*FeynArts*

Great independent agreement up to 14 digits! : limit in double precision

Previous agreement with *FeynArts*: 11 digits hep-ph/0307132, SANC: 10 digits hep-ph/0207156

Thanks to T. Hahn, numbers supplied with *FeynArts + FormCalc + LoopTools*

Zinnowitz, 28/04/2004 A. Lorca —Automatization and width effects with aITALC

Bhabha scattering

Bardin,Hollik,T.R., Z.PhysikC49(1991)485

Table 2:
The differential Bhabha cross section in nbarn as function of the scattering angle and the cms-energy.
$M_Z = 91.16\ GeV$, $m_t = 150\ GeV$, $M_H = 100\ GeV$.
Upper rows: DZ, lower rows: H.
δ_m: *largest relative deviation in per mille.*

$\sqrt{s}$ (GeV) / θ	60	89	91.16	93	200
15°	129.6 129.6	65.11 65.11	57.93 57.93	49.00 49.00	11.82 11.82
45°	1.451 1.451	1.376 1.377	1.755 1.756	.4833 .4837	11.67 11.68
60°	.4303 .4305	.6124 .6129	1.125 1.126	.2697 .2699	.03075 .03077
75°	.1717 .1718	.3627 .3630	.8718 .8720	.2232 .2233	.01072 .01072
90°	.08873 .08876	.2768 .2769	.7790 .7787	.2088 .2087	.004862 .004855
105°	.05917 .05918	.2690 .2690	.8082 .8074	.2157 .2157	.002858 .002853
120°	.04906 .04906	.3053 .3051	.9323 .9309	.2429 .2426	.002077 .002074
135°	.04671 .04672	.3626 .3624	1.111 1.109	.2838 .2833	.001743 .001742
165°	.04839 .04839	.4638 .4635	1.425 1.422	.3590 .3584	.001539 .001540
δ_m	0.6	0.8	1.8	2.0	1.7

Status by end of 2004

Established: 10^{-3} MC programs for LEP, ILC

Introduction to NLLBHA by Trentadue and to BHLUMI by Jadach in:
Proc. of Loops and Legs, Rheinsberg, Germany, 1996

Recent mini-review: Jadach, "Theoretical error of luminosity cross section at LEP", hep-ph/0306083 [1]

- **BHLUMI v.4.04: Jadach, Placzek, Richter-Was, Was: CPC 1997**
- **see also: Jadach, Melles, Ward, Yost: PLB 1996, thesis Melles 1996 [2]**
- **NLLBHA: Arbuzov, Fadin, Kuraev, Lipatov, Merenkov, Trentadue: NPB 1997, CERN 96-01**
- **SAMBHA: Arbuzov, Haidt, Matteuzzi, Paganoni, Trentadue: hep-ph/0402211**

See e.g.: Table 1 of [1] and Figure 3.1 of [2] $\rightarrow$ Conclude:

The nonlogarithmic $O(\alpha^2)$ terms, originating
from pure QED radiative 1-loop and from 2-loop diagrams are not completely covered.
They have to be calculated and integrated into the MC programs.
Beware: $m_e, m_\gamma, (d-4), E_\gamma$

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Status 2005

Know the constant term ($m_e = 0$) from 2-loop Bhabha scattering

A. Penin, Two-Loop Corrections to Bhabha Scattering, hep-ph/0501120 v.3, $\rightarrow$ PRL

Transform the massless 2-loop results of Bern, Dixon, Ghinculov (2002) with InfraRed (IR) regulation by $D = 4 - 2\epsilon$ into the on-mass-shell renormalization with $m_e \rightarrow 0$ and IR regulation by $\lambda = m_\gamma \neq 0$

Use IR-properties of amplitudes (see Penin):

[A] Exponentiation of the IR logarithms (Sudakov 1956,...)

[B] Factorization of the collinear logarithms into expernal legs (Frenkel, Taylor 1976)

[C] Non-renormalization of the IR exponents (YFS 1961, ...)

Isolate the closed fermion loop contribution (does not fulfil [C]) and add it separately (Burgers 1985, Bonciani et al. 2005, Penin)

If all this is correct, the constant term in m_e is known for the MCs (but the radiative one-loops with 5-point functions).

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The massive 2-loop contributions

We are interested in a calculation of the virtual second order corrections to

$$\frac{d\sigma}{d\cos\vartheta}(e^+e^- \to e^+e^-)$$

We are using a scheme with

(1) $m_e \neq 0$ (good with the MC's)

(2) $m_\gamma = 0$ (bad with the MC's; $\to$ Mastrolia, Remiddi 2003)

(3) dim.reg. for UV and IR divergences

Also:

Argeri, Bonciani, Ferroglia, Mastrolia, Remiddi, v.d.Bij: all but many 2-boxes
Heinrich, Smirnov: Calculation of selected complicated Feynman integrals

.

There are few **topologies** only:

- 1-loop: 5
- 2-loop self energies: 5 (3 for external legs)
- 2-loop vertices: 5
- 2-loop boxes: 6 $\rightarrow$ next slides

The many Feynman integrals may be reduced to 'few' master integrals (Remiddi-Laporta algorithm, 1996/2000 $\rightarrow$ IdSolver (Czakon 2003)).

The massive diagrams (See also webpage)

Assume 3 leptonic flavors, do not look at loops in external legs.

