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Some recent results on evaluating Feynman integrals

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- Part I: Evaluating triple boxes by Mellin–Barnes representation
  Z. Bern, L.J. Dixon, and V.A. Smirnov, hep-th/0505205
  "Iteration of Planar Amplitudes in Maximally Supersymmetric Yang-Mills Theory at Three Loops and Beyond"
- Part II: Applying Gröbner Bases to Solve Reduction Problems for Feynman Integrals
  A.V. Smirnov and V.A. Smirnov, hep-th/0509...

A given Feynman graph Γ $\rightarrow$ tensor reduction $\rightarrow$ various scalar Feynman integrals that have the same structure of the integrand with various distributions of powers of propagators.

$$F_\Gamma(a_1, a_2, \ldots) = \int \ldots \int \frac{\mathrm{d}^d k_1 \mathrm{d}^d k_2 \ldots}{(p_1^2 - m_1^2)^{a_1} (p_2^2 - m_2^2)^{a_2} \ldots}$$

$d = 4 - 2\epsilon$

Methods: analytical, numerical, semianalytical ...

A straightforward analytical strategy:
to evaluate, by some methods, every scalar Feynman integral generated by the given graph.

An advanced strategy:
to derive, without calculation, and then apply integration by parts (IBP) and Lorentz-invariance (LI) identities between the given family of Feynman integrals as *recurrence relations*.

A general integral from the given family is expressed as a linear combination of some basic (*master*) integrals.

The whole problem of evaluation$\rightarrow$

- constructing a reduction procedure
- evaluating master integrals

Evaluating master integrals:
Feynman/alpha parameters, Mellin–Barnes (MB) representation [V.A. Smirnov, J.B Tausk], DE (method of differential equations [A.V. Kotikov, Phys. Lett. **B254** (1991) 158; **B259** (1991) 314; **B267** (1991) 123; E. Remiddi, Nuovo Cim. **110A** (1997) 1435; T. Gehrmann and E. Remiddi, Nucl. Phys. **B580** (2000) 485])

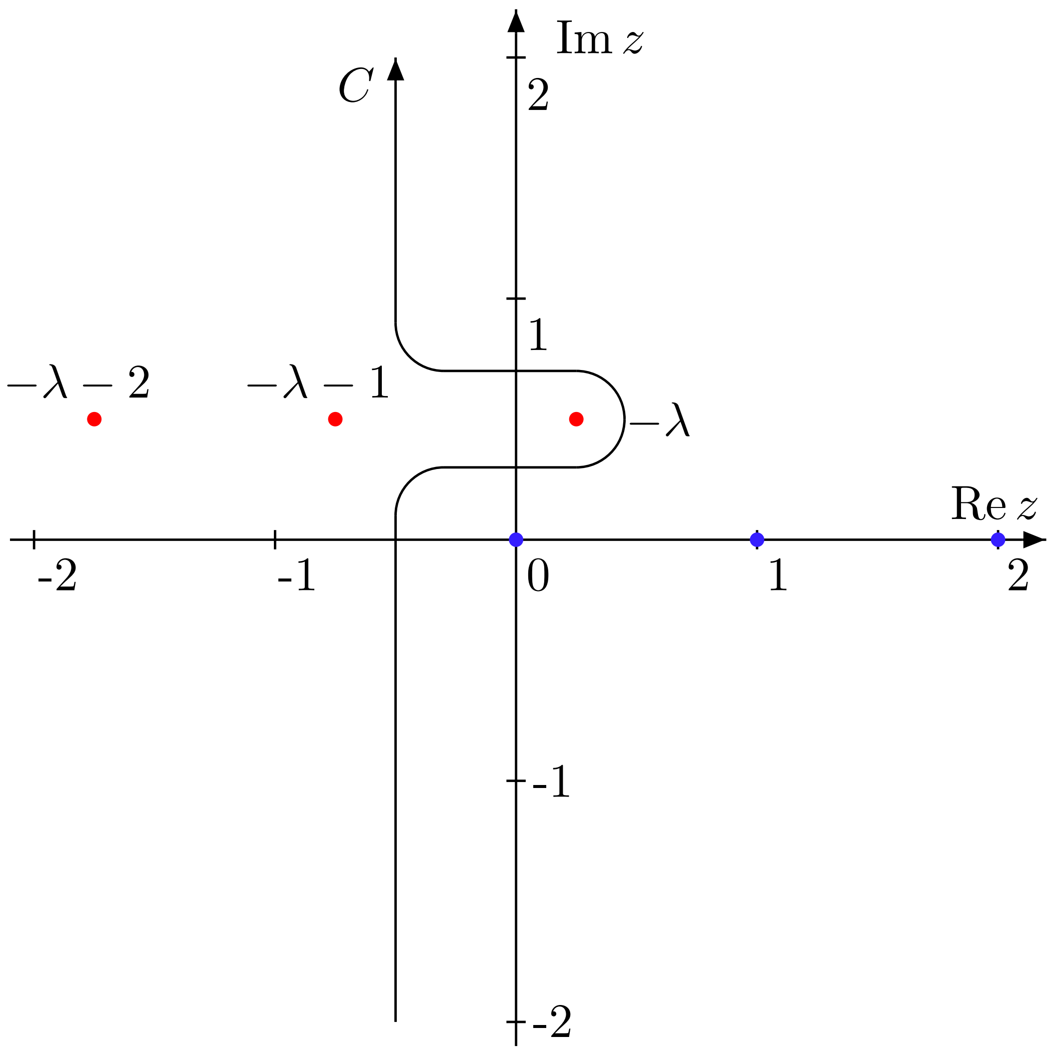
Part I.

Mellin–Barnes representation as a tool to evaluate master integrals

The basic formula:

$$\frac{1}{(X+Y)^{\lambda}} = \frac{1}{\Gamma(\lambda)}\frac{1}{2\pi \mathrm{i}}\int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \mathrm{d}z \frac{Y^z}{X^{\lambda+z}}\Gamma(\lambda+z)\Gamma(-z) \; .$$

The poles with a $\Gamma(\ldots+z)$ dependence are to the left of the contour and the poles with a $\Gamma(\ldots-z)$ dependence are to the right:

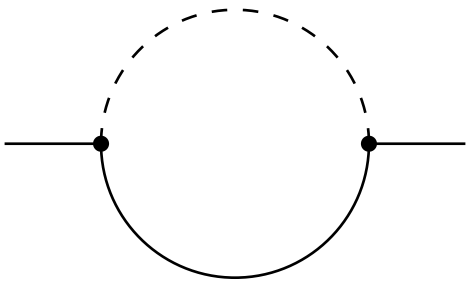


The simplest possibility:

$$\frac{1}{(m^2-k^2)^\lambda} = \frac{1}{\Gamma(\lambda)}\frac{1}{2\pi\mathrm{i}}\int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \mathrm{d}z \frac{(m^2)^z}{(-k^2)^{\lambda+z}}\Gamma(\lambda+z)\Gamma(-z) \ .$$

Example

$$F_\Gamma(q^2, m^2; a_1, a_2, d) = \int \frac{\mathrm{d}^d k}{(k^2-m^2)^{a_1}[(q-k)^2]^{a_2}}$$

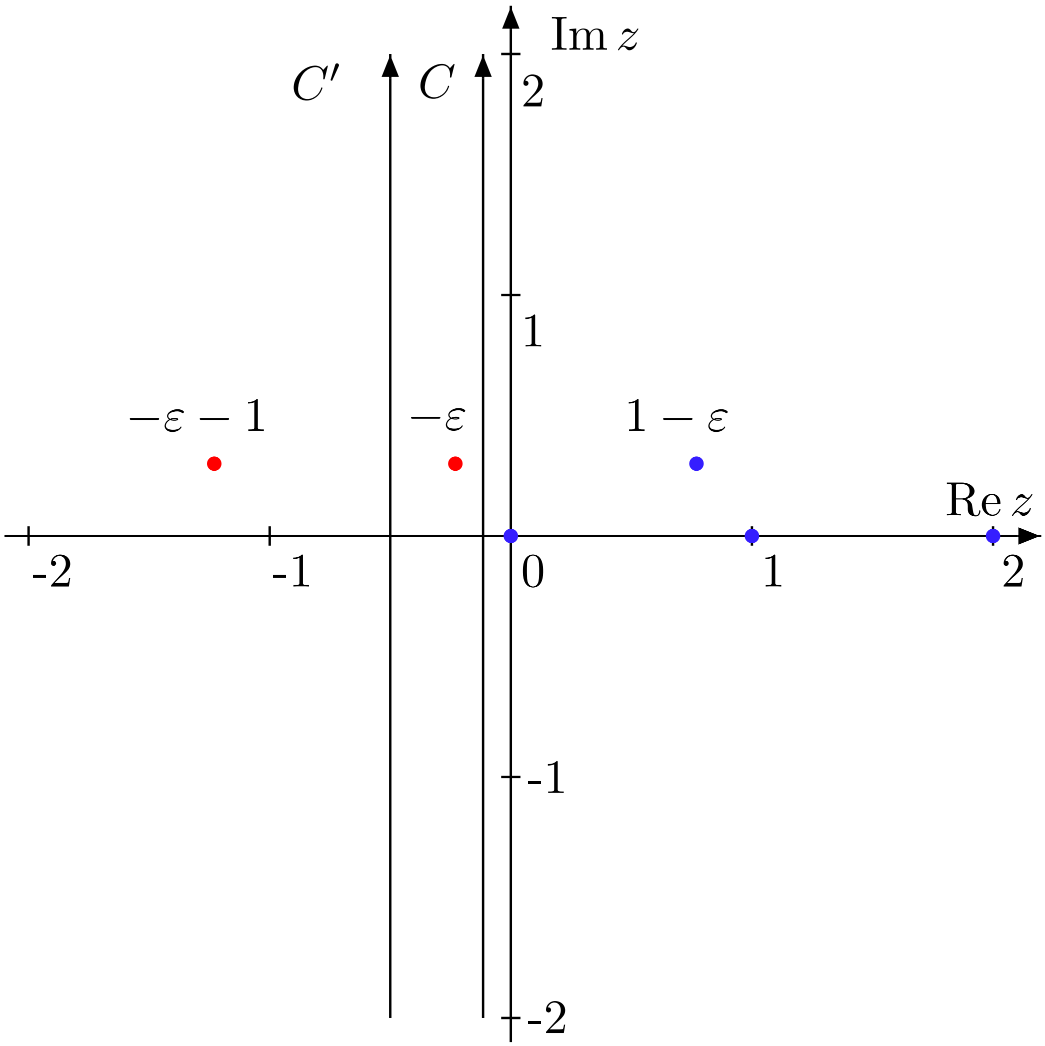


$$\begin{aligned} F_\Gamma(q^2, m^2; a_1, a_2, d) \ &= \ \frac{\mathrm{i}\pi^{d/2}(-1)^{a_1+a_2}\Gamma(2-\epsilon-a_2)}{\Gamma(a_1)\Gamma(a_2)(-q^2)^{a_1+a_2+\epsilon-2}} \\ &\times \frac{1}{2\pi\mathrm{i}}\int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \mathrm{d}z \left(\frac{m^2}{-q^2}\right)^z \Gamma(a_1+a_2+\epsilon-2+z) \\ &\times \frac{\Gamma(2-\epsilon-a_1-z)\Gamma(-z)}{\Gamma(4-2\epsilon-a_1-a_2-z)} \end{aligned}$$

In particular,

$$\begin{aligned} F_\Gamma(q^2, m^2; 1, 1, d) \ &= \ \frac{\mathrm{i}\pi^2\Gamma(1-\epsilon)}{(-q^2)^\epsilon} \\ &\times \frac{1}{2\pi\mathrm{i}}\int_C \mathrm{d}z \left(\frac{m^2}{-q^2}\right)^z \frac{\Gamma(\epsilon+z)\Gamma(-z)\Gamma(1-\epsilon-z)}{\Gamma(2-2\epsilon-z)} \end{aligned}$$

$\Gamma(\epsilon+z)\Gamma(-z) \longrightarrow$ a singularity in ϵ



Take a residue at $z = -\epsilon$:

$$\mathrm{i}\pi^2 \frac{\Gamma(\epsilon)}{(m^2)^\epsilon (1-\epsilon)}$$

and shift the contour:

$$\mathrm{i}\pi^2 \frac{1}{2\pi\mathrm{i}} \int_{C'} \mathrm{d}z \left(\frac{m^2}{-q^2}\right)^z \frac{\Gamma(z)\Gamma(-z)}{1-z}$$

Multiple MB Integrals

Two parts of the procedure:

- resolution of singularities in ϵ,
- evaluating MB integrals after expansion in ϵ

For part 1: shift contours and take residues

E.g., the product $\Gamma(1+z)\Gamma(-1-\epsilon-z)$ generates a pole of the type $\Gamma(-\epsilon)$.

