Apparent Horizons Associated with Dynamical Black Hole Entropy

P3

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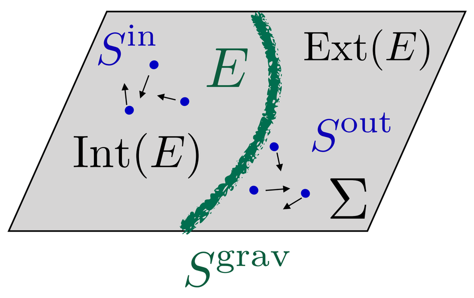
Abstract

We **define entropic marginally outer trapped surfaces (E-MOTSs)** as a **generalization of apparent horizons.** We show that **the dynamical black hole (BH) entropy** proposed by Hollands, Wald, and Zhang(HWZ), on a background Killing horizon **is coincide with the Wall entropy on an E-MOTS** associated with it, in the first-order perturbations around a stationary BH. Our discussion is **applicable in any diffeomorphism invariant theory of gravity.**

Entropies of Spacetime

Generalized entropy with entangling surface

▷ **Generalized entropy have been generalized!**



BH ⟶ (Q.)Extremal surface ⟶ Entangling surface

$$S^{\rm gen}(E) = S^{\rm grav}(E) + S^{\rm out}(E) + \text{counterterms}$$

E : Codim.-2 spatial surface dividing a Cauchy surface into two pieces

Leads to **Q. Focusing Conjecture, Q. Null Energy Condition, ・・・**

Bousso-Fisher-Leichenauer-Wall '15 , ⋯

Now, **what should be the gravitational part** $S^{\rm grav}$ **?**

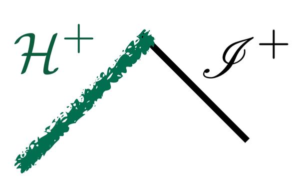
Bekenstein-Hawking entropy

Bekenstein '73
Hawking '75

BH entropy as area of event horizon cross-section

$$S_{\rm BH} := \frac{k_B}{4G\hbar} A_{\rm event}$$

$\mathcal{H}^+$: Boundary of causal past of future null infinity $\mathscr{I}^+$



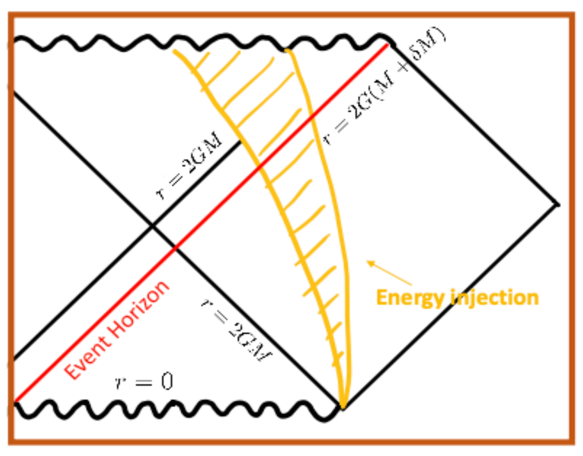
Based on thermodynamical properties :

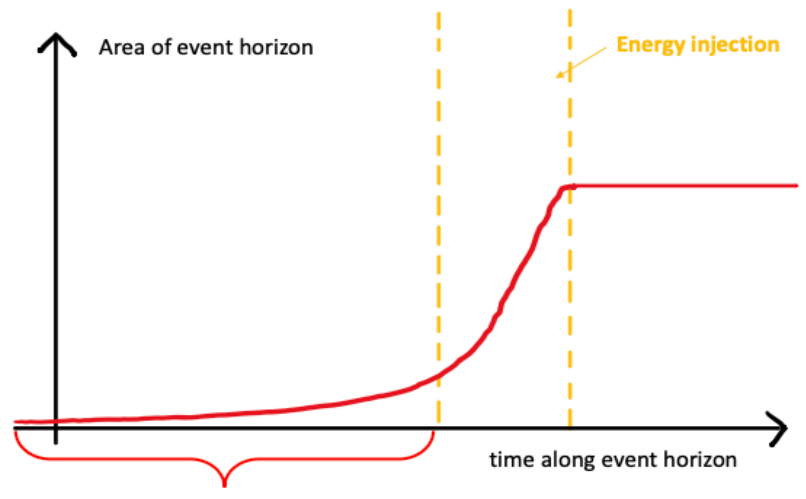
- **The four laws of BH thermodynamics** Bardeen-Carter-Hawking '73 , Iyer-Wald '94
- **Hawking radiation with thermal spectrum** Hawking '75