Not too many QED diagrams:

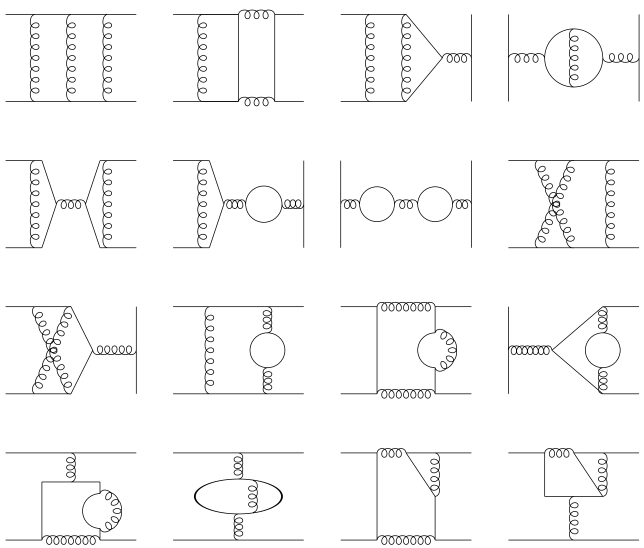
- Born diagrams: 2
- 1-loop diagrams: 14
- 2-loop diagrams: 154 (with 68 double-boxes) interfere with Born

$m = 0$

Two Loop Bhabha Scattering

To calculate Bhabha scattering it is best to first compute $e^+e^- \to \mu^+\mu^-$, since it's closely related but has less diagrams.

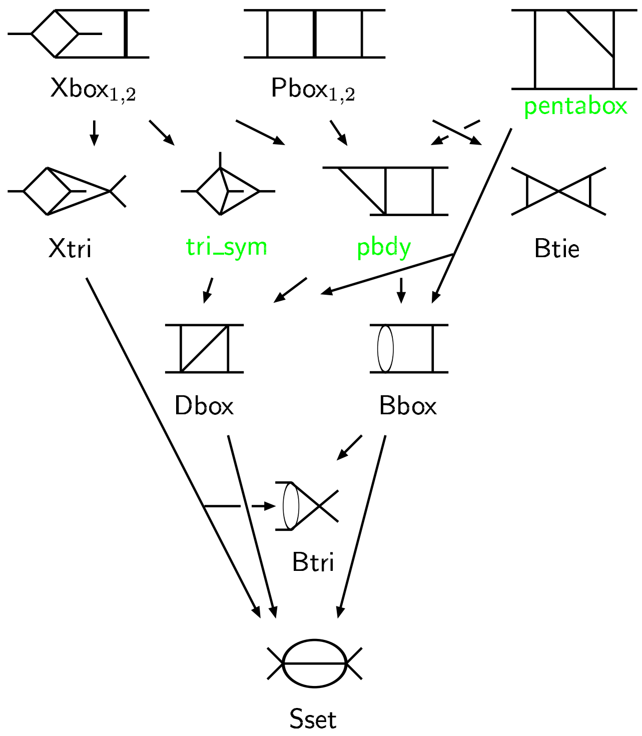
There are 47 QED diagrams contributing to $e^+e^- \to \mu^+\mu^-$.



In this calculation all particles massless.

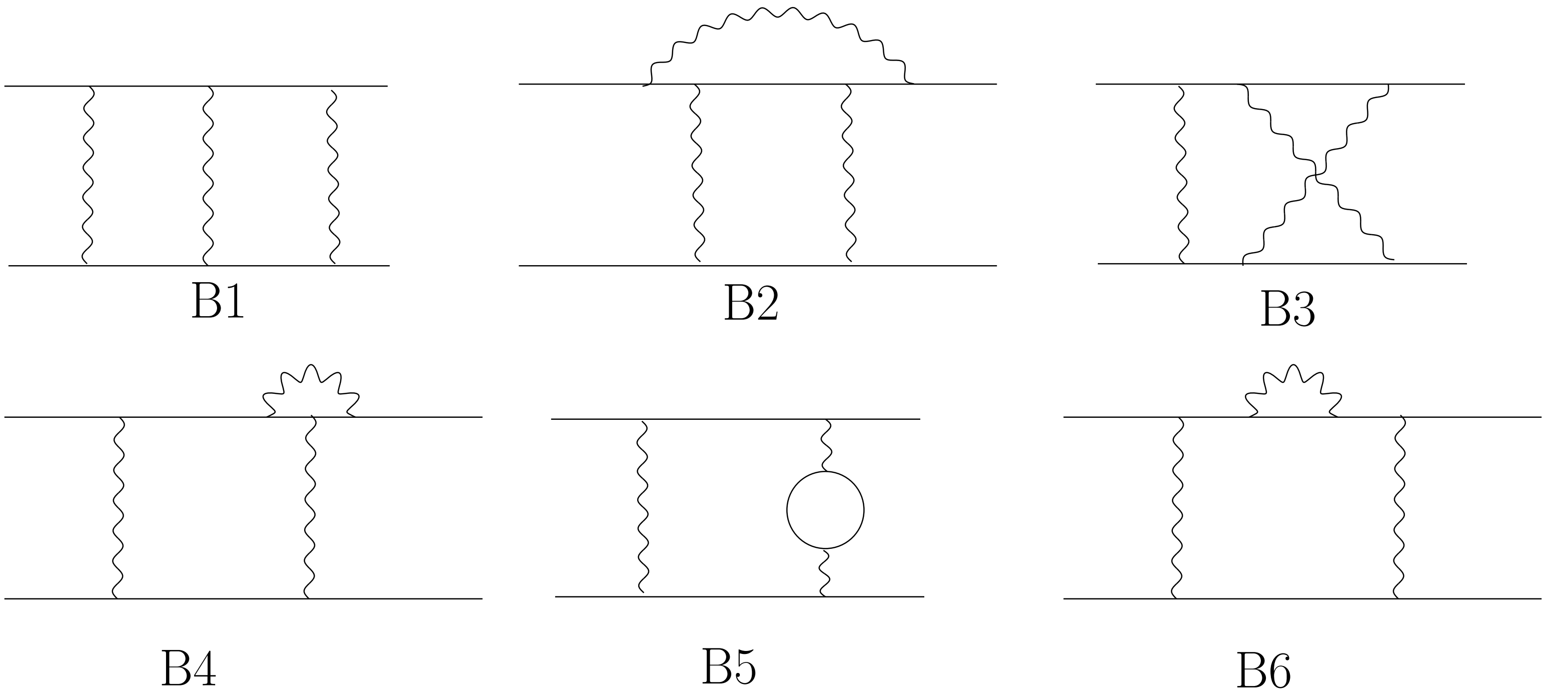
The Bhabha scattering amplitude can be obtained from $e^+e^- \to \mu^+\mu^-$ simply by summing it with the crossed amplitude (including fermi minus sign).

