Time as a Measurement-Induced Operator:
The Lee–Tsutsui Formalism with a Gaussian Wave-Packet Basis

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QR code for position momentum paper →

Message

Usually, time is **external parameter**, not self-adjoint operator, in quantum mechanics
In **Lee-Tsutsui formalism**, we can construct quantum operators from classical functions
Time-dependent bases provide POVMs: point/Gaussian detector
Commutator between **pullback time operator** and free Hamiltonian is **very analogous to CCR**
Non-existence of partial inverse of pullback of free Hamiltonian; **Lee error diverges**

1. Problem setting

- Usually, time is **parameter**, not operator in QM
- **LT formalism**: constructing quantum operators by **pullback**
  - Already applied to position and momentum with Gaussian wavepacket basis → How about time?
- **How about commutator with free Hamiltonian?**
- How about error of free Hamiltonian under considered situation?

2. Time in QM

Usually, time is parameter, not operator in QM

Pauli's theorem: self-adjoint operators canonically-conjugate with Hamiltonian bounded from below do not exist

Pauli 1933

Time-energy uncertainty relation:

$$\Delta E \Delta t \gtrsim h$$

Time is not self-adjoint operator, so this has become controversial issue (not Robertson-type inequality?)

Attempts to formulate time as operator:

Aharonov-Bohm time operator:

$$\widehat{T_{\mathrm{AB}}} := \frac{m}{2}\left(\widehat{p_x}^{-1}\hat{x} + \hat{x}\widehat{p_x}^{-1}\right)$$

CCR

$$\left[\widehat{T_{\mathrm{AB}}}, \widehat{H_{\mathrm{free}}}\right] = -i\hat{1}$$

$$\widehat{H_{\mathrm{free}}} := \frac{\widehat{p_x}^2}{2m}$$

※ $\widehat{T_{\mathrm{AB}}}$ is **symmetric operator**, not self-adjoint operator
※ Physical meaning of $\widehat{T_{\mathrm{AB}}}$ is viewed as arrival time of particle

Aharonov and Bohm 1961

3. Lee-Tsutsui formalism

	expectation value	seminorm	semi-inner product
classical	$\langle \mathrm{f}\rangle_{\mathrm{p}} := \int_\Omega d\omega\, \mathrm{f}(\omega)\mathrm{p}(\omega)$	$\lVert \mathrm{f}\rVert_p^2 := \langle \mathrm{f}^{\,2}\rangle_{\mathrm{p}}$	$\langle \mathrm{f},\mathrm{g}\rangle_p := \langle \mathrm{fg}\rangle_{\mathrm{p}}$
quantum	$\langle \hat{A}\rangle_{\hat{\rho}} := \mathrm{Tr}[\hat{A}\hat{\rho}]$	$\lVert \hat{A}\rVert_{\hat{\rho}}^2 := \langle \hat{A}^2\rangle_{\hat{\rho}}$	$\langle \hat{A},\hat{B}\rangle_{\hat{\rho}} := \left\langle \frac{\{\hat{A},\hat{B}\}}{2}\right\rangle_{\hat{\rho}}$

LT adjoint (pullback):

$$\left\langle \widehat{M^{\star}\mathrm{f}}\right\rangle_{\hat{\rho}} = \langle \mathrm{f}\rangle_{M\hat{\rho}}$$

pushforward:

$$\left\langle \hat{A}, \widehat{M^{\star}\mathrm{f}}\right\rangle_{\hat{\rho}} = \left\langle M_{\star}\hat{A}, \mathrm{f}\right\rangle_{M\hat{\rho}}$$