Generally, $\Gamma(a+z)\Gamma(b-z)$, where a and b depend on the rest of the variables, generates a pole of the type $\Gamma(a+b)$.

'Key' gamma functions

An alternative strategy [Tausk]

Try to have a minimal number of MB integrations.

Derive MB representations for *general* powers of the propagators $\rightarrow$

- unambiguous prescriptions for choosing integration contours,
- various partial simple cases for check and various non-trivial partial cases

IBP is also possible, e.g.

$$\int_C \mathrm{d}z \frac{f(z)}{z^2} = \int_C \mathrm{d}z \frac{f'(z)}{z}$$

For part 2: apply the first and the second Barnes lemmas

$$\frac{1}{2\pi \mathrm{i}}\int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \mathrm{d}z\, \Gamma(\lambda_1+z)\Gamma(\lambda_2+z)\Gamma(\lambda_3-z)\Gamma(\lambda_4-z) = \frac{\Gamma(\lambda_1+\lambda_3)\Gamma(\lambda_1+\lambda_4)\Gamma(\lambda_2+\lambda_3)\Gamma(\lambda_2+\lambda_4)}{\Gamma(\lambda_1+\lambda_2+\lambda_3+\lambda_4)}$$

$$\frac{1}{2\pi \mathrm{i}}\int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \mathrm{d}z\, \frac{\Gamma(\lambda_1+z)\Gamma(\lambda_2+z)\Gamma(\lambda_3+z)\Gamma(\lambda_4-z)\Gamma(\lambda_5-z)}{\Gamma(\lambda_6+z)} = \frac{\Gamma(\lambda_1+\lambda_4)\Gamma(\lambda_2+\lambda_4)\Gamma(\lambda_3+\lambda_4)\Gamma(\lambda_1+\lambda_5)}{\Gamma(\lambda_1+\lambda_2+\lambda_4+\lambda_5)\Gamma(\lambda_1+\lambda_3+\lambda_4+\lambda_5)} \times \frac{\Gamma(\lambda_2+\lambda_5)\Gamma(\lambda_3+\lambda_5)}{\Gamma(\lambda_2+\lambda_3+\lambda_4+\lambda_5)}\,,$$

where $\lambda_6 = \lambda_1+\lambda_2+\lambda_3+\lambda_4+\lambda_5$

and their multiple corollaries, e.g.,

$$\frac{1}{2\pi \mathrm{i}}\int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \mathrm{d}z\, \Gamma(\lambda_1+z)\Gamma^*(\lambda_2+z)\Gamma(-\lambda_2-z)\Gamma(\lambda_3-z) = \Gamma(\lambda_1-\lambda_2)\Gamma(\lambda_2+\lambda_3)\left[\psi(\lambda_1-\lambda_2)-\psi(\lambda_1+\lambda_3)\right]$$

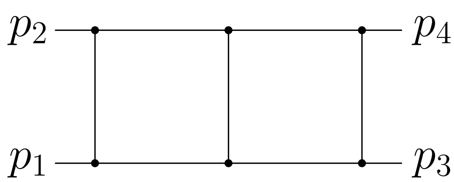
Closing contour in the complex plane and summation in terms of HPL [E. Remiddi and J.A.M. Vermaseren, Int. J. Mod. Phys. **A15** (2000) 725]

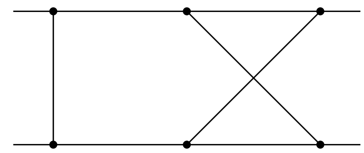
$$H(a_1,a_2,\ldots,a_n;x) = \int_0^x f(a_1;t)H(a_2,\ldots,a_n;t)\,,$$

where

$$f(\pm 1;t) = 1/(1\mp t),\quad f(0;t)=1/t,$$
$$H(\pm 1;x) = \mp\ln(1\mp x),\quad H(0;x)=\ln x\,,$$

with $a_i = 1, 0, -1$.





Massless on-shell ($p_i^2 = 0,\ i = 1, 2, 3, 4$) double boxes: done in 1999-2000, with multiple subsequent applications.

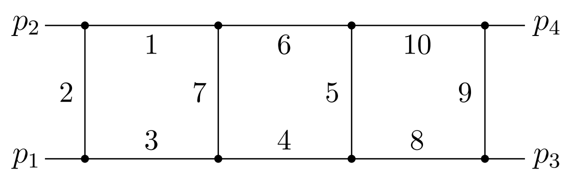
Reduction using shifting dimension

Master integrals: MB.

more loops, more legs, more parameters. . .

triple boxes

#loops + #legs $= 3 + 4 = 7 \gg 1$.



The general planar triple box Feynman integral

$$T(a_1, \ldots, a_{10}; s, t; \epsilon) \ = \ \int\int\int \frac{\mathrm{d}^d k \, \mathrm{d}^d l \, \mathrm{d}^d r}{[k^2]^{a_1}[(k+p_2)^2]^{a_2}[(k+p_1+p_2)^2]^{a_3}}$$
$$\times \frac{1}{[(l+p_1+p_2)^2]^{a_4}[(r-l)^2]^{a_5}[l^2]^{a_6}[(k-l)^2]^{a_7}}$$
$$\times \frac{1}{[(r+p_1+p_2)^2]^{a_8}[(r+p_1+p_2+p_3)^2]^{a_9}[r^2]^{a_{10}}} \, ,$$

where $k^2 = k^2 + i0$, $s = s + i0$, etc., $s = (p_1 + p_2)^2$, $t = (p_2 + p_3)^2$, and k, l and r are loop momenta.

Sevenfold MB representation of the general planar triple box

$$T(a_1,\ldots,a_8;s,t,m^2;\epsilon) \;=\; \frac{\left(i\pi^{d/2}\right)^3(-1)^a}{\prod_{j=2,5,7,8,9,10}\Gamma(a_j)\Gamma(4-a_{589(10)}-2\epsilon)(-s)^{a-6+3\epsilon}}$$
$$\times\frac{1}{(2\pi i)^7}\int_{-i\infty}^{+i\infty}\mathrm{d}w\prod_{j=2}^{7}\mathrm{d}z_j\left(\frac{t}{s}\right)^w\frac{\Gamma(a_2+w)\Gamma(-w)\Gamma(z_2+z_4)\Gamma(z_3+z_4)}{\Gamma(a_1+z_3+z_4)\Gamma(a_3+z_2+z_4)}$$
$$\times\frac{\Gamma(2-a_1-a_2-\epsilon+z_2)\Gamma(2-a_2-a_3-\epsilon+z_3)\Gamma(a_7+w-z_4)}{\Gamma(4-a_1-a_2-a_3-2\epsilon+w-z_4)\Gamma(a_6-z_5)\Gamma(a_4-z_6)}$$
$$\times\Gamma(+a_1+a_2+a_3-2+\epsilon+z_4)\Gamma(w+z_2+z_3+z_4-z_7)\Gamma(-z_5)\Gamma(-z_6)$$
$$\times\Gamma(2-a_5-a_9-a_{10}-\epsilon-z_5-z_7)\Gamma(2-a_5-a_8-a_9-\epsilon-z_6-z_7)$$
$$\times\Gamma(a_4+a_6+a_7-2+\epsilon+w-z_4-z_5-z_6-z_7)\Gamma(a_9+z_7)$$
$$\times\Gamma(4-a_4-a_6-a_7-2\epsilon+z_5+z_6+z_7)$$
$$\times\Gamma(2-a_6-a_7-\epsilon-w-z_2+z_5+z_7)\Gamma(2-a_4-a_7-\epsilon-w-z_3+z_6+z_7)$$
$$\times\Gamma(a_5+z_5+z_6+z_7)\Gamma(a_5+a_8+a_9+a_{10}-2+\epsilon+z_5+z_6+z_7)\,,$$

where $a_{589(10)}=a_5+a_8+a_9+a_{10}, a_{13}=a_1+a_3,\ \ldots,$ and $a=a_{12\ldots(10)}$.

The master triple box:

$$T(1,1,\ldots,1;s,t;\epsilon)$$
$$=\frac{\left(i\pi^{d/2}\right)^3}{\Gamma(-2\epsilon)(-s)^{4+3\epsilon}}\frac{1}{(2\pi i)^7}\int_{-i\infty}^{+i\infty}\mathrm{d}w\prod_{j=2}^{7}\mathrm{d}z_j\left(\frac{t}{s}\right)^w\frac{\Gamma(1+w)\Gamma(-w)}{\Gamma(1-2\epsilon+w-z_4)}$$
$$\times\frac{\Gamma(-\epsilon+z_2)\Gamma(-\epsilon+z_3)\Gamma(1+w-z_4)\Gamma(-z_2-z_3-z_4)\Gamma(1+\epsilon+z_4)}{\Gamma(1+z_2+z_4)\Gamma(1+z_3+z_4)}$$
$$\times\frac{\Gamma(z_2+z_4)\Gamma(z_3+z_4)\Gamma(-z_5)\Gamma(-z_6)\Gamma(w+z_2+z_3+z_4-z_7)}{\Gamma(1-z_5)\Gamma(1-z_6)\Gamma(1-2\epsilon+z_5+z_6+z_7)}$$
$$\times\Gamma(-1-\epsilon-z_5-z_7)\Gamma(-1-\epsilon-z_6-z_7)\Gamma(1+z_7)$$
$$\times\Gamma(1+\epsilon+w-z_4-z_5-z_6-z_7)\Gamma(-\epsilon-w-z_2+z_5+z_7)$$
$$\times\Gamma(-\epsilon-w-z_3+z_6+z_7)\Gamma(1+z_5+z_6+z_7)\Gamma(2+\epsilon+z_5+z_6+z_7)\,.$$

Result [V.S., Phys. Lett. **B567** (2003) 193]:

$$T(1,1,\ldots,1;s,t;\epsilon) = -\frac{\left(i\pi^{d/2}\mathrm{e}^{-\gamma_{\mathrm{E}}\epsilon}\right)^3}{s^3(-t)^{1+3\epsilon}} \sum_{i=0}^{6} \frac{c_j(x,L)}{\epsilon^j} \,,$$

where $x=-t/s$, $L=\ln(s/t)$, and

$$c_6 = \frac{16}{9}\,,\quad c_5 = -\frac{5}{3}L\,,\quad c_4 = -\frac{3}{2}\pi^2\,,$$

$$c_3 = 3(H_{0,0,1}(x)+LH_{0,1}(x)) + \frac{3}{2}(L^2+\pi^2)H_1(x) - \frac{11}{12}\pi^2 L - \frac{131}{9}\zeta_3\,,$$

$$\begin{aligned} c_2 = & -3\left(17H_{0,0,0,1}(x)+H_{0,0,1,1}(x)+H_{0,1,0,1}(x)+H_{1,0,0,1}(x)\right) \\ & -L\left(37H_{0,0,1}(x)+3H_{0,1,1}(x)+3H_{1,0,1}(x)\right) - \frac{3}{2}(L^2+\pi^2)H_{1,1}(x) \\ & -\left(\frac{23}{2}L^2+8\pi^2\right)H_{0,1}(x) - \left(\frac{3}{2}L^3+\pi^2 L-3\zeta_3\right)H_1(x) + \frac{49}{3}\zeta_3 L - \frac{1411}{1080}\pi^4\,, \end{aligned}$$