Two-loop integral inheritance chart



From Talk of Zvi Bern, LoopFest 2002

The two-loop box diagrams for massive Bhabha scattering



- B5: Completely known (2004)
  Bonciani, Ferroglia, Mastrolia, Remiddi, van der Bij: hep-ph/0405275, hep-ph/0411321
  Czakon, Gluza, Riemann: *http://www-zeuthen.desy.de/.../MastersBhabha.m* (unpubl.)
- B1–B3: Few masters known (Smirnov, Heinrich 2002,2004)
- B4, B6: Not much known (Czakon et al. 2004)

The basic planar 2-box master of B1, B7l4m, was a breakthrough (Smirnov, 2002)

The two-loop Feynman integrals

One has to solve many Feynman integrals with $L = 2$ loops and
$N \leq 7$ internal lines having indices $a_i \geq 1$, and
with a variety of numerators:

$$G(X) = \frac{1}{\left(i\pi^{d/2}\right)^2} \int \frac{d^D k_1 d^D k_2 \; X}{(q_1^2 - m_1^2)^{a_1} \ldots (q_j^2 - m_j^2)^{a_j} \ldots (q_N^2 - m_N^2)^{a_N}},$$

$$X = 1, (k_1 P), (k_1 k_2), (k_2 P), \cdots$$

where P is some external momentum: $p_1, ..p_4$

A completely numerical approach might be possible Passarino 2004.

For checks in the Euclidean region $(s < 0, t < 0)$ this has been proven to be a powerful tool Binoth, Heinrich 2000/03

We prefer to calculate the integrals analytically (where possible)
Derive a minimal set of so-called master integrals and algebraic expressions in terms of them for all the other Feynman integrals

.

We need a A table of master integrals

We use IdSolver with the Laporta/Remiddi algorithm:

Derive with integration-by-parts (and Lorentz-invariance) identities a system of *algebraic* equations for the Feynman integrals and solve the system.

- 1-loop: 5 masters (all known)
- 2-loop self energies: 6 masters (all known)
- 2-loop vertices: 19 masters (all known)
- 2-loop boxes: 33 masters → (O(5) published, maybe more known) see table

The calculation of the master integrals is mainly done with two methods:

- derive and solve (systems of) differential equations (with boundary conditions)
- derive and solve (up to 8-dimensional) Mellin-Barnes integral representations for single Feynman integrals

Miyamoto Musashi (1584 – 1645)

The Five Rings

The Broad Principles of Musashi's Strategy

- Do not think dishonestly.
- The Way is in training.
- Become aquainted with every art.
- Know the Ways of all professions.
- Distinguish between gain and loss in worldly matters.
- Develop intuitive judgement and understanding for everything.
- Perceive those things which cannot be seen.
- Pay attention even to trifles.
- Do nothing which is of no use.

It is important to start by setting these broad principles in your heart, and train in the Way of strategy. If you do not look at things on a large scale it will be difficult for you to master strategy. If you learn and attain this strategy you will never lose even to twenty or thirty enemies.

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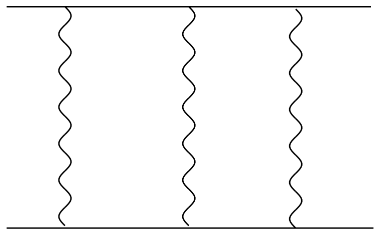
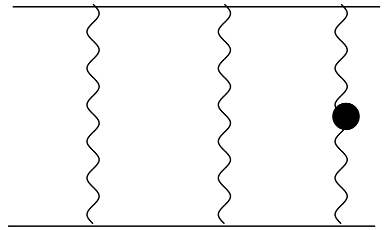
From Czakon et al., PRD 71 (2004): 4-point MIs entering basic two-loop box diagrams. An asterisk denotes one-loop MI. MIs with a dagger: know singular parts only