LT error, Lee error (clear operational meaning):

$$\varepsilon_{\hat{\rho}}[\hat{A};M] := \sqrt{\underbrace{\lVert\hat{A}\rVert_{\hat{\rho}}^2}_{\text{pre-measurement}} - \underbrace{\lVert M_{\star}\hat{A}\rVert_{M\hat{\rho}}^2}_{\text{post-measurement}}}$$

$$\tilde{\varepsilon}_{\hat{\rho}}[\hat{A};M] := \sqrt{\underbrace{\lVert M^{\star-1}\hat{A}\rVert_{M\hat{\rho}}^2}_{\text{post-measurement}} - \underbrace{\lVert\hat{A}\rVert_{\hat{\rho}}^2}_{\text{pre-measurement}}}$$

cf) **Ozawa error**

$$\varepsilon^{O}[\hat{A}] := \sqrt{\left\langle\left(\hat{\mathbb{A}} - \hat{A}\otimes\hat{1}\right)^2\right\rangle}$$

$$\tilde{\varepsilon}[\hat{A}] \geq \varepsilon[\hat{A}]$$

$$\tilde{\varepsilon}[\hat{A}] = +\infty \left(\hat{A} \notin \mathrm{ran}M^{\star}\right)$$

LT inequality, Lee inequality (some sorts of CS inequalities):

$$\varepsilon[\hat{A}]\varepsilon[\hat{B}] \geq \sqrt{\mathcal{J}_{\hat{\rho}}^2[\hat{A},\hat{B}] + \mathcal{R}_{\hat{\rho}}^2[\hat{A},\hat{B}]}$$

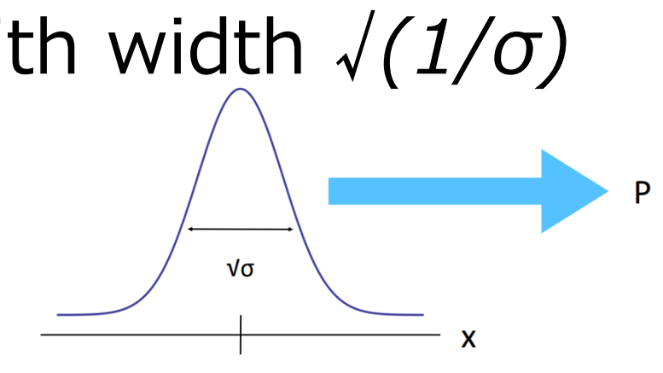
$$\tilde{\varepsilon}[\hat{A}]\tilde{\varepsilon}[\hat{B}] \geq \sqrt{\mathcal{J}_{0\hat{\rho}}^2[\hat{A},\hat{B}] + \tilde{\mathcal{R}}_{\hat{\rho}}^2[\hat{A},\hat{B}]}$$

Lee and Tsutsui 2020; Lee 2022

4. Gaussian wavepacket basis

momentum rep.: localizing around center position X_i with width $\surd\sigma$, around center momentum P_i with width $\surd(1/\sigma)$

$$\langle \boldsymbol{p}|\boldsymbol{X},\boldsymbol{P};\sigma\rangle := \left(\frac{\sigma}{\pi}\right)^{\frac{d}{4}} \exp\left(-i\boldsymbol{p}\cdot\boldsymbol{X} - \frac{\sigma}{2}(\boldsymbol{p}-\boldsymbol{P})^2\right)$$



non-orthogonality normalizability

$$\langle \boldsymbol{X},\boldsymbol{P};\sigma|\boldsymbol{X}',\boldsymbol{P}';\sigma\rangle = \exp\left(i\frac{\boldsymbol{P}+\boldsymbol{P}'}{2}\cdot(\boldsymbol{X}-\boldsymbol{X}') - \frac{(\boldsymbol{X}-\boldsymbol{X}')^2}{4\sigma} - \frac{\sigma}{4}(\boldsymbol{P}-\boldsymbol{P}')^2\right)$$