⟶ **Equilibrium system of more fundamental d.o.f.s of gravity?**

▷ **Causal definition causes peculiar behavior in dynamical situation**

Transition from Schwartzschild BH with mass M to $M+\delta M$





Entropy increase BEFORE matter falls into the BH !

⟶ **Is it really corresponds to microscopic equilibrium system??**

⟶ **Need to entropy properly respond to physical operations**

It may leads to :
- Where the fundamental d.o.f.s lived in ?
- Non-equilibrium aspects of BH thermodynamics ?

Hollands-Wald-Zhang entropy

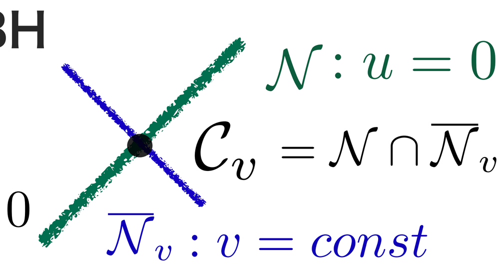
HWZ '24
Visser-Yan '24

▷ **Non-stationary perturbation around stationary BH**

(Affine) Gaussian Null Coordinate $x^\mu = (u, v, \varphi^A)$

$k^a = (\partial_v)^a,\ \ell^a = (\partial_u)^a$: affine null vector fields on $\mathcal{N}: u=0$

$g_{\mu\nu}(\epsilon) = \mathring{g}_{\mu\nu} + \epsilon\delta g_{\mu\nu} + \mathcal{O}(\epsilon^2)$ $\quad \mathring{g}_{\mu\nu}$: background stationary BH metric



Identification for perturbation : $\mathcal{N} \equiv \mathcal{H}^+$: background Killing horizon

In this poster, we **only consider first-order analysis**

▷ **Noether charge approach of covariant phase space formalism**

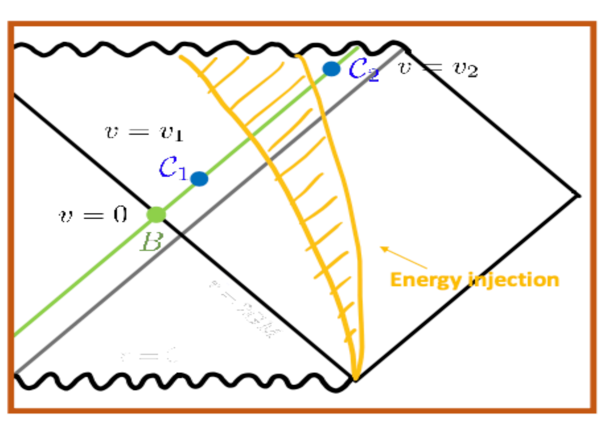
⟶ Applicable to any theory with diffeomorphism invariant Lagrangian

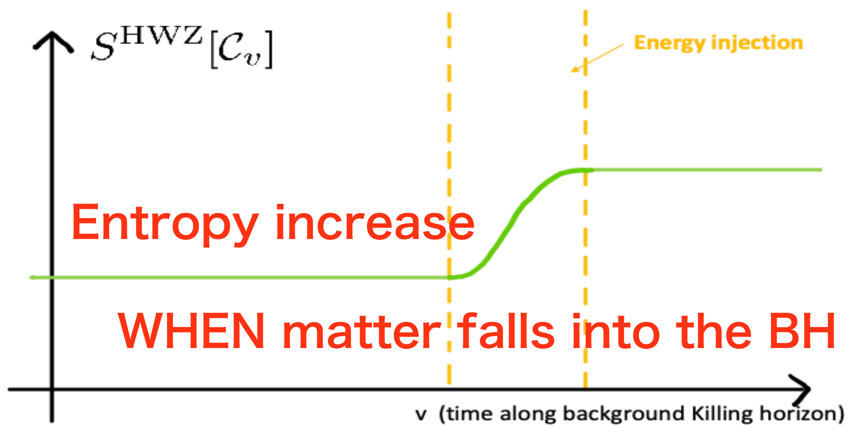
$$s^{\rm HWZ} := \frac{2\pi}{\kappa\hbar}\left[\boldsymbol{Q}_\xi - \iota_\xi \boldsymbol{B}_{\mathcal{H}^+}\right]$$