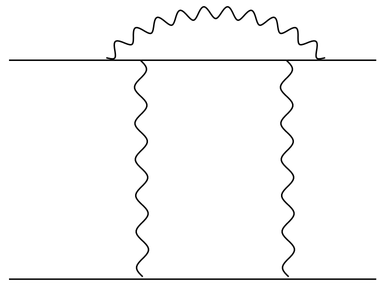
MI	B1	B2	B3	B4	B5	B6	ref.
B714m1	+	−	−	−	−	−	Smirnov:2001cm
B714m1N	+	−	−	−	−	−	Heinrich:2004iq
B714m2	−	+	−	−	−	−	Heinrich:2004iq†
B714m2[d1--d3]	−	+	−	−	−	−	
B714m3	−	−	+	−	−	−	Heinrich:2004iq†
B714m3[d1--d2]	−	−	+	−	−	−	
B613m1	+	−	+	−	−	−	
B613m1d	+	−	+	−	−	−	
B613m2	−	+	−	+	−	−	
B613m2d	−	+	−	+	−	−	
B613m3	−	−	+	−	−	−	
B613m3[d1--d5]	−	−	+	−	−	−	
B512m1	+	−	+	−	−	−	Czakon:2004tg
B512m2	−	+	−	+	−	+	Sec. IIIE1†
B512m2[d1--d2]	−	+	−	+	−	+	Sec. IIIE1†
B512m3	+	−	+	−	−	−	
B512m3[d1--d3]	+	−	+	−	−	−	Sec. IIIE1†
B513m	−	+	+	+	−	−	
B513m[d1--d3]	−	+	+	+	−	−	
B514m	−	+	+	+	+	−	Bonciani:2003cj
B514md	−	+	+	+	+	−	Sec. IIIE
B412m*	−	−	−	+	+	+	'tHooft:1972fi,Bonciani:2003cj
total = 33+1*	9	15	22	11+1*	2+1*	3+1*	



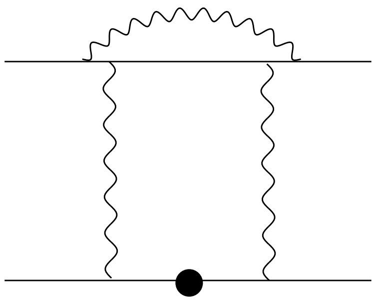
B7l4m1



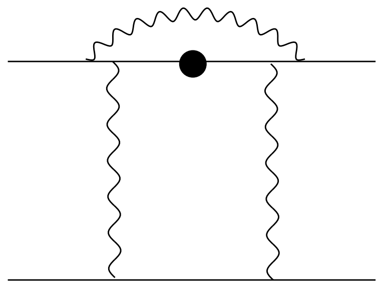
B7l4m1d



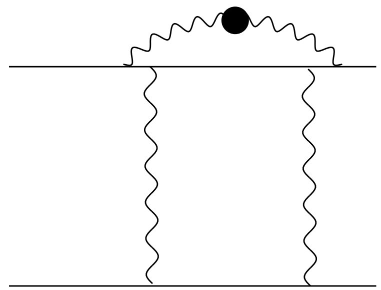
B7l4m2



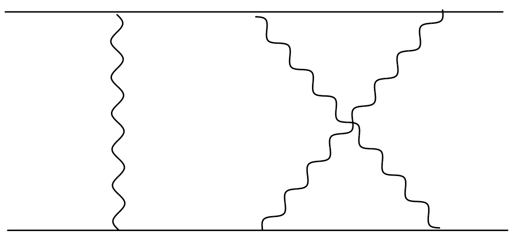
B7l4m2d1



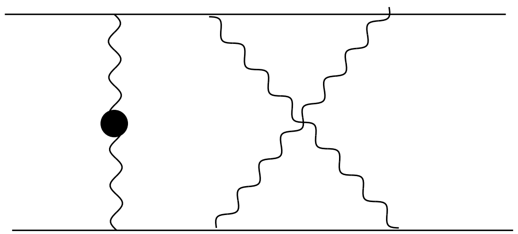
B7l4m2d2



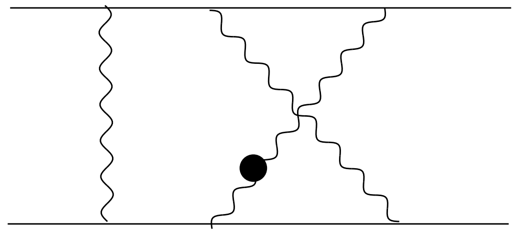
B7l4m2d3



B7l4m3



B7l4lm3d1

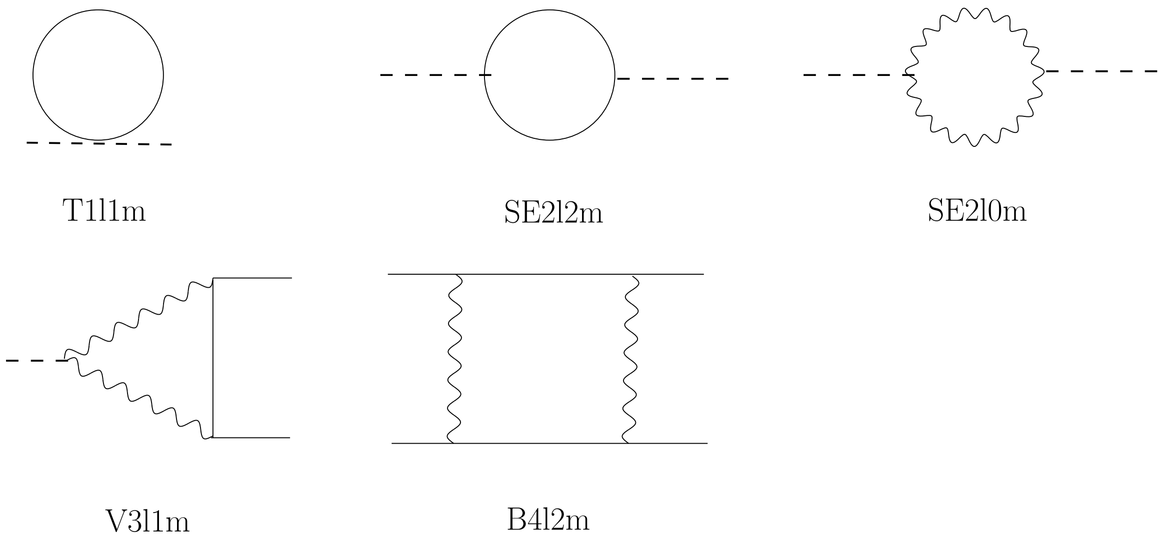


B7l4m3d2

The nine two-loop box MIs with seven internal lines.