normalizable, and hence being element of Hilbert space

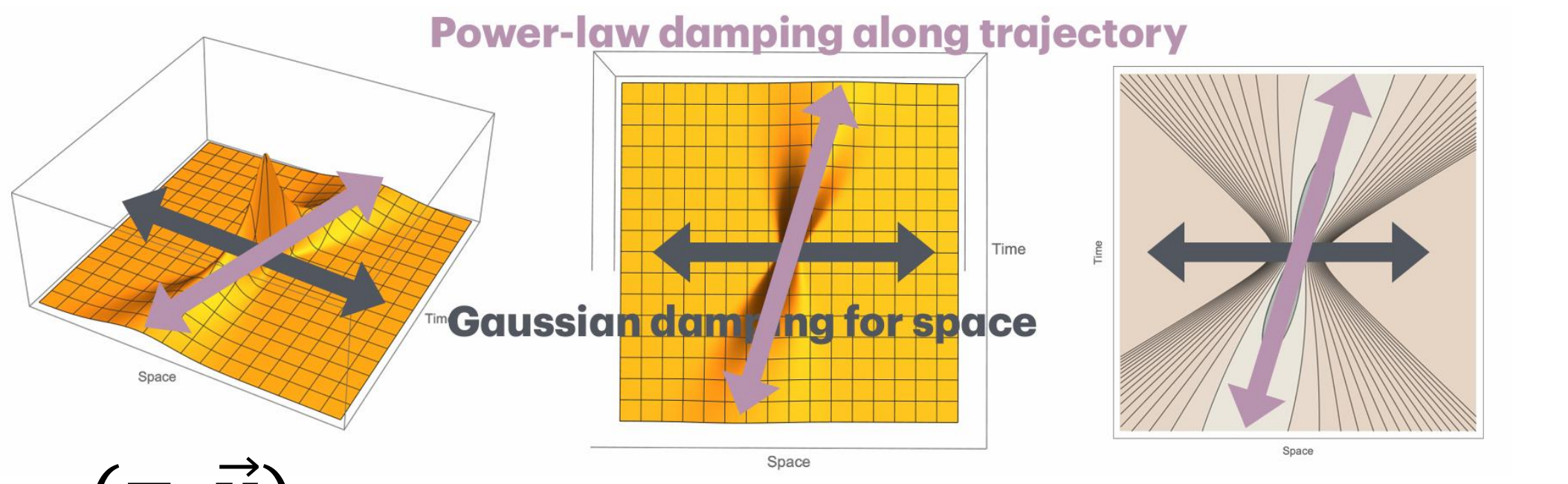
GWB gives POVM:

positivity + completeness

$$|\boldsymbol{X},\boldsymbol{P};\sigma\rangle\langle\boldsymbol{X},\boldsymbol{P};\sigma|/(2\pi)^d \geq 0\ ,\ \int_{\mathbb{R}^{2d}} \mathrm{d}^d\boldsymbol{X}\mathrm{d}^d\boldsymbol{P}\frac{|\boldsymbol{X},\boldsymbol{P};\sigma\rangle\langle\boldsymbol{X},\boldsymbol{P};\sigma|}{(2\pi)^d} = \hat{1}$$

Time-shifted GWB

Localizing around $\boldsymbol{X}(t) := \boldsymbol{X} + \frac{\boldsymbol{P}}{m}(t-T)$ with width $\sigma + \frac{(t-T)^2}{\sigma m^2}$



momentum rep.: $\vec{X} := (T, \vec{X})$

$$\langle \boldsymbol{p}|\vec{X},\boldsymbol{P};\sigma\rangle := \left(\frac{\sigma}{\pi}\right)^{\frac{d}{4}} \exp\left(\mathrm{i}\mathrm{E}(\boldsymbol{p})T - \mathrm{i}\boldsymbol{p}\cdot\boldsymbol{X} - \frac{\sigma}{2}(\boldsymbol{p}-\boldsymbol{P})^2\right)$$

5. Position and momentum

initial state: $\hat{\rho}_{\mathrm{in}} := |X_{\mathrm{in}},P_{\mathrm{in}};\sigma_{\mathrm{in}}\rangle\langle X_{\mathrm{in}},P_{\mathrm{in}};\sigma_{\mathrm{in}}|$
LT inequality: RHS is 0 (trivial)
Lee inequality: RHS is ½, always saturated