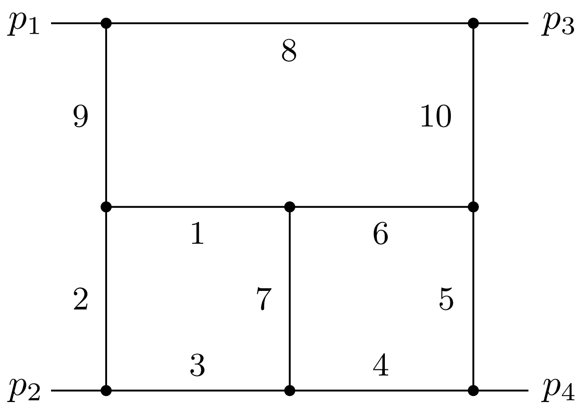
$$\begin{aligned} c_1 = & \,3\left(81H_{0,0,0,0,1}(x)+41H_{0,0,0,1,1}(x)+37H_{0,0,1,0,1}(x)+H_{0,0,1,1,1}(x)\right. \\ & +33H_{0,1,0,0,1}(x)+H_{0,1,0,1,1}(x)+H_{0,1,1,0,1}(x)+29H_{1,0,0,0,1}(x) \\ & \left.+H_{1,0,0,1,1}(x)+H_{1,0,1,0,1}(x)+H_{1,1,0,0,1}(x)\right) + L\left(177H_{0,0,0,1}(x)+85H_{0,0,1,1}(x)\right. \\ & \left.+73H_{0,1,0,1}(x)+3H_{0,1,1,1}(x)+61H_{1,0,0,1}(x)+3H_{1,0,1,1}(x)+3H_{1,1,0,1}(x)\right) \\ & +\left(\frac{119}{2}L^2+\frac{139}{12}\pi^2\right)H_{0,0,1}(x) + \left(\frac{47}{2}L^2+20\pi^2\right)H_{0,1,1}(x) \\ & +\left(\frac{35}{2}L^2+14\pi^2\right)H_{1,0,1}(x) + \frac{3}{2}\left(L^2+\pi^2\right)H_{1,1,1}(x) \\ & +\left(\frac{23}{2}L^3+\frac{83}{12}\pi^2 L-96\zeta_3\right)H_{0,1}(x) + \left(\frac{3}{2}L^3+\pi^2 L-3\zeta_3\right)H_{1,1}(x) \\ & +\left(\frac{9}{8}L^4+\frac{25}{8}\pi^2 L^2-58\zeta_3 L+\frac{13}{8}\pi^4\right)H_1(x) - \frac{503}{1440}\pi^4 L + \frac{73}{4}\pi^2\zeta_3 - \frac{301}{15}\zeta_5\,, \end{aligned}$$

$$\begin{aligned} c_0 = & -\left(951H_{0,0,0,0,0,1}(x)+819H_{0,0,0,0,1,1}(x)+699H_{0,0,0,1,0,1}(x)+195H_{0,0,0,1,1,1}(x)\right. \\ & +547H_{0,0,1,0,0,1}(x)+231H_{0,0,1,0,1,1}(x)+159H_{0,0,1,1,0,1}(x)+3H_{0,0,1,1,1,1}(x) \\ & +363H_{0,1,0,0,0,1}(x)+267H_{0,1,0,0,1,1}(x)+195H_{0,1,0,1,0,1}(x)+3H_{0,1,0,1,1,1}(x) \\ & +123H_{0,1,1,0,0,1}(x)+3H_{0,1,1,0,1,1}(x)+3H_{0,1,1,1,0,1}(x)+147H_{1,0,0,0,0,1}(x) \\ & +303H_{1,0,0,0,1,1}(x)+231H_{1,0,0,1,0,1}(x)+3H_{1,0,0,1,1,1}(x)+159H_{1,0,1,0,0,1}(x) \end{aligned}$$

$$
\begin{aligned}
&+3H_{1,0,1,0,1,1}(x)+3H_{1,0,1,1,0,1}(x)+87H_{1,1,0,0,0,1}(x)+3H_{1,1,0,0,1,1}(x)\\
&+3H_{1,1,0,1,0,1}(x)+3H_{1,1,1,0,0,1}(x))\\
&-L\,(729H_{0,0,0,0,1}(x)+537H_{0,0,0,1,1}(x)+445H_{0,0,1,0,1}(x)+133H_{0,0,1,1,1}(x)\\
&+321H_{0,1,0,0,1}(x)+169H_{0,1,0,1,1}(x)+97H_{0,1,1,0,1}(x)+3H_{0,1,1,1,1}(x)\\
&+165H_{1,0,0,0,1}(x)+205H_{1,0,0,1,1}(x)+133H_{1,0,1,0,1}(x)+3H_{1,0,1,1,1}(x)\\
&+61H_{1,1,0,0,1}(x)+3H_{1,1,0,1,1}(x)+3H_{1,1,1,0,1}(x))\\
&-\left(\frac{531}{2}L^2+\frac{89}{4}\pi^2\right)H_{0,0,0,1}(x)-\left(\frac{311}{2}L^2+\frac{619}{12}\pi^2\right)H_{0,0,1,1}(x)\\
&-\left(\frac{247}{2}L^2+\frac{307}{12}\pi^2\right)H_{0,1,0,1}(x)-\left(\frac{71}{2}L^2+32\pi^2\right)H_{0,1,1,1}(x)\\
&-\left(\frac{151}{2}L^2-\frac{197}{12}\pi^2\right)H_{1,0,0,1}(x)-\left(\frac{107}{2}L^2+50\pi^2\right)H_{1,0,1,1}(x)\\
&-\left(\frac{35}{2}L^2+14\pi^2\right)H_{1,1,0,1}(x)-\frac{3}{2}\left(L^2+\pi^2\right)H_{1,1,1,1}(x)\\
&-\left(\frac{119}{2}L^3+\frac{317}{12}\pi^2L-455\zeta_3\right)H_{0,0,1}(x)-\left(\frac{47}{2}L^3+\frac{179}{12}\pi^2L\right.\\
&\left.-120\zeta_3\right)H_{0,1,1}(x)-\left(\frac{35}{2}L^3+\frac{35}{12}\pi^2L-156\zeta_3\right)H_{1,0,1}(x)-\left(\frac{3}{2}L^3+\pi^2L\right.\\
&\left.-3\zeta_3\right)H_{1,1,1}(x)-\left(\frac{69}{8}L^4+\frac{101}{8}\pi^2L^2-291\zeta_3L+\frac{559}{90}\pi^4\right)H_{0,1}(x)\\
&-\left(\frac{9}{8}L^4+\frac{25}{8}\pi^2L^2-58\zeta_3L+\frac{13}{8}\pi^4\right)H_{1,1}(x)\\
&-\left(\frac{27}{40}L^5+\frac{25}{8}\pi^2L^3-\frac{183}{2}\zeta_3L^2+\frac{131}{60}\pi^4L-\frac{37}{12}\pi^2\zeta_3+57\zeta_5\right)H_1(x)\\
&+\left(\frac{223}{12}\pi^2\zeta_3+149\zeta_5\right)L+\frac{167}{9}\zeta_3^2-\frac{624607}{544320}\pi^6\,.
\end{aligned}
$$

Studying cross order relations in $N=4$ supersymmetric gauge theories [C. Anastasiou, L.J. Dixon, Z. Bern and D.A. Kosower, Phys. Rev. Lett. **91** (2003) 251602]:

"One more triple box is needed" ('tennis court' diagram)



with numerator $(l_1+l_3)^2$ [Z. Bern, L.J. Dixon, and V.A. Smirnov, hep-th/0505205, to be published in PRD]

$$T(s,t;1,\ldots,1,-1,\epsilon) = -\frac{\left(\mathrm{i}\pi^{d/2}\right)^3}{\Gamma(-2\epsilon)(-s)^{1+3\epsilon}t^2}$$

$$\times\frac{1}{(2\pi\mathrm{i})^8}\int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty}\ldots\int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty}\mathrm{d}w\,\mathrm{d}z_1\prod_{j=2}^{7}\mathrm{d}z_j\Gamma(-z_j)\left(\frac{t}{s}\right)^w\Gamma(1+3\epsilon+w)$$

$$\times\frac{\Gamma(-3\epsilon-w)\Gamma(1+z_1+z_2+z_3)\Gamma(-1-\epsilon-z_1-z_3)\Gamma(1+z_1+z_4)}{\Gamma(1-z_2)\Gamma(1-z_3)\Gamma(1-z_6)\Gamma(1-2\epsilon+z_1+z_2+z_3)}$$

$$\times\frac{\Gamma(-1-\epsilon-z_1-z_2-z_4)\Gamma(2+\epsilon+z_1+z_2+z_3+z_4)}{\Gamma(-1-4\epsilon-z_5)\Gamma(1-z_4-z_7)\Gamma(2+2\epsilon+z_4+z_5+z_6+z_7)}$$

$$\times\Gamma(-\epsilon+z_1+z_3-z_5)\Gamma(2-w+z_5)\Gamma(-1+w-z_5-z_6)$$

$$\times\Gamma(z_5+z_7-z_1)\Gamma(1+z_5+z_6)\Gamma(-1+w-z_4-z_5-z_7)$$

$$\times\Gamma(-\epsilon+z_1+z_2-z_5-z_6-z_7)\Gamma(1-\epsilon-w+z_4+z_5+z_6+z_7)$$

$$\times\Gamma(1+\epsilon-z_1-z_2-z_3+z_5+z_6+z_7)$$

$$\Gamma(-1+w-z_5-z_6)\Gamma(-1+w-z_4-z_5-z_7)\,.$$

are crucial for the generation of poles in ϵ

$$T = T_{00} + T_{01} + T_{10} + T_{11}$$

T_{01}: residue at $z_7 = -1 + w - z_4 - z_5$ and changing the nature of the first pole of $\Gamma(-1+w-z_5-z_6)$
T_{10}: residue at $z_6 = -1 + w - z_5$ and changing the nature of the first pole of $\Gamma(-1+w-z_4-z_5-z_7)$
T_{11}: taking both residues
T_{00}: changing the nature of both poles

124 contributions (e.g. $T_{00} \to 11$ contributions)