The simplest diagram is the tadpole:

$$
\begin{aligned}
\texttt{T1l1m} &= \frac{e^{\epsilon\gamma_E}}{i\pi^{D/2}} \int \frac{d^D q}{q^2 - 1} \\
&= \frac{1}{\epsilon} + 1 + \left(1 + \frac{\zeta_2}{2}\right)\epsilon + \left(1 + \frac{\zeta_2}{2} - \frac{\zeta_3}{3}\right)\epsilon^2 + \ldots
\end{aligned}
$$



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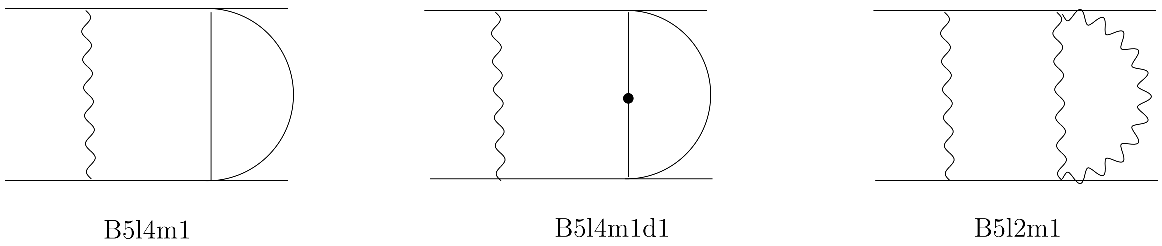
How to calculate 2-loop Bhabha masters?

- **Self-energies and vertices and (very few) 2-boxes:**
  use differential equations and Harmonic Polylogarithms, introduced by Remiddi, Vermaseren, plus . . .)
- **Some 7-line 2-boxes**
  use Mellin-Barnes technique, sum up multiple series, use numerical checks in Euclidean space (s, t negative)
- **For the unsolved 2-boxes:**
  Combination of both methods: present study

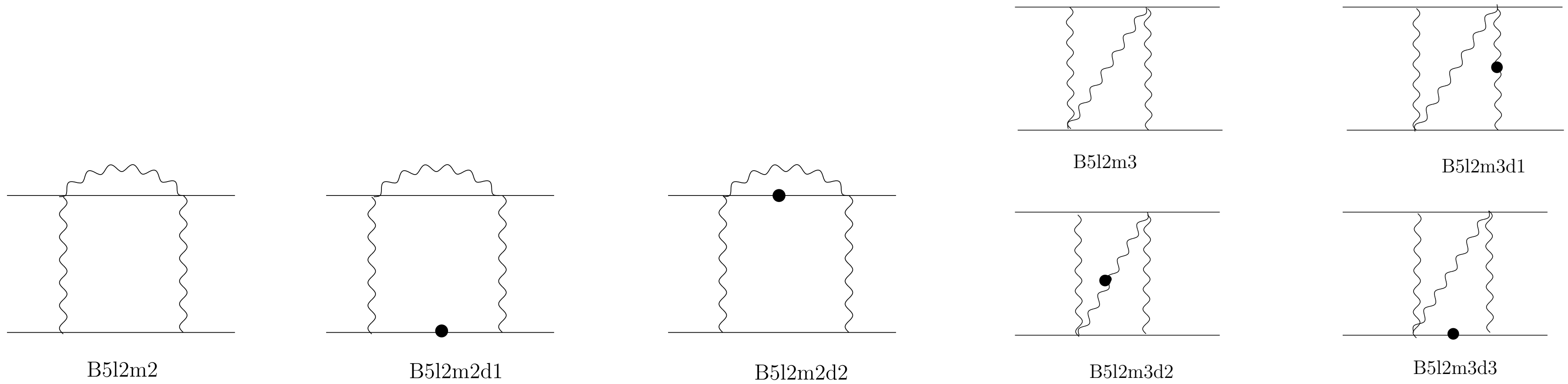
There are other methods not used here:
difference equations
pure numerical approaches
. . .

The 2-boxes with 5 lines

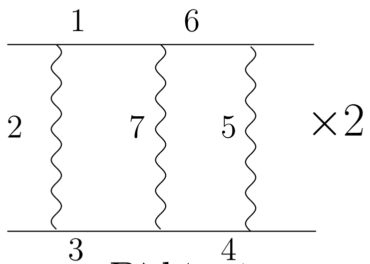
The completely known 2-boxes with 5 lines are B5l4m (Bonciani et al., Czakon et al. 2004), B5l2m1 (Czakon et al. 2004) :



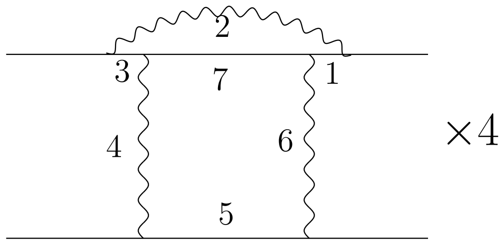
The divergent parts of the B5l2m2 and B5l2m3 type are known (Czakon et al. 2004):



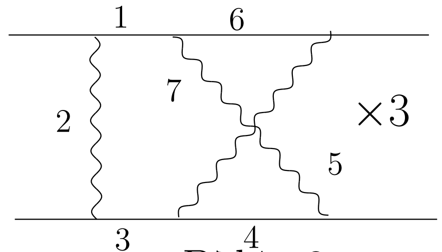
B5l3m: The divergences in $D - 4 = -2\epsilon$