$\boldsymbol{Q}_\xi$: Noether charge codim2-form , ι : interior product

$\boldsymbol{B}_{\mathcal{H}^+}$: Boundary correction from symplectic potential

HWZ entropy $\quad S_{\rm HWZ}(\mathcal{C}_v) := \int_{\mathcal{C}_v} s^{\rm HWZ} =: \int s^{\rm HWZ}[\mathcal{C}_v] d^{D-2}\varphi$





▷ **HWZ entropy in Einstein-Hilbert action**

$$S_{\rm HWZ}(\mathcal{C}_v) = S_{\rm Area}(\mathcal{C}_v) - \frac{1}{4G\hbar}\int_{\mathcal{C}_v} \sqrt{\gamma}\, v\vartheta_+ d^{D-2}\varphi$$

$\vartheta_+ := \nabla_\mu n_+^\mu$: expansion

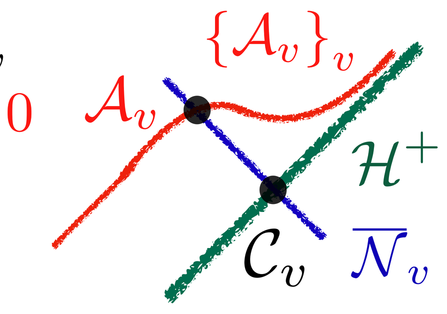
n_+^μ : Outward null normal on $\mathcal{C}_v$

dynamical correction

It satisfies $S_{\rm HWZ}(\mathcal{C}_v) = S_{\rm Area}(\mathcal{A}_v)$, $\mathcal{A}_v$: MOTS on $\overline{\mathcal{N}}_v$

Surface where $\vartheta_+ = 0$

How about in general?



Our Work

▷ **Our statement:** $S_{\rm HWZ}(\mathcal{C}_v) = S_{\rm Wall}(\mathcal{T}_v)$ **E-MOTS** : generalization of MOTS

▷ **Wall entropy** $\quad \partial_v^2 \delta S_{\rm Wall}(\mathcal{C}_v) \equiv -\frac{2\pi}{\hbar}\int \boldsymbol{\varepsilon}^{\mathcal{C}_v}\delta T_{ab}k^a k^b$, $\quad \boldsymbol{\varepsilon}^{\mathcal{C}_v} \overset{\mathcal{C}_v}{=} \iota_k \iota_\ell \boldsymbol{\varepsilon}$ $\quad$ Wall '15

Wall entropy coincide with (linearized) Dong entropy in $f(\text{Riemann})$ theories

Holographic entanglement entropy (HEE) with curvature correction $\quad$ Dong '13

▷ **(Outward/Inward) entropic expansion** $\Theta_\pm$ associated with $S(\mathcal{C})$

Assume there exist entropy density $s[\mathcal{C}]$ s.t. $\quad S(\mathcal{C}) = \int_{\mathcal{C}} \boldsymbol{s} = \int s[\mathcal{C}] d^{D-2}\varphi$

$$\Theta_\pm := \frac{1}{s}\partial_\mu(s n_\pm^\mu) = \vartheta_\pm + n_\pm^\mu \nabla_\mu \log(s[\mathcal{C}]/\sqrt{\gamma}[\mathcal{C}])$$, Then $\mathcal{C}$ is :

- **Entropic Trapped Surface** if $\Theta_+\Theta_- > 0$
- **Entropic Marginally Outer Trapped Surface (E-MOTS)** if $\Theta_+ = 0$