Result:

$$T(s,t;1,\ldots,1,-1,\epsilon) = -\frac{\left(\mathrm{i}\pi^{d/2}\mathrm{e}^{-\gamma_{\mathrm{E}}\epsilon}\right)^3}{(-s)^{1+3\epsilon}\,t^2} \sum_{i=0}^{6} \frac{c_j}{\epsilon^j}\,,$$

where

$$\begin{aligned}
c_6 &= \frac{16}{9}\,,\quad c_5 = -\frac{13}{6}\ln x\,,\quad c_4 = -\frac{19}{12}\pi^2 + \frac{1}{2}\ln^2 x\\
c_3 &= \frac{5}{2}\left[\mathrm{Li}_3\left(-x\right) - \ln x\,\mathrm{Li}_2\left(-x\right)\right] + \frac{7}{12}\ln^3 x - \frac{5}{4}\ln^2 x\ln(1+x)\\
&\quad + \frac{157}{72}\pi^2\ln x - \frac{5}{4}\pi^2\ln(1+x) - \frac{241}{18}\zeta(3)\,,
\end{aligned}$$

$$
\begin{aligned}
c_2 = {} & \frac{1}{2}\left[11H_{0,0,0,1}(x) - 5H_{0,0,1,1}(x) - 5H_{0,1,0,1}(x) - 5H_{1,0,0,1}(x)\right] \\
& + \frac{1}{2}L\left[14H_{0,0,1}(x) - 5H_{0,1,1}(x) - 5H_{1,0,1}(x)\right] + \frac{1}{4}L^2\left[17H_{0,1}(x) - 5H_{1,1}(x)\right] \\
& + \frac{4}{3}\pi^2 H_{0,1}(x) - \frac{5}{4}\pi^2 H_{1,1}(x) + \frac{5}{3}L^3 H_1(x) + \frac{25}{12}L\pi^2 H_1(x) \\
& - \frac{41}{3}L\zeta_3 + \frac{5}{2}H_1(x)\zeta_3 - \frac{1}{3}L^4 - \frac{1}{4}L^2\pi^2 + \frac{2429}{6480}\pi^4 , \\
c_1 = {} & \frac{1}{2}\left[-55H_{0,0,0,0,1}(x) - 59H_{0,0,0,1,1}(x) - 31H_{0,0,1,0,1}(x) + 5H_{0,0,1,1,1}(x)\right. \\
& - 3H_{0,1,0,0,1}(x) + 5H_{0,1,0,1,1}(x) + 5H_{0,1,1,0,1}(x) + 25H_{1,0,0,0,1}(x) \\
& \left. + 5H_{1,0,0,1,1}(x) + 5H_{1,0,1,0,1}(x) + 5H_{1,1,0,0,1}(x)\right] \\
& + \frac{1}{2}L\left[22H_{0,0,0,1}(x) - 46H_{0,0,1,1}(x) - 18H_{0,1,0,1}(x) + 5H_{0,1,1,1}(x)\right. \\
& \left. + 10H_{1,0,0,1}(x) + 5H_{1,0,1,1}(x) + 5H_{1,1,0,1}(x)\right] \\
& + \frac{1}{4}L^2\left[64H_{0,0,1}(x) - 33H_{0,1,1}(x) - 5H_{1,0,1}(x) + 5H_{1,1,1}(x)\right] \\
& + \frac{1}{24}\pi^2\left[25H_{0,0,1}(x) - 128H_{0,1,1}(x) + 40H_{1,0,1}(x) + 30H_{1,1,1}(x)\right] \\
& + \frac{1}{12}L^3\left[71H_{0,1}(x) - 20H_{1,1}(x)\right] \\
& + \frac{1}{24}L\pi^2\left[153H_{0,1}(x) - 50H_{1,1}(x)\right] + \frac{1}{2}\left[8H_{0,1}(x) - 5H_{1,1}(x)\right]\zeta_3 \\
& + \frac{43}{48}L^4 H_1(x) + \frac{71}{48}L^2\pi^2 H_1(x) - \frac{5}{144}\pi^4 H_1(x) - \frac{5}{2}LH_1(x)\zeta_3 + \frac{7}{48}L^5 \\
& + \frac{227}{144}L^3\pi^2 + \frac{13}{4}L^2\zeta_3 + \frac{10913}{8640}L\pi^4 + \frac{3257}{216}\pi^2\zeta_3 - \frac{889}{10}\zeta_5 , \\
c_0 = {} & \frac{1}{2}\left[379H_{0,0,0,0,0,1}(x) + 343H_{0,0,0,0,1,1}(x) + 419H_{0,0,0,1,0,1}(x)\right. \\
& + 347H_{0,0,0,1,1,1}(x) + 355H_{0,0,1,0,0,1}(x) + 175H_{0,0,1,0,1,1}(x) \\
& + 223H_{0,0,1,1,0,1}(x) - 5H_{0,0,1,1,1,1}(x) + 151H_{0,1,0,0,0,1}(x) + 3H_{0,1,0,0,1,1}(x) \\
& + 51H_{0,1,0,1,0,1}(x) - 5H_{0,1,0,1,1,1}(x) + 99H_{0,1,1,0,0,1}(x) - 5H_{0,1,1,0,1,1}(x) \\
& - 5H_{0,1,1,1,0,1}(x) - 193H_{1,0,0,0,0,1}(x) - 169H_{1,0,0,0,1,1}(x) - 121H_{1,0,0,1,0,1}(x) \\
& - 5H_{1,0,0,1,1,1}(x) - 73H_{1,0,1,0,0,1}(x) - 5H_{1,0,1,0,1,1}(x) - 5H_{1,0,1,1,0,1}(x)
\end{aligned}
$$

$$-25H_{1,1,0,0,0,1}(x) - 5H_{1,1,0,0,1,1}(x) - 5H_{1,1,0,1,0,1}(x) - 5H_{1,1,1,0,0,1}(x)]$$
$$+\frac{1}{2}L\,[98H_{0,0,0,0,1}(x) - 22H_{0,0,0,1,1}(x) + 98H_{0,0,1,0,1}(x) + 238H_{0,0,1,1,1}(x)$$
$$+78H_{0,1,0,0,1}(x) + 66H_{0,1,0,1,1}(x) + 114H_{0,1,1,0,1}(x) - 5H_{0,1,1,1,1}(x)$$
$$-82H_{1,0,0,0,1}(x) - 106H_{1,0,0,1,1}(x) - 58H_{1,0,1,0,1}(x) - 5H_{1,0,1,1,1}(x)$$
$$-10H_{1,1,0,0,1}(x) - 5H_{1,1,0,1,1}(x) - 5H_{1,1,1,0,1}(x)]$$
$$+\frac{1}{4}L^2\,[124H_{0,0,0,1}(x) - 208H_{0,0,1,1}(x) - 44H_{0,1,0,1}(x) + 129H_{0,1,1,1}(x)$$
$$-20H_{1,0,0,1}(x) - 43H_{1,0,1,1}(x) + 5H_{1,1,0,1}(x) - 5H_{1,1,1,1}(x)]$$
$$+\frac{1}{24}\pi^2\,[183H_{0,0,0,1}(x) - 121H_{0,0,1,1}(x) + 375H_{0,1,0,1}(x) + 704H_{0,1,1,1}(x)$$
$$+31H_{1,0,0,1}(x) - 328H_{1,0,1,1}(x) - 40H_{1,1,0,1}(x) - 30H_{1,1,1,1}(x)]$$
$$+\frac{1}{12}L^3\,[260H_{0,0,1}(x) - 215H_{0,1,1}(x) - 7H_{1,0,1}(x) + 20H_{1,1,1}(x)]$$
$$+\frac{1}{24}L\pi^2\,[326H_{0,0,1}(x) - 633H_{0,1,1}(x) + 127H_{1,0,1}(x) + 50H_{1,1,1}(x)]$$
$$-\frac{1}{2}\,[-3LH_{0,1}(x) - 5LH_{1,1}(x) + 165H_{0,0,1}(x) + 104H_{0,1,1}(x)$$
$$-68H_{1,0,1}(x) - 5H_{1,1,1}(x)]\,\zeta_3$$
$$+\frac{1}{48}L^4\,[309H_{0,1}(x) - 43H_{1,1}(x)] + \frac{1}{48}L^2\pi^2\,[725H_{0,1}(x) - 71H_{1,1}(x)]$$
$$+\frac{1}{720}\pi^4\,[1848H_{0,1}(x) + 25H_{1,1}(x)]$$
$$+\frac{37}{120}L^5H_1(x) + \frac{11}{8}L^3\pi^2H_1(x) + \frac{641}{720}L\pi^4H_1(x) + \frac{38}{3}L^3\zeta_3 + \frac{479}{18}L\pi^2\zeta_3$$
$$-2L^2H_1(x)\zeta_3 - \frac{269}{24}\pi^2H_1(x)\zeta_3 + \frac{129}{2}H_1(x)\zeta_5 + \frac{151}{720}L^6 + \frac{373}{288}L^4\pi^2$$
$$+\frac{3163}{2880}L^2\pi^4 - \frac{1054}{5}L\zeta_5 + \frac{1391417}{3265920}\pi^6 + \frac{197}{6}\zeta_3^2$$

no numerical checks ;-(

Independent check: evaluating the leading orders of the asymptotic behavior in the limit $s/t \to 0$, using the strategy of expansion by regions [M. Beneke and V.A. Smirnov, Nucl. Phys. B522, 321 (1998)]

(1c-1c-1c) and (4c-4c-4c) contributions as well as the (2c-4c-us) contribution (the momentum of the line between the external vertices with momenta p_1 and p_3 is considered ultrasoft (us), the loop momentum of the left box subgraph is considered 2-collinear and the loop momentum of the right box subgraph is considered 4-collinear)

$$T^{c-c-us}(s,t) = \frac{\left(\mathrm{i}\pi^{d/2}\right)^3}{(-s)^{1+4\epsilon}(-t)^{2-\epsilon}\,\epsilon}\Gamma(-\epsilon)^3\Gamma(\epsilon)^2\Gamma(1+2\epsilon)^2$$

The leading asymptotic behavior when $s/t \to 0$:

$$\begin{aligned}
T(s,t) \;&=\; -\frac{1}{(-s)^{1+3\epsilon}\,t^2} \\
\times \Bigg\{ & \frac{16}{9}\frac{1}{\epsilon^6} + \frac{13}{6}L\frac{1}{\epsilon^5} + \left[\frac{1}{2}L^2 - \frac{19}{12}\pi^2\right]\frac{1}{\epsilon^4} + \left[-\frac{1}{6}L^3 - \frac{67}{72}\pi^2 L - \frac{241}{18}\zeta_3\right]\frac{1}{\epsilon^3} \\
+ & \left[\frac{1}{24}L^4 + \frac{13}{24}\pi^2 L^2 - \frac{67}{6}\zeta_3 L - \frac{19}{6480}\pi^4\right]\frac{1}{\epsilon^2} \\
+ & \left[-\frac{1}{120}L^5 - \frac{13}{72}\pi^2 L^3 - \frac{5}{2}\zeta_3 L^2 - \frac{6523}{8640}\pi^4 L + \frac{1385}{216}\pi^2\zeta_3 - \frac{1129}{10}\zeta_5\right]\frac{1}{\epsilon} \\
+ & \frac{1}{720}L^6 + \frac{13}{288}\pi^2 L^4 + \frac{5}{6}\zeta_3 L^3 + \frac{331}{960}\pi^4 L^2 + \left(\frac{317}{72}\pi^2\zeta_3 - \frac{1203}{10}\zeta_5\right)L \\
- & \frac{180631}{3265920}\pi^6 - \frac{163}{6}\zeta_3^2 + O\left(\frac{s}{t}\right)\Bigg\}
\end{aligned}$$

C. Anastasiou, Z. Bern, L. J. Dixon and D. A. Kosower, Phys. Rev. Lett. **91**, 251602 (2003); Z. Bern, L. J. Dixon and D. A. Kosower, JHEP **0408**, 012 (2004): for the planar MHV n-point amplitude in $N = 4$ SUSY YM in two loops, one has

$$M_4^{(2)}(\epsilon) = \frac{1}{2}(M_4^{(1)}(\epsilon))^2 + f^{(2)}(\epsilon)\, M_4^{(1)}(2\epsilon) + C^{(2)} + O(\epsilon)\,,$$

where

$$f^{(2)}(\epsilon) = -(\zeta_2 + \zeta_3\epsilon + \zeta_4\epsilon^2 + \cdots)\,, \quad C^{(2)} = -\frac{1}{2}\zeta_2^2\,.$$