B7l4m1

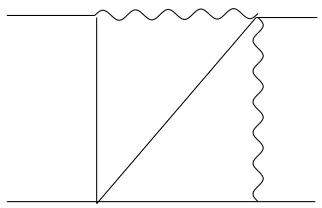


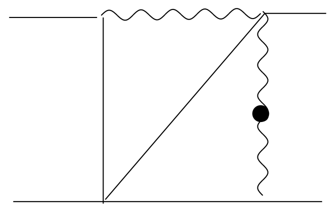
B7l4m2

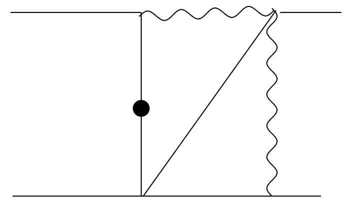


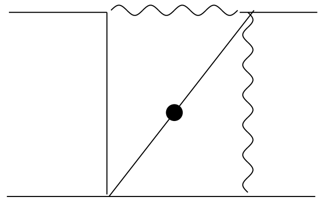
B7l4m3

The B5l3m boxes contribute to B2 (2nd planar 2-box) (shrink lines 1 and 4 ...) and to B3 and B4


B5l3m


B5l3md1


B5l3md2


B5l3md3

The B5l3md2 topology appears twice as a master
but the B5l3md1 does not!

The B5l3m topology: Gross features

$$MB5l3m[x,y] = Sum[B5l3m[k,x,y]*ep^k, k, 0, 1]; \quad (1)$$

$$MB5l3md1[x,y] = Sum[B5l3md1[k,x,y]*ep^k, k, -2, 1]; \quad (2)$$

$$MB5l3md2[x,y] = Sum[B5l3md2[k,x,y]*ep^k, k, -2, 1]; \quad (3)$$

$$MB5l3md2a[x,y] = Sum[B5l3md2a[k,x,y]*ep^k, k, -2, 1]; \quad (4)$$

$$MB5l3md3[x,y] = Sum[B5l3md3[k,x,y]*ep^k, k, -1, 1]; \quad (5)$$

Note:

- B5l3m – the basic master is finite
- B5l3md2 – use 4-dim. MB-Representation
- B5l3md2' – the same, but $(s \leftrightarrow t)$
- B5l3md1, B5l3md3 – system of 2 coupled differential eqns
  Only BLB5l3md1 has $1/\epsilon^2$ (so decouples), and last step is the two $1/\epsilon$ coefficients of B5l3md1 and B5l3md3.
  The first one is found by algebraic manipulations (see Czakon et al. LCWS Paris 2004), the second then fulfils a diff.eqn

Differential equations

$$\frac{\partial B5l3md3[-1]}{\partial x} = \frac{1+x^2}{x(1-x^2)} B5l3md3[-1] - \frac{yH[0,y]}{(1-x^2)(1-y^2)} \tag{6}$$

with $s = -(1-x)^2/x$, $t = -(1-y)^2/y$

Solution:

$$B5l3md3[-1] = -\frac{xy}{(-1+x^2)(-1+y^2)} H[0,x] H[0,y] \tag{7}$$

with

$$H[0,x] = \ln(x) \tag{8}$$

The coefficients in the equation are of the form

$$\frac{A_1}{x-B_1} + \frac{A_2}{x-B_2} + .. \tag{9}$$

One may derive (systems of) differential equations for the masters, with inhomogeneity composed of simpler masters (Kotikov, Laporta, Remiddi)

$$\frac{\partial M_n}{\partial x} = A(x,y)\ M_n + I(x,y) \tag{10}$$

$$I(x,y) = \sum_{k=0,n-1} c_k\ M_k \tag{11}$$

Expand in ϵ ($D = 4 - 2\epsilon$):

$$M_n = \sum_{i=-2,i_m} M_{n,i}\epsilon^i \qquad \text{etc.} \tag{12}$$

General solution for homogeneous eqn. ($M_h' = A\ M_h$):

$$M_h'/M_h = A \tag{13}$$

$$\int (M_h'/M_h) = \ln M_h = \int A \tag{14}$$

$$= \int \sum \frac{a_i}{x - x_i} \sim \ln(x - x_i) \tag{15}$$

so:

$$M_h \sim \text{Polynomials} \tag{16}$$

Then the inhomogeneous solution is:

$$M(x,y) = M_h(x,y) \left(\text{const(y)} + \int \frac{I(x',y)}{M_h(x',y)} \right) \tag{17}$$

Result:

nested integrals over 'simple' iterated integrands

The method leads to the HPLs $H(\{a\},x)$ and GPLs $G(\{a(y)\},x)$

Harmonic Polylogarithms $H(x)$

$$H[-1,1,x] = \int_0^x \frac{dx''}{(1+x'')} \int_0^{x''} \frac{dx'}{(1-x')} \tag{18}$$

$$= Li_2\left(\frac{1+x}{2}\right) + \ldots \tag{19}$$

Generalized Harmonic Polylogarithms $G(x,y)$ ...

but it works only if the polynomial structure is simple enough for a solution with this class of functions

Method is absolutely 'super' if it works.