(Future/Past) indicates $\Theta_- < 0 \,/\, \Theta_- > 0$, resp. $\quad$ Terminologies: Hayward '94 ⋯

▷ **Outline of a proof of the statement**

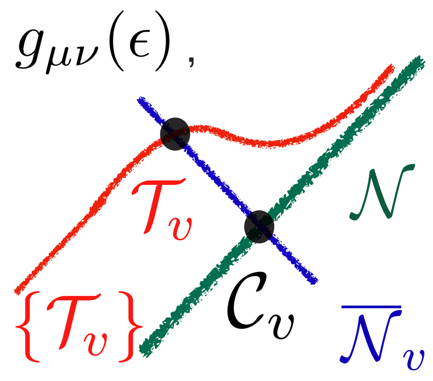
Lemma $\quad$ General statement on E-MOTS associated with an $S(\epsilon;\mathcal{C})$

Note: Assume $\boldsymbol{s}$ is defined on $\mathcal{C}$ near $\mathcal{N}$ below

Let $\mathcal{N}$ be a bg. future or past E-MOTT, foliated by bg. E-MOTSs $\{\mathcal{C}_v\}_v$

Then, for a given v, **there exists an E-MOTS** $\mathcal{T}_v$ on $\overline{\mathcal{N}}_v$ for $g_{\mu\nu}(\epsilon)$,

and the entropy satisfies $\quad S(\epsilon;\mathcal{T}_v) = (1 - v\partial_v)S(\epsilon;\mathcal{C}_v)$.



Remarks:
HWZ, Visser-Yan proved $S_{\rm HWZ} = (1 - v\partial_v)S_{\rm Wall}$
HWZ proposed $\boldsymbol{s}^{\rm Wall}$ on $\mathcal{C}_v$ ⟶ Need extension

We provide a possible extension of $\boldsymbol{s}^{\rm Wall}$ away from $\mathcal{N}$,
under the assumption there exists an E-MOTS near $\mathcal{N}$

⟶ **We can apply lemma and prove the statement** □

▷ **Summary**

- HWZ proposed BH entropy applicable to dynamical situations
  Satisfy physical process version of BH thermodynamic laws
- HWZ entropy coincide with Wall entropy on E-MOTS
  Surface where outer E-expansion $\Theta_+ = \vartheta_+ + n_+^\mu\nabla_\mu \log(s^{\rm Wall}/s^{\rm BH}) = 0$

▷ **Discussion**

- We did not prove existence of E-MOTS associated with $S_{\rm Wall}$
  It may relate to covariant definition of $\boldsymbol{s}^{\rm Wall}$ away from horizon
  Property of (generalized) stability operator? Andersson-Mars-Simon '05
- Extension of $\boldsymbol{s}^{\rm Wall}$ is not unique (though statement hold in any case)
  Is there some preferable extension?
  Extension determines the value of Wall entropy and location of E-MOTSs
- Implications for **holography** and discussion w/ **Q-expansion**? Bousso '25
  E-MOTS connects BH entropy and HEE e.g., Semiclassical singularity theorem using Q-trapped surfaces

Cf. **Quantum expansion :** $\quad \Theta^{\rm BFLW}(\varphi^A) := \frac{4G\hbar}{\sqrt{\gamma_c}}\frac{\delta}{\delta V(\varphi^A)}S_{\rm gen}[\mathcal{C}] \overset{\hbar\to 0}{\approx} (s^{\rm Wall}/\sqrt{\gamma_c})\Theta_+$

Bousso-Fisher-Leichenauer-Wall '15 , ⋯

$\Rightarrow$ **Q-screen :** $\left\{\tilde{\mathcal{T}} \in \mathcal{M}, \text{where } \Theta^{\rm BFLW}\Big|_{\tilde{\mathcal{T}}} = 0\right\} \overset{\hbar\to 0}{\approx} \{\mathcal{T}_v\}_v$ **: E-MOTT**

Bousso-Engelhardt '15

- Second-order analysis? $\quad$ Similar entropy relation still hold? Irreversible thermodynamics analog? ⟵ **Upcoming!**