Z. Bern, L.J. Dixon, and V.A. Smirnov, hep-th/0505205: taking into account the results for the ladder triple box and the tennis court diagram up to ϵ^0, for planar double box up to ϵ^2, and for the box up to ϵ^4, we obtain, in three loops,

$$\begin{aligned} M_4^{(3)}(\epsilon) \; &= \; -\frac{1}{3}[M_4^{(1)}(\epsilon)]^3 + M_4^{(1)}(\epsilon)\, M_4^{(2)}(\epsilon) \\ &\quad + f^{(3)}(\epsilon)\, M_4^{(1)}(3\,\epsilon) + C^{(3)} + O(\epsilon)\,, \end{aligned}$$

where

$$f^{(3)}(\epsilon) = \frac{11}{2}\,\zeta_4 + \epsilon(6\zeta_5 + 5\zeta_2\zeta_3) + \epsilon^2(c_1\zeta_6 + c_2\zeta_3^2)\,,$$

$$C^{(3)} = \left(\frac{341}{216} + \frac{2}{9}c_1\right)\zeta_6 + \left(-\frac{17}{9} + \frac{2}{9}c_2\right)\zeta_3^2\,.$$

An exponentiation of the planar MHV n-point amplitudes in $N = 4$ SUSY YM at L loops:

$$\begin{aligned}\mathcal{M}_n &\equiv 1 + \sum_{L=1}^{\infty} a^L M_n^{(L)}(\epsilon) \\ &= \exp\left[\sum_{l=1}^{\infty} a^l \left(f^{(l)}(\epsilon) M_n^{(1)}(l\epsilon) + C^{(l)} + E_n^{(l)}(\epsilon)\right)\right].\end{aligned}$$

where

$$a \equiv \frac{N_c \alpha_s}{2\pi}(4\pi e^{-\gamma})^{\epsilon},$$

$M_n^{(1)}(l\epsilon)$ is the all-orders-in-ϵ one-loop amplitude (with $\epsilon \to l\epsilon$), and

$$f^{(l)}(\epsilon) = f_0^{(l)} + \epsilon f_1^{(l)} + \epsilon^2 f_2^{(l)}.$$

The constants $f_k^{(l)}$ and $C^{(l)}$ are independent of the number of legs n.

The $E_n^{(l)}(\epsilon)$ are non-iterating $O(\epsilon)$ contributions to the l-loop amplitudes (with $E_n^{(l)}(0) = 0$).

By definition, the all-orders-in-ϵ one-loop amplitude is absorbed into $M_n^{(1)}(\epsilon)$:

$$f^{(1)}(\epsilon) = 1, \qquad C^{(1)} = 0, \qquad E_n^{(1)}(\epsilon) = 0.$$

Advantages of the method based on MB representation:

- An appropriate multiple MB representation for a given class of integrals is derived for general powers of the propagators and irreducible numerators. In order to achieve the minimal number of MB integrations it is recommended to derive an MB representation for a sub-loop integral, insert it in the given integral over the loop momenta, etc.
- There is always the possibility to check multiple MB representations, which are sometimes rather cumbersome, by using simple partial cases.
- Multiple MB integrals are very flexible for the resolution of the singularities in ϵ. This procedure reduces to shifting contours and taking residues.
- After the resolution of the singularities in ϵ, at least some of the integrations can be performed explicitly by corollaries of the first and the second Barnes lemmas, with results in terms of gamma and psi functions.
- One can usually have an easy numerical control on finite (in ϵ) MB integrals: it is enough to integrate from $-5i$ to $+5i$ along the imaginary axis to have a very good accuracy.
- When the integration in multiple MB integrals is hardly performed explicitly, one can convert them into multiple series and apply such packages as `SUMMER`for summation.

Part II.

Gröbner Basis as a tool to Solve Reduction Problems

Recent attempts to make the reduction procedure systematic:

- Laporta's idea → [S. Laporta, Int. J. Mod. Phys. **A15** (2000) 5087; T. Gehrmann and E. Remiddi [Nucl. Phys. **B601** (2001) 248, 287]: "When increasing the total dimension of the denominator and numerator in Feynman integrals of a given family, the total number of IBP and LI equations grows faster than the number of independent Feynman integrals (labelled by the powers of propagators and powers of independent scalar products in the numerators). Therefore this system of equations sooner or later becomes overdetermined, and one obtains the possibility to perform a reduction to master integrals"
- shifting dimension [O.V. Tarasov, Nucl. Phys. **B 480** (1996) 397; Phys. Rev. D54 (1996) 6479]
- Baikov's method [P.A. Baikov, Phys. Lett. **B385** (1996) 404; Nucl. Instrum. Methods **A389** (1997) 347; V.A. Smirnov and M. Steinhauser, Nucl. Phys. **B 672** (2003) 199]
- Gröbner basis [O.V. Tarasov, Acta Phys. Polon. **B29** (1998) 2655; Nucl. Instrum. Meth. A534 (2004) 293; V.P. Gerdt, Nucl. Phys. B (Proc. Suppl) **135** (2004) 2320]

Classical Gröbner basis

Let $\mathcal{A} = \mathbb{C}[x_1, \ldots, x_n]$ be the ring of polynomials of n variables $x_1, \ldots, x_n$ and $\mathcal{I} \subset \mathcal{A}$ be an ideal.
(A ring is a set with multiplication and addition. A non-empty subset $\mathcal{I}$ of a ring R is called a left ideal if (i) for any $a, b \in \mathcal{I}$ one has $a+b \in \mathcal{I}$ and (ii) for any $a \in \mathcal{I}, c \in R$ one has $ca \in \mathcal{I}$.)

A classical problem is to construct an algorithm that shows whether a given element $g \in \mathcal{A}$ is a member of $\mathcal{I}$ or not.

A finite set of polynomials in $\mathcal{I}$ is said to be a *basis* of $\mathcal{I}$ if any element of $\mathcal{I}$ can be represented as a linear combination of its elements, where the coefficients are some elements of $\mathcal{A}$.
Let $\{f_1, f_2, \ldots, f_k\}$ be a basis of $\mathcal{I}$.

Problem. If $g \in \mathcal{A}$, find out whether there are $r_1, \ldots, r_k \in \mathcal{A}$ such that $g = r_1 f_1 + \ldots + r_k f_k$.

NB: The problem is solved easily if we have a Gröbner basis

Ordering of monomials $cx_1^{i_1} \ldots x_n^{i_n}$, with $c \in \mathbb{C}$

E.g., *lexicographical* ordering:
A set $(i_1, \ldots, i_n)$ is *higher* than a set $(j_1, \ldots, j_n)$,

$(i_1, \ldots, i_n) \succ (j_1, \ldots, j_n)$

if there is $l \leq n$ such that $i_1 = j_1$, $i_2 = j_2$, $\ldots$, $i_{l-1} = j_{l-1}$ and $i_l > j_l$.

Degree-lexicographical ordering: $(i_1, \ldots, i_n) \succ (j_1, \ldots, j_n)$ if $\sum i_k > \sum j_k$, or $\sum i_k = \sum j_k$ and $(i_1, \ldots, i_n) \succ (j_1, \ldots, j_n)$ in the sense of the lexicographical ordering.

Two axioms:

1 is the only minimal element under this ordering,

if $f_1 \succ f_2$ then $gf_1 \succ gf_2$ for any g.

Fix an ordering. The *leading term* of a polynomial

$$P(x_1, \dots, x_n) = \sum c_{i_1,\dots,i_n} x_1^{i_1} \dots x_n^{i_n}$$

is the non-zero monomial $c_{i_1^0,\dots,i_n^0} x_1^{i_1^0} \dots x_n^{i_n^0}$ such that the degree $(i_1^0, \dots, i_n^0)$ is greater than the degrees of other monomials in P. Let us denote it by $\hat{P}$. We have $P = \hat{P} + \tilde{P}$.

Suppose that the leading term of the given polynomial g is divisible by the leading term of some polynomial of the basis, i.e. $\hat{g} = Q\hat{f}_i$ where Q is a monomial. Let $g_1 = g - Qf_i$.
The leading term of g_1 is lower than the leading term of g and that $g_1 \in \mathcal{I}$ if and only if $g \in \mathcal{I}$.

Then proceed with g_1 as with g, using the same f_i or some other element f_k of the initial basis, and obtain similarly $g_2, g_3, \dots$.

Obtain $g_k \equiv 0$ or an element g_k such that $\hat{g}_k$ is not divisible by any leading term $\hat{f}_i$.

g is reduced to g_k modulo the basis $\{f_1, f_2, \dots, f_k\}$

A basis $\{f_1, f_2, \dots, f_k\}$ is called a *Gröbner basis* of $\mathcal{I}$ if any $g \in \mathcal{I}$ is reduced by the described procedure to zero.
Gröbner basis $\to$ an algorithm to verify whether an element $g \in \mathcal{A}$ is a member of $\mathcal{I}$.

Let $\{f_1, f_2, \ldots, f_k\}$ be a basis of $\mathcal{I} \rightarrow$
construct a Gröbner basis starting from it and using *Buchberger algorithm*

Let $\hat{f}_i = wq_i$ and $\hat{f}_j = wq_j$ where w, q_i and q_j are monomials and w is not a constant.

Define $f_{i,j} = f_i q_j - f_j q_i$.

Reduce $f_{i,j}$ modulo the set $\{f_i\}$. If one obtains a non-zero polynomial due to this reduction, add it to the set $\{f_i\}$. Consider then the other elements with $\hat{f}'_i = wq'_i$ and $\hat{f}'_j = w'q'_j$ with w' which is not a constant etc. If there is nothing to do according to this procedure one obtains a Gröbner basis.

Buchberger: "Such a procedure stops after a finite number of steps"

A family of dimensionally regularized Feynman integrals

$$F(a_1,\ldots,a_n) \;=\; \int\cdots\int \frac{\mathrm{d}^d k_1\ldots \mathrm{d}^d k_h}{E_1^{a_1}\ldots E_n^{a_n}}\,,$$

where k_i, $i=1,\ldots,h$, are loop momenta and

$$E_r \;=\; \sum_{i\geq j\geq 1} A_r^{ij}\, p_i\cdot p_j - m_r^2\,,$$

with $r = 1,\ldots,n$; $p_i = k_i$, $i = 1,\ldots,h$, or independent external momenta $p_{h+1}=q_1,\ldots,p_{h+n}=q_N$ of the graph.

Irreducible polynomials in the numerator can be represented as denominators raised to negative powers.

IBP:

$$\int\ldots\int \mathrm{d}^d k_1 \mathrm{d}^d k_2 \ldots \frac{\partial}{\partial k_i}\left(p_j \frac{1}{E_1^{a_1}\ldots E_N^{a_N}}\right) = 0$$

$\rightarrow$

$$\sum_{r,r',i'} \bar{A}_r^{i'i}\tilde{A}_{r'}^{ji'}\left(\mathbf{r}'^{-} + m_{r'}^2\right) a_r \mathbf{r}^{+} \;=\; (d-h-1)\delta_{ij}/2\;,$$

where $\bar{A}_r^{ij} = A_r^{ij}$ for $i=j$, $A_r^{ij}/2$ for $i>j$ and $A_r^{ji}/2$ for $i<j$.

$$\sum_{r=1}^{N} A_r^{ij}\,(A^{-1})_r^{i'j'} \;=\; \delta_{ii'}\delta_{jj'}\;.$$

Then $\tilde{A}_r^{ij}$ is the symmetrical extension of $(A^{-1})_r^{ij}$ to all values i,j.

$$\mathbf{r}^{\pm}F(\ldots,a_r,\ldots) = F(\ldots,a_r\pm 1,\ldots)$$

Let $\mathcal{A}$ be the algebra generated by $Y_i = \mathbf{i}^+, Y_i^- = \mathbf{i}^-$ and the operators of multiplication by the indices a_i.