But:

one needs complete chains of masters of lower complexity, and there are systems of up to 6 (!) potentially coupled 1st order equations

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Mellin-Barnes representations

Boos, Davydychev 1991, Smirnov 1999, Tausk 1999, Smirnov book 2004

$$\frac{1}{(A+B)^{\nu}} = \frac{B^{-\nu}}{(1-(-A/B))^{-\nu}} = \frac{B^{-\nu}}{2\pi i \Gamma(\nu)} \int\limits_{-i\infty}^{i\infty} d\sigma A^{\sigma} \; B^{-\sigma} \; \Gamma(-\sigma)\Gamma(\nu+\sigma) \qquad (20)$$

Is special case of a well-known Mellin-Barnes integral for hypergeometric functions

$$\frac{1}{(1-z)^{\nu}} = {}_2F_1(\nu, b, b', z)|_{b=b'} \qquad (21)$$

$$= \frac{1}{2\pi i \Gamma(\nu)} \frac{\Gamma(b')}{\Gamma(b)} \int_{-i\infty}^{+i\infty} d\sigma (-z)^{\sigma} \Gamma(\nu+\sigma)\Gamma(-\sigma) \frac{\Gamma(b+\sigma)}{\Gamma(b'+\sigma)} \qquad (22)$$

with $-z = A/B$**.**

How can this be made useful here?

Introduce Feynman parameters

The momentum integrals of a Feynman diagram may be performed with Feynman parameters, one for each line.
In 2-loops, consider two subsequent sub-loops (the first: off-shell 1-loop, second on-shell 1-loop) and get e.g. for B7l4m2, the planar 2nd type 2-box:
allow for propagators with indices, $1/(k_1^2 - m_1^2)^{a_1}$ etc.

$$K_{\text{1-loop Box,off}} = \frac{(-1)^{a_{4567}}\Gamma(a_{4567}-d/2)}{\Gamma(a_4)\Gamma(a_5)\Gamma(a_6)\Gamma(a_7)} \int_0^\infty \Pi_{j=4}^{7} dx_j x_j^{a_j-1} \frac{\delta(1-x_4-x_5-x_6-x_7)}{F^{a_{4567}-d/2}} \tag{23}$$

where $a_{4567} = a_4 + a_5 + a_6 + a_7$ and the function F is characteristic of the diagram; here for the off-shell 1-box (2nd type):

$$F = [-t]x_4x_7 + [-s]x_5x_6 + m^2(x_5+x_6)^2 \tag{24}$$

$$+(m^2 - Q_1^2)x_7(x_4 + 2x_5 + x_6) + (m^2 - Q_2^2)x_7x_5 \tag{25}$$

We want to apply now:

$$\int_0^1 \prod_i^4 dx_i\ x_i^{\alpha_i - 1}\ \delta(1 - x_1 - x_2 - x_3 - x_4) = \frac{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)\Gamma(\alpha_4)}{\Gamma(\alpha_1 + \alpha_2 + \alpha_3 + +\alpha_4)} \tag{26}$$

with coefficients α_i dependent on a_i and on F

For this, we have to apply several MB-integrals here.
And repeat the procedure for the 2nd subloop.

For the 2nd planar 2-box, B7l4m2, one gets (Smirnov book 4.73):

$$B_{\mathrm{pl},2} = \frac{\mathrm{const}}{(2\pi i)^6} \int_{-i\infty}^{+i\infty} \left[\frac{m^2}{-s}\right]^{z_5+z_6} \left[\frac{-t}{-s}\right]^{z_1} \prod_{j=1}^{6} [dz_j \Gamma(-z_j)] \frac{\prod_{k=7}^{18} \Gamma_k(\{z_i\})}{\prod_{l=19}^{24} \Gamma_l(\{z_i\})} \tag{27}$$

with $a = a_1 + ... + a_7$ and

$$z_i = \mathrm{const} + i\, \Im m(z_i) \tag{28}$$

$$d = 4 - 2\epsilon \tag{29}$$

$$\mathrm{const} = \frac{(i\pi^{d/2})^2 (-1)^a (-s)^{d-a}}{\Gamma(a_2)\Gamma(a_4)\Gamma(a_5)\Gamma(a_6)\Gamma(a_7)\Gamma(d - a_{4567})} \tag{30}$$

The integrand includes e.g.:

$$\Gamma_2 = \Gamma(-z_2) \tag{31}$$

$$\Gamma_4 = \Gamma(-z_4) \tag{32}$$

$$\Gamma_7 = \Gamma(a_4 + z_2 + z_4) \tag{33}$$

$$\Gamma_8 = \Gamma(D - a_{445667} - z_2 - z_3 - 2z_4) \tag{34}$$

$$... \tag{35}$$

We now derive from B7l4m2 the MB-integral for B5l3m by setting $a_1 = 0$ (trivial, gives B6l3m2) and $a_4 = 0$.

The latter do with care because of

$$\frac{1}{\Gamma(a_4)} \to \frac{1}{\Gamma(0)} = 0 \tag{36}$$

See by inspection that we will get factor $\Gamma(a_4)$ if $z_2, z_4 \to 0$.

$\to$ Start with the z_2, z_4 integrations by taking the residues for closing the integration contours to the right:

$$I_{2,4} = \frac{(-1)^2}{(2\pi i)^2} \int dz_2 \Gamma(-z_2) \int dz_4 \frac{\Gamma(a_4 + z_2 + z_4)}{\Gamma(a_4)} \Gamma(-z_4) R(z_i) \tag{37}$$

$$= \frac{1}{(2\pi i)} \int dz_2 \Gamma(-z_2) \sum_{n=0,1,..} \frac{-(-1)^n}{n!} \frac{\Gamma(a_4 + z_2 + n)}{\Gamma(a_4)} R(z_i) \tag{38}$$

$$= \sum_{n,m=0,1,..} \frac{(-1)^{n+m}}{n!m!} \frac{\Gamma(a_4 + n + m)}{\Gamma(a_4)} R(z_i) \to_{a=0} 1 \tag{39}$$

So, setting $a_1 = a_4 = 0$ and eliminating $\int dz_2 dz_4$ with setting $z_2 = z_4 = 0$
we got a 4-fold Mellin-Barnes integral for B5l3m
with $24 - 3 = 21$ z_i-dependent Γ-functions which may yield residua within four-fold sums.

As mentioned:

This formula has to be calculated now explicitly for the case

$$B5l3md2 = \frac{B_2}{\epsilon^2} + \frac{B_1}{\epsilon} + B_0 \qquad (40)$$

(B5l3md2 is a dotted master, with index $a_2 = 2$, all other $a_i = 1$)

Next tasks:

- Find a region of definiteness of the n-fold MB-integral

$$\Re(z_1) = -1/80, \Re(z_3) = -33/40, \Re(z_5) = -21/20, \Re(z_6) = -59/160, \Re(\epsilon) = -1/10! \qquad (41)$$

- Then go to the physical region where $\epsilon << 1$ by distorting the integration path step by step (adding each crossed residuum – per residue this means one integral less!!!)
- Take integrals by sums over residua, i.e. introduce infinite sums
- Sum these infinite multiple series into some known functions of a given class, e.g. Nielsen polylogs, Harmonic polylogs or whatever is appropriate.

Here this means:

$$B5l3md2 \rightarrow MB(\text{4-dim}, \epsilon^0) + MB_3(\text{3-dim}, \epsilon^0) \quad (42)$$

$$+ MB_{36}(\text{2-dim}, \epsilon^{-1}, \epsilon^0) + MB_{365}(\text{1-dim}, \epsilon^{-2}, \epsilon^{-1}, \epsilon^0) \quad (43)$$

$$+ MB_5(\text{3-dim}, \epsilon^0) \quad (44)$$

After these preparations e.g.:

$$MB_{365}(\text{1-dim}, \epsilon^{-2}) \sim \frac{1}{\epsilon^2} \int dz_6 \frac{(-s)^{(z_6-1)}\Gamma(-z_6)^3\Gamma(1+z_6)}{8\Gamma(-2z_6)} \quad (45)$$

$$\sim \frac{1}{\epsilon^2} \sum_{n=0,\infty} -\frac{(-1)^n(-s)^n\Gamma(1+n)^3)}{8n!\Gamma(-2(-1-n))} \quad (46)$$

$$= -\frac{1}{\epsilon^2} \frac{ArcSin(\sqrt{s}/2)}{2\sqrt{4-s}\sqrt{s}} \quad (47)$$

$$= \frac{1}{\epsilon^2} \frac{x}{4(1-x^2)} H[0,x] \quad (48)$$

Here the residua were taken at $z_6 = -n-1, n = 0, 1, ..$

The divergent parts of the masters B5l3m are:

```
B513m[-2,x_,y_] = B513m[-1,x_,y_] = 0;

B513md1[-2,x_,y_] = ((-1 + x)^2*y*(-1 + y^2 + 2*y*H[0, y]))/
                    (8*x*(-1 + y)*(1 + y)^3);
B513md1[-1,x_,y_] = ((y*(6*(-1 + x - x^2 + x^3)*H[0, x]*
  (-1 + y^2 + 2*y*H[0, y]) - 6*(1 + x)*(-2 - 2*x^2 + 2*y^2 + 2*x^2*y^2 +
  y*z2 - 2*x*y*z2 + x^2*y*z2 + 2*(-2*x - y + 2*x*y - x^2*y - 2*x*y^2 +
  (-1 + x)^2*y*H[-1, -y] + 3*(-1 + x)^2*y*H[-1, y])*H[0, y] -
  6*(-1 + x)^2*y*H[0, -1, y] - 4*y*H[0, 0, y] + 8*x*y*H[0, 0, y] -
  4*x^2*y*H[0, 0, y] + 2*y*H[0, 1, y] - 4*x*y*H[0, 1, y] +
  2*x^2*y*H[0, 1, y])))/(24*x*(1 + x)*(-1 + y)*(1 + y)^3));

B513md2[-2,x_,y_] = -x/(1 - x^2)/4 H[0, x];
B513md2[-1,x_,y_] = ((x*(2*(1 + y^2)*H[0, x]*H[0, y] -
   (-1 + y^2)*(z2 + 6*H[-1, 0, x] - 4*H[0, 0, x] -
       2*H[1, 0, x])))/(4*(-1 + x^2)*(-1 + y^2)));

B513md2a[-a,x_,y_] = B513md2[-a,y,x],   a=-2,-1;

B513md3[-2,x_,y_] = 0;
B513md3[-1,x_,y_] = -((x*y*H[0, x]*H[0, y])/((-1 + x^2)*(-1 + y^2)));
```

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Summary

A calculation of the constant 2-loop term for Bhabha scattering is now available from massless calculations Penin, Bonciani et al.

A complete, massive 2-loop calculation is not finished so far.

- A complete list of MASSIVE masters was derived (2004)
- Huge files with algebraic relations for all the reducible Feynman integrals needed for the interferences of boxes with Born exist (not complete, but fully understood)
- Essential progress for the massive 2-box master integral determination.
  Underway: Determination of all 2-box masters in a systematic approach
  use Generalized Harmonic Polylogarithms Remiddi, Vermaseren
  plus potentially . . .)
- An unsolved problem is the systematic summation of the massive multiple sums after the MB-integral evaluation

It is possible to do the massive 2-loop calculation with present computers.

Miyamoto Musashi (1584 – 1645)

The Five Rings

The Broad Principles of Musashi's Strategy

- Do not think dishonestly.
- The Way is in training.
- Become aquainted with every art.
- Know the Ways of all professions.
- Distinguish between gain and loss in worldly matters.
- Develop intuitive judgement and understanding for everything.
- Perceive those things which cannot be seen.
- Pay attention even to trifles.
- Do nothing which is of no use.