The left-hand sides of IBP relations are determined by some operators $f_i \in \mathcal{A}$. The ideal $\mathcal{I}$ of IBP relations generated by the elements f_i.

Let us think of $a_i > 0$,

$$F(a_1, a_2, \ldots, a_n) = Y_1^{a_1-1} \ldots Y_n^{a_n-1} F(1, 1, \ldots, 1)$$

The reduction problem $\longrightarrow$
reduce the monomial $Y_1^{a_1-1} \ldots Y_n^{a_n-1}$ modulo the ideal of the IBP relations

$$Y_1^{a_1-1} \ldots Y_n^{a_n-1} = \sum r_i f_i + \sum c_{i_1,\ldots,i_n} Y_1^{i_1-1} \ldots Y_n^{i_n-1}$$

Apply to $F(1, 1, \ldots, 1)$ to obtain

$$F(a_1, a_2, \ldots, a_n) = \sum c_{i_1,\ldots,i_n} F(i_1, i_2, \ldots, i_n) \, .$$

If a given basis is a Gröbner basis then the number of different integrals on the right-hand side of these equations for various a_i is finite $\longrightarrow$ reduction to master integrals

This set can be, however, non-minimal.

A complication: the indices a_i can be not only positive but also negative.

2^n regions (*sectors*) labelled by subsets $\nu \subseteq \{1, \ldots, n\}$:
$\sigma_\nu = \{(a_1, \ldots, a_n) : a_i > 0 \;\text{ if }\; i \in \nu \, , \;\; a_i \leq 0 \;\text{ if }\; i \notin \nu\}$
In the sector $\sigma_{\{1,\ldots,n\}}$, consider Y_i as basic operators.

In the sector σ_ν, consider Y_i for $i \in \nu$ and Y_i^- for other i as basic operators.

Construct a Gröbner basis for all non-trivial sectors. (A sector is called *trivial* if all the given Feynman integrals are identically zero in it due to boundary conditions. At least $\sigma_\emptyset$ is trivial.)

Problems:

- too many master integrals
- leading coefficients are polynomials in a_i, so that they can be equal to zero at certain points
- practical implementations fail to work already in low-dimensional examples because of huge computer time

Our algorithm [A.S.& V.S, hep-th/0509...]:

- Construct *sector bases* (s-bases), rather than Gröbner bases for all the sectors.

  An s-basis for a sector σ_ν is a set of elements of a basis which provides the possibility of a reduction to master integrals *and* integrals whose indices lie in *lower* sectors, i.e. $\sigma_{\nu'}$ for $\nu' \subset \nu$. (It is most complicated to construct s-bases for minimal sectors, i.e. which are non-trivial but whose lower sectors are trivial.)

- The construction can be terminated when the Gröbner basis is not yet constructed but the 'current' basis already provides us the needed reduction.

Example 1.

$$F(a_1,a_2) = \int \frac{\mathrm{d}^d k}{(k^2-m^2)^{a_1}[(q-k)^2]^{a_2}}\;.$$

$F(a_1,a_2)=0$ if $a_1\leq 0$.

IBP relations $\rightarrow$

$$\begin{aligned}
f_1 &= d-2a_1-a_2-2m^2a_1Y_1-m^2a_2Y_2+q^2a_2Y_2-a_2Y_2Y_1^{-1}\\
f_2 &= a_2-a_1-m^2a_1Y_1-q^2a_1Y_1-m^2a_2Y_2+q^2a_2Y_2\\
&\quad -a_2Y_2Y_1^{-1}+a_1Y_1Y_2^{-1}\,.
\end{aligned}$$

Two sectors. $\sigma_{\{1,2\}} \rightarrow$

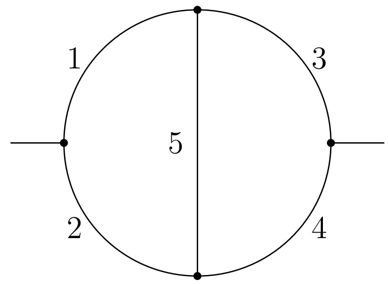
$$\begin{aligned}
g_{11} &= Y_1^2+a_1Y_1^2+3Y_1Y_2-dY_1Y_2+a_1Y_1Y_2+2a_2Y_1Y_2\\
&\quad +m^2Y_1^2Y_2-q^2Y_1^2Y_2+m^2a_1Y_1^2Y_2-q^2a_1Y_1^2Y_2\;,\\
g_{12} &= -3Y_1Y_2+dY_1Y_2-2a_1Y_1Y_2-a_2Y_1Y_2-2m^2Y_1^2Y_2\\
&\quad -2m^2a_1Y_1^2Y_2-Y_2^2-a_2Y_2^2-m^2Y_1Y_2^2\\
&\quad +q^2Y_1Y_2^2-m^2a_2Y_1Y_2^2+q^2a_2Y_1Y_2^2\,.
\end{aligned}$$

$\sigma_1 \rightarrow$

$$\begin{aligned}
g_{21} &= 1-a_2+m^2Y_1-q^2Y_1-m^2a_2Y_1+q^2a_2Y_1-Y_1Y_2^{-1}\\
&\quad +dY_1Y_2^{-1}-2a_1Y_1Y_2^{-1}-a_2Y_1Y_2^{-1}-2m^2Y_1^2Y_2^{-1}\\
&\quad -2m^2a_1Y_1^2Y_2^{-1}\;,\\
g_{22} &= -2m^2+2m^2a_2-2m^4Y_1+2m^2q^2Y_1+2m^4a_2Y_1\\
&\quad -2m^2q^2a_2Y_1-2Y_2^{-1}+a_2Y_2^{-1}+2m^2Y_1Y_2^{-1}+2q^2Y_1Y_2^{-1}\\
&\quad +2m^2a_1Y_1Y_2^{-1}-m^2a_2Y_1Y_2^{-1}-q^2a_2Y_1Y_2^{-1}\\
&\quad +2m^4Y_1^2Y_2^{-1}+2m^2q^2Y_1^2Y_2^{-1}+2m^4a_1Y_1^2Y_2^{-1}\\
&\quad +2m^2q^2a_1Y_1^2Y_2^{-1}-dY_1Y_2^{-2}+2a_1Y_1Y_2^{-2}+a_2Y_1Y_2^{-2}\,.
\end{aligned}$$

Two master integrals, $F(1,1)$ and $F(1,0)$

Example 2. Two-loop massless propagator integrals



$$F(a_1, a_2, a_3, a_4, a_5)$$
$$= \int\int \frac{\mathrm{d}^d k\, \mathrm{d}^d l}{(k^2)^{a_1}[(q-k)^2]^{a_2}(l^2)^{a_3}[(q-l)^2]^{a_4}[(k-l)^2]^{a_5}}$$

Boundary conditions:
$F(a_1, a_2, a_3, a_4, a_5) = 0$, if $a_i, a_5 \leq 0$ for $i = 1, \ldots, 4$, or $a_1, a_2 \leq 0$, or $a_3, a_4 \leq 0$, or $a_1, a_3 \leq 0$, or $a_2, a_4 \leq 0$

Symmetry:

$$F(a_1, a_2, a_3, a_4, a_5) = F(a_2, a_1, a_4, a_3, a_5) = F(a_3, a_4, a_1, a_2, a_5)\,.$$

IBP relations $\rightarrow$

$$\begin{aligned}
f_1 &= (d - 2a_1 - a_2 - a_5) + a_2 Y_2 (q^2 - Y_1^{-1}) - a_5 Y_5 (Y_1^{-1} - Y_3^{-1})\,, \\
f_2 &= (d - a_2 - 2a_3 - a_5) + a_4 Y_4 (q^2 - Y_3^{-1}) - a_5 Y_5 (Y_3^{-1} - Y_1^{-1})\,, \\
f_3 &= (d - a_1 - a_2 - 2a_5) + a_1 Y_1 (Y_3^{-1} - Y_5^{-1}) + a_2 Y_2 (Y_4^{-1} - Y_5^{-1})\,, \\
f_4 &= (d - a_3 - a_4 - 2a_5) + a_3 Y_3 (Y_1^{-1} - Y_5^{-1}) + a_4 Y_4 (Y_2^{-1} - Y_5^{-1})\,, \\
f_5 &= (d - a_1 - 2a_2 - a_5) + a_1 Y_1 (q^2 - Y_2^{-1}) - a_5 Y_5 (Y_2^{-1} - Y_4^{-1})\,, \\
f_6 &= (d - a_3 - 2a_4 - a_5) + a_3 Y_3 (q^2 - Y_4^{-1}) - a_5 Y_5 (Y_4^{-1} - Y_2^{-1})
\end{aligned}$$

Triangle rule → reduce some index to zero → iterated integration in terms of gamma functions → probably, there is no need to reduce such simple integrals to true master integrals

Let us try to obtain a minimal number of the master integrals.

Construct s-bases corresponding to the sectors $\sigma_{\{1,2,3,4,5\}}$, $\sigma_{\{2,3,4,5\}}$ as well as three more (symmetrical) sectors, the minimal sector $\sigma_{\{1,2,3,4\}}$, the minimal sector $\sigma_{\{2,3,5\}}$ as well as one more (symmetrical) sector.

The degree-lexicographical ordering in the first three cases, and the ordering corresponding to the linear combinations: $a_1 + a_3,\ a_3,\ a_2 + a_4,\ a_2,\ a_5$ in the last case

For example, this is the s-basis associated with $\sigma_{\{1234\}}$:

$$
\begin{aligned}
g_1 \;=\;& 2Y_1Y_2Y_3Y_5^{-1} - a_5Y_1Y_2Y_3Y_5^{-1} - 2Y_1Y_2Y_4Y_5^{-1} \\
&+a_5Y_1Y_2Y_4Y_5^{-1} - 2Y_1Y_3Y_4Y_5^{-1} + a_5Y_1Y_3Y_4Y_5^{-1} \\
&+2Y_2Y_3Y_4Y_5^{-1} - a_5Y_2Y_3Y_4Y_5^{-1} + 2q^2Y_1Y_2Y_3Y_4Y_5^{-1} \\
&-q^2a_5Y_1Y_2Y_3Y_4Y_5^{-1} + 2q^2Y_1Y_2Y_4^2Y_5^{-1} - q^2a_5Y_1Y_2Y_4^2Y_5^{-1} \\
&-2q^2Y_1Y_3Y_4^2Y_5^{-1} + q^2a_5Y_1Y_3Y_4^2Y_5^{-1} - 2q^2Y_2Y_3Y_4^2Y_5^{-1} \\
&+q^2a_5Y_2Y_3Y_4^2Y_5^{-1} - (q^2)^2Y_1Y_2Y_3Y_4^2Y_5^{-1} + d(q^2)^2Y_1Y_2Y_3Y_4^2Y_5^{-1} \\
&-(q^2)^2a_3Y_1Y_2Y_3Y_4^2Y_5^{-1} - (q^2)^2a_4Y_1Y_2Y_3Y_4^2Y_5^{-1} \\
&-2(q^2)^2a_5Y_1Y_2Y_3Y_4^2Y_5^{-1} + (q^2)^2Y_2Y_3^2Y_4^2Y_5^{-1} \\
&+(q^2)^2a_3Y_2Y_3^2Y_4^2Y_5^{-1} + 2(q^2)^2Y_1Y_3Y_4^3Y_5^{-1} \\
&+(q^2)^2a_4Y_1Y_3Y_4^3Y_5^{-1} - Y_1Y_2Y_3^2Y_5^{-2} - a_3Y_1Y_2Y_3^2Y_5^{-2} \\
&+a_3Y_1Y_2Y_3Y_4Y_5^{-2} - a_4Y_1Y_2Y_3Y_4Y_5^{-2} + Y_1Y_2Y_4^2Y_5^{-2} \\
&+a_4Y_1Y_2Y_4^2Y_5^{-2} - 6q^2Y_1Y_2Y_3Y_4^2Y_5^{-2} + 2dq^2Y_1Y_2Y_3Y_4^2Y_5^{-2} \\
&-3q^2a_3Y_1Y_2Y_3Y_4^2Y_5^{-2} - 4q^2a_4Y_1Y_2Y_3Y_4^2Y_5^{-2}
\end{aligned}
$$

$$
\begin{aligned}
& -2q^2a_5Y_1Y_2Y_3Y_4^2Y_5^{-2} - 2q^2Y_1Y_2Y_4^3Y_5^{-2} - q^2a_4Y_1Y_2Y_4^3Y_5^{-2} \ , \\
g_2 \ = \ & Y_1Y_2Y_4^2 - a_5Y_1Y_2Y_4^2 - Y_2Y_3Y_4^2 + a_5Y_2Y_3Y_4^2 \\
& -3Y_1Y_2Y_3Y_4^2Y_5^{-1} + dY_1Y_2Y_3Y_4^2Y_5^{-1} - 2a_3Y_1Y_2Y_3Y_4^2Y_5^{-1} \\
& -a_4Y_1Y_2Y_3Y_4^2Y_5^{-1} - a_5Y_1Y_2Y_3Y_4^2Y_5^{-1} - 2Y_1Y_2Y_4^3Y_5^{-1} \\
& -a_4Y_1Y_2Y_4^3Y_5^{-1} + 2q^2Y_1Y_2Y_3Y_4^3Y_5^{-1} + q^2a_4Y_1Y_2Y_3Y_4^3Y_5^{-1} \ , \\
g_3 \ = \ & -Y_1Y_2Y_3Y_4 + a_5Y_1Y_2Y_3Y_4 + Y_1Y_2Y_4^2 - a_5Y_1Y_2Y_4^2 + Y_1Y_3Y_4^2 \\
& -a_5Y_1Y_3Y_4^2 - Y_2Y_3Y_4^2 + a_5Y_2Y_3Y_4^2 + Y_1Y_2Y_3^2Y_4Y_5^{-1} \\
& +a_3Y_1Y_2Y_3^2Y_4Y_5^{-1} + Y_1Y_2Y_3Y_4^2Y_5^{-1} - a_3Y_1Y_2Y_3Y_4^2Y_5^{-1} \\
& +a_4Y_1Y_2Y_3Y_4^2Y_5^{-1} - q^2Y_1Y_2Y_3^2Y_4^2Y_5^{-1} - q^2a_3Y_1Y_2Y_3^2Y_4^2Y_5^{-1} \\
& -2Y_1Y_2Y_4^3Y_5^{-1} - a_4Y_1Y_2Y_4^3Y_5^{-1} + 2q^2Y_1Y_2Y_3Y_4^3Y_5^{-1} \\
& +q^2a_4Y_1Y_2Y_3Y_4^3Y_5^{-1} \ , \\
g_4 \ = \ & -Y_1Y_2Y_4^2 + a_5Y_1Y_2Y_4^2 + Y_2Y_3Y_4^2 - a_5Y_2Y_3Y_4^2 \\
& -2Y_1Y_2Y_3Y_4^2Y_5^{-1} + dY_1Y_2Y_3Y_4^2Y_5^{-1} - 2a_1Y_1Y_2Y_3Y_4^2Y_5^{-1} \\
& -a_2Y_1Y_2Y_3Y_4^2Y_5^{-1} - a_5Y_1Y_2Y_3Y_4^2Y_5^{-1} - Y_2^2Y_3Y_4^2Y_5^{-1} \\
& -a_2Y_2^2Y_3Y_4^2Y_5^{-1} + q^2Y_1Y_2^2Y_3Y_4^2Y_5^{-1} + q^2a_2Y_1Y_2^2Y_3Y_4^2Y_5^{-1} \ , \\
g_5 \ = \ & Y_1Y_2Y_3Y_4 - a_5Y_1Y_2Y_3Y_4 - Y_1Y_2Y_4^2 + a_5Y_1Y_2Y_4^2 - Y_1Y_3Y_4^2 \\
& +a_5Y_1Y_3Y_4^2 + Y_2Y_3Y_4^2 - a_5Y_2Y_3Y_4^2 + Y_1^2Y_3Y_4^2Y_5^{-1} \\
& +a_1Y_1^2Y_3Y_4^2Y_5^{-1} - a_1Y_1Y_2Y_3Y_4^2Y_5^{-1} + a_2Y_1Y_2Y_3Y_4^2Y_5^{-1} \\
& -q^2Y_1^2Y_2Y_3Y_4^2Y_5^{-1} - q^2a_1Y_1^2Y_2Y_3Y_4^2Y_5^{-1} - Y_2^2Y_3Y_4^2Y_5^{-1} \\
& -a_2Y_2^2Y_3Y_4^2Y_5^{-1} + q^2Y_1Y_2^2Y_3Y_4^2Y_5^{-1} + q^2a_2Y_1Y_2^2Y_3Y_4^2Y_5^{-1} \ , \\
g_6 \ = \ & -q^2Y_1Y_2Y_4^2 + q^2a_5Y_1Y_2Y_4^2 + q^2Y_2Y_3Y_4^2 - q^2a_5Y_2Y_3Y_4^2 \\
& +2Y_1Y_2Y_3Y_4Y_5^{-1} - a_5Y_1Y_2Y_3Y_4Y_5^{-1} - 2Y_1Y_2Y_4^2Y_5^{-1} \\
& +a_5Y_1Y_2Y_4^2Y_5^{-1} - 2Y_1Y_3Y_4^2Y_5^{-1} + a_5Y_1Y_3Y_4^2Y_5^{-1} \\
& +2Y_2Y_3Y_4^2Y_5^{-1} - a_5Y_2Y_3Y_4^2Y_5^{-1} + 2q^2Y_1Y_2Y_3Y_4^2Y_5^{-1}
\end{aligned}
$$

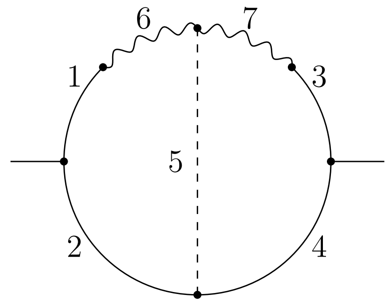
$$
\begin{aligned}
& +q^2 a_3 Y_1 Y_2 Y_3 Y_4^2 Y_5^{-1} - q^2 a_5 Y_1 Y_2 Y_3 Y_4^2 Y_5^{-1} + q^2 Y_2 Y_3^2 Y_4^2 Y_5^{-1} \\
& +q^2 a_3 Y_2 Y_3^2 Y_4^2 Y_5^{-1} + 2q^2 Y_1 Y_2 Y_4^3 Y_5^{-1} + q^2 a_4 Y_1 Y_2 Y_4^3 Y_5^{-1} \\
& +2q^2 Y_1 Y_3 Y_4^3 Y_5^{-1} + q^2 a_4 Y_1 Y_3 Y_4^3 Y_5^{-1} - 2(q^2)^2 Y_1 Y_2 Y_3 Y_4^3 Y_5^{-1} \\
& -(q^2)^2 a_4 Y_1 Y_2 Y_3 Y_4^3 Y_5^{-1} - Y_1 Y_2 Y_3^2 Y_4 Y_5^{-2} - a_3 Y_1 Y_2 Y_3^2 Y_4 Y_5^{-2} \\
& -Y_1 Y_2 Y_3 Y_4^2 Y_5^{-2} + a_3 Y_1 Y_2 Y_3 Y_4^2 Y_5^{-2} - a_4 Y_1 Y_2 Y_3 Y_4^2 Y_5^{-2} \\
& +2 Y_1 Y_2 Y_4^3 Y_5^{-2} + a_4 Y_1 Y_2 Y_4^3 Y_5^{-2} - 4q^2 Y_1 Y_2 Y_3 Y_4^3 Y_5^{-2} \\
& -2q^2 a_4 Y_1 Y_2 Y_3 Y_4^3 Y_5^{-2} \,,
\end{aligned}
$$

$$
\begin{aligned}
g_7 \;=\; & q^2 Y_1 Y_2 Y_4^2 - q^2 a_5 Y_1 Y_2 Y_4^2 - q^2 Y_2 Y_3 Y_4^2 + q^2 a_5 Y_2 Y_3 Y_4^2 \\
& -2 Y_1 Y_2 Y_3 Y_4 Y_5^{-1} + a_5 Y_1 Y_2 Y_3 Y_4 Y_5^{-1} + q^2 Y_1 Y_2^2 Y_3 Y_4 Y_5^{-1} \\
& +q^2 a_2 Y_1 Y_2^2 Y_3 Y_4 Y_5^{-1} + 2 Y_1 Y_2 Y_4^2 Y_5^{-1} - a_5 Y_1 Y_2 Y_4^2 Y_5^{-1} \\
& +q^2 Y_1^2 Y_2 Y_4^2 Y_5^{-1} + q^2 a_1 Y_1^2 Y_2 Y_4^2 Y_5^{-1} + 2 Y_1 Y_3 Y_4^2 Y_5^{-1} \\
& -a_5 Y_1 Y_3 Y_4^2 Y_5^{-1} - 2 Y_2 Y_3 Y_4^2 Y_5^{-1} + a_5 Y_2 Y_3 Y_4^2 Y_5^{-1} \\
& +2q^2 Y_1 Y_2 Y_3 Y_4^2 Y_5^{-1} + q^2 a_1 Y_1 Y_2 Y_3 Y_4^2 Y_5^{-1} - q^2 a_5 Y_1 Y_2 Y_3 Y_4^2 Y_5^{-1} \\
& +q^2 Y_2^2 Y_3 Y_4^2 Y_5^{-1} + q^2 a_2 Y_2^2 Y_3 Y_4^2 Y_5^{-1} - (q^2)^2 Y_1 Y_2^2 Y_3 Y_4^2 Y_5^{-1} \\
& -(q^2)^2 a_2 Y_1 Y_2^2 Y_3 Y_4^2 Y_5^{-1} - Y_1^2 Y_3 Y_4^2 Y_5^{-2} - a_1 Y_1^2 Y_3 Y_4^2 Y_5^{-2} \\
& +a_1 Y_1 Y_2 Y_3 Y_4^2 Y_5^{-2} - a_2 Y_1 Y_2 Y_3 Y_4^2 Y_5^{-2} + Y_2^2 Y_3 Y_4^2 Y_5^{-2} \\
& +a_2 Y_2^2 Y_3 Y_4^2 Y_5^{-2} - 2q^2 Y_1 Y_2^2 Y_3 Y_4^2 Y_5^{-2} - 2q^2 a_2 Y_1 Y_2^2 Y_3 Y_4^2 Y_5^{-2} \,.
\end{aligned}
$$

Three master integrals:

$F(1,1,1,1,0)$, and $F(0,1,1,0,1) = F(1,0,0,1,1)$

Example 3. Two-loop Feynman integrals for the heavy quark static potential



n_f contributions to the three-loop quark static potential $\rightarrow$ two-loop diagrams with the index of the central line $a_5 + \epsilon$.

$$F(a_1, a_2, a_3, a_4, a_5, a_6, a_7) = \int\int \frac{\mathrm{d}^d k \, \mathrm{d}^d l}{(-k^2)^{a_1}(-l^2)^{a_2}[-(k-q)^2]^{a_3}} \times \frac{1}{[-(l-q)^2]^{a_4}[-(k-l)^2]^{a_5+\epsilon}(-v\cdot k)^{a_6}(-v\cdot l)^{a_7}} .$$

with $v \cdot q = 0$

Boundary conditions:

$F(a_1, \ldots, a_7) = 0$if $a_1, a_3 \leq 0$, or $a_2, a_4 \leq 0$, or $a_1, a_2, a_6 \leq 0$, or $a_3, a_4, a_7 \leq 0$

Symmetry:

$$\begin{aligned} F(a_1, a_2, a_3, a_4, a_5, a_6, a_7) &= F(a_2, a_1, a_4, a_3, a_5, a_6, a_7) \\ &= F(a_3, a_4, a_1, a_2, a_5, a_7, a_6) \end{aligned}$$

Let a_5 be not shifted by ϵ

[M. Peter, Phys. Rev. Lett. **78** (1997) 602; Nucl. Phys. B **501** (1997) 471; Y. Schröder, Phys. Lett. B **447** (1999) 321; B.A. Kniehl, A.A. Penin, V.A. Smirnov, and M. Steinhauser, Phys. Rev. D **65** (2002) 091503]

IBP $\rightarrow$

$$
\begin{aligned}
f_1 &= d - a_{11256} - a_2 Y_2 (q^2 + Y_1^{-1}) - a_5 Y_5 (Y_1^{-1} - Y_3^{-1})\,, \\
f_2 &= d - a_{23357} - a_4 Y_4 (q^2 + Y_3^{-1}) - a_5 Y_5 (Y_3^{-1} - Y_1^{-1})\,, \\
f_3 &= d - a_{12556} + a_1 Y_1 (Y_3^{-1} - Y_5^{-1}) + a_2 Y_2 (Y_4^{-1} - Y_5^{-1}) + a_6 Y_6 Y_7^{-1}\,, \\
f_4 &= d - a_{34557} + a_3 Y_3 (Y_1^{-1} - Y_5^{-1}) + a_4 Y_4 (Y_2^{-1} - Y_5^{-1}) + a_7 Y_6^{-1} Y_7\,, \\
f_5 &= d - a_{12256} - a_1 Y_1 (q^2 + Y_2^{-1}) - a_5 Y_5 (Y_2^{-1} - Y_4^{-1})\,, \\
f_6 &= d - a_{34457} - a_3 Y_3 (q^2 + Y_4^{-1}) - a_5 Y_5 (Y_4^{-1} - Y_2^{-1})\,, \\
f_7 &= 2a_1 Y_1 Y_6^{-1} + 2a_2 Y_2 Y_6^{-1} + a_5 Y_5 (Y_6^{-1} - Y_7^{-1}) - v^2 a_6 Y_6\,, \\
f_8 &= 2a_3 Y_3 Y_7^{-1} + 2a_4 Y_4 Y_7^{-1} - a_5 Y_5 (Y_6^{-1} - Y_7^{-1}) - v^2 a_7 Y_7\,,
\end{aligned}
$$

where $a_{11256} = 2a_1 + a_2 + a_5 + a_6$ etc.

For Example 3, just make the replacement $a_5 \rightarrow a_5 + \epsilon$.

Construct s-bases corresponding to

$\sigma_{\{1,2,3,4,5,6,7\}}$, $\sigma_{\{2,3,4,5,6,7\}}$, $\sigma_{\{1,2,3,4,5,7\}}$, $\sigma_{\{3,4,5,6,7\}}$, $\sigma_{\{2,3,5,6,7\}}$, $\sigma_{\{2,3,4,5,7\}}$, $\sigma_{\{2,3,4,5,6\}}$, $\sigma_{\{1,2,3,4,5\}}$, $\sigma_{\{2,3,4,5\}}$, $\sigma_{\{2,3,5,6\}}$.

Master integrals:

$I_1 = F(1,1,1,1,0,1,1), I_{21} = F(1,1,1,1,0,0,1)$,
$I_{22} = F(1,1,1,1,0,1,0), I_3 = F(1,1,1,1,0,0,0)\,;$
$(I_{21} = I_{12} = I_2$ because of the symmetry);
$I_{51} = F(1,0,0,1,1,1,1), I_{71} = F(1,0,0,1,1,0,1)$,
$I_{81} = F(1,0,0,1,1,1,0), I_{41} = F(1,0,0,1,1,0,0)\,;$
$(I_{71} = I_{81} = I_7)$;
$I_{52}, I_{72}, I_{82}, I_{42}$, obtained by $1 \leftrightarrow 2, 3 \leftrightarrow 4$;

$I_{61} = F(0,0,1,1,1,1,0), \bar{I}_{61} = F(0,0,1,1,1,2,0)$ and the corresponding symmetrical family.

$$
\begin{aligned}
F(a_1,\ldots,a_7) \;=\; & \frac{\left(\mathrm{i}\pi^{d/2}\right)^2 2^{a_7-1}(v^2)^{-(a_{67}/2}(Q^2)^{4-a_{12345}-a_{67}/2-2\epsilon}}{\prod_{l=3,4,5,7}\Gamma(a_l)\Gamma(4-a_{3457}-2\epsilon)} \\
\times & \frac{1}{(2\pi\mathrm{i})^3}\int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty}\ldots\int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty}\mathrm{d}z_1\mathrm{d}z_2\mathrm{d}z_3\frac{\Gamma(a_{12345}+a_{67}/2+2\epsilon-4+z_3)}{\Gamma(a_1-z_1)\Gamma(a_2-z_2)} \\
\times & \frac{\Gamma(a_{345}+a_7/2+\epsilon-2+z_1+z_2+z_3)\Gamma(-z_1)\Gamma(-z_2)\Gamma(-z_3)}{\Gamma(a_{345}+a_{67}/2+\epsilon-3/2+z_1+z_2+z_3)} \\
\times & \frac{\Gamma(a_3+z_1+z_3)\Gamma(a_4+z_2+z_3)\Gamma(2-a_{345}-\epsilon-z_1-z_2-z_3)}{\Gamma(8-a_{1267}-2a_{345}-4\epsilon-z_1-z_2-2z_3)} \\
\times & \Gamma(a_{345}+a_7/2+\epsilon-3/2+z_1+z_2+z_3) \\
\times & \Gamma(4-a_{1345}-a_{67}/2-2\epsilon-z_2-z_3) \\
\times & \Gamma(4-a_{2345}-a_{67}/2-2\epsilon-z_1-z_3) \\
\times & \Gamma(4-2a_{34}-a_{57}-2\epsilon-z_1-z_2-2z_3)\;,
\end{aligned}
$$

where $a_{12345} = a_1 + a_2 + a_3 + a_4 + a_5$ etc.

Master integrals:

$$
\begin{aligned}
I_1 &= \frac{\left(i\pi^{d/2}e^{-\gamma_{\mathrm{E}}\epsilon}\right)^2}{Q^{4+6\epsilon}v^2}\left[-\frac{8\pi^2}{9\epsilon}-\frac{16\pi^2}{9}+\frac{40\zeta(3)}{3}+O(\epsilon)\right]\;, \\
I_2 &= \frac{\left(i\pi^{d/2}e^{-\gamma_{\mathrm{E}}\epsilon}\right)^2}{Q^{3+6\epsilon}v}\left[\frac{\pi^4}{3}+O(\epsilon)\right]\;, \\
I_3 &= \frac{\left(i\pi^{d/2}e^{-\gamma_{\mathrm{E}}\epsilon}\right)^2}{Q^{2+6\epsilon}}\left[6\zeta(3)+\left(\frac{\pi^4}{10}+12\zeta(3)\right)\epsilon+O(\epsilon^2)\right]\;, \\
I_4 &= \left(i\pi^{d/2}\right)^2 Q^{2-6\epsilon}\frac{\Gamma\left(1-2\epsilon\right)\Gamma\left(1-\epsilon\right)^2\Gamma\left(3\epsilon-1\right)}{\Gamma\left(3-4\epsilon\right)\Gamma\left(1+\epsilon\right)}\;, \\
I_5 &= \frac{\left(i\pi^{d/2}e^{-\gamma_{\mathrm{E}}\epsilon}\right)^2}{Q^{6\epsilon-2}v^2}\left[\frac{4\pi^2}{9\epsilon}+\frac{32\pi^2}{9}-\frac{8\zeta(3)}{3}+O(\epsilon)\right]\;,
\end{aligned}
$$

$$
\begin{aligned}
I_6 &= \left(i\pi^{d/2}\right)^2 \frac{\sqrt{\pi}\Gamma\left(3/2-3\epsilon\right)^2\Gamma\left(1-2\epsilon\right)\Gamma\left(3\epsilon-1/2\right)\Gamma\left(4\epsilon-1\right)}{4^{2\epsilon-1}\Gamma\left(3-6\epsilon\right)\Gamma\left(2\epsilon\right)\Gamma\left(1+\epsilon\right)Q^{6\epsilon-1}v} \, , \\
\bar{I}_6 &= \left(i\pi^{d/2}\right)^2 \frac{4^{1-2\epsilon}\sqrt{\pi}\Gamma\left(1-3\epsilon\right)^2\Gamma\left(1-2\epsilon\right)\Gamma\left(3\epsilon\right)\Gamma\left(4\epsilon\right)}{\Gamma\left(2-6\epsilon\right)\Gamma\left(1+\epsilon\right)\Gamma\left(1/2+2\epsilon\right)Q^{6\epsilon-2}v^2} \, , \\
I_7 &= \left(i\pi^{d/2}\right)^2 \frac{\sqrt{\pi}\Gamma\left(3/2-3\epsilon\right)\Gamma\left(1-2\epsilon\right)\Gamma\left(1/2-\epsilon\right)\Gamma\left(1-\epsilon\right)}{Q^{6\epsilon-1}v\Gamma\left(2-4\epsilon\right)\Gamma\left(2-3\epsilon\right)\Gamma\left(1+\epsilon\right)} \, , \\
&\quad \times\Gamma\left(3\epsilon-1/2\right) \, , \\
I_8 &= I_7 \, ,
\end{aligned}
$$

where $Q=\sqrt{-q^2}$ and $v=\sqrt{v^2}$.

For example, we obtain

$$
\begin{aligned}
F(1,1,1,1,1,1,-1) &= -\frac{2Q^2v^2}{(3d-10)}\bar{I}_2 - 3I_3 \\
&-\frac{8(d-3)(2d-7)(11d-46)}{(d-4)^2(3d-14)Q^4}I_4 + \frac{4(3d-11)(7d-30)v^2}{(d-4)(3d-14)(3d-10)Q^2}\bar{I}_6 \, , \\
F(2,1,1,1,1,1,1) &= -\frac{3d-14}{2Q^2}I_1 \\
&-\frac{4(d-3)(d-2)(2d-7)(3d-10)(9d-40)}{(d-5)(d-4)(2d-11)(3d-16)(3d-14)Q^8v^2}I_4 \\
&-\frac{3(d-4)(4d-17)(4d-15)}{(2(d-5)(2d-11)Q^6}I_5 \\
&-\frac{16(3d-13)(3d-11)}{(2d-11)(3d-16)(3d-14)Q^6}\bar{I}_6
\end{aligned}
$$

The status and perspectives

- Our generalization of the Buchberger algorithm works at the level of modern calculations.
- Computer time: seconds for Example 2 and minutes for Example 3 on slow computers.
- There are interesting mathematical problems.
- Further improvements are necessary for more sophisticated calculations.
- Combining our algorithm with other ideas. Janet basis? [Gerdt et al.]

to be continued