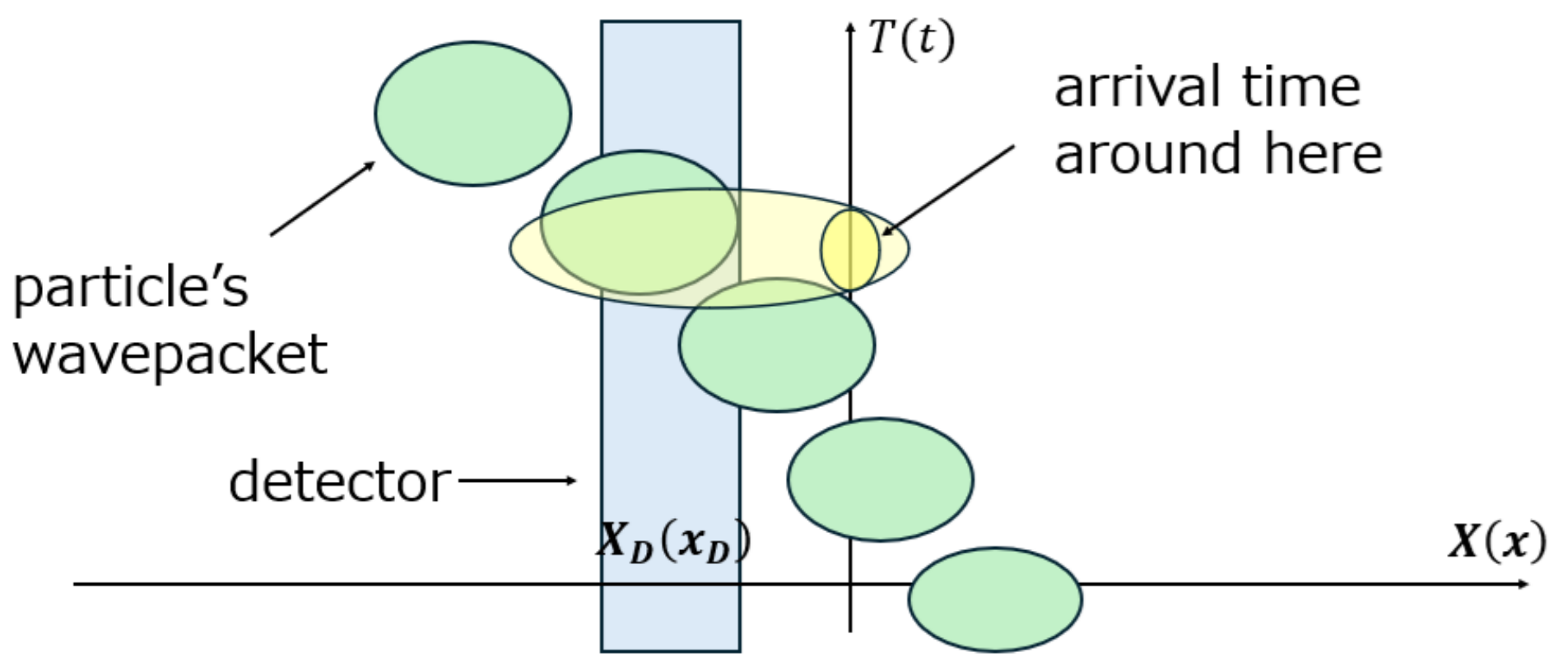
$$\left(\varepsilon[\hat{x}]\varepsilon[\widehat{p_x}] =\right)\frac{1}{2}\sqrt{\frac{\sigma_{\mathrm{red}}}{\sigma_{\mathrm{sum}}}} \geq 0$$

$$\left(\tilde{\varepsilon}[\hat{x}]\,\tilde{\varepsilon}[\hat{p}] =\right)\frac{1}{2} \geq \frac{1}{2}$$

Oda and Ogawa 2024

6. Method and Results

outcome space: $\Omega = \mathbb{R} \sqcup \{ND\}$
ND: no-detection
finite interval: $\mathcal{J} = [T_{\mathrm{i}}, T_{\mathrm{f}}]$
γ: coupling constant dependent on details of detector
differs by detector
mass dimension: $[\gamma] = -(d-1)$



point detector M_x (POVM):

$$\widehat{\Pi_{x_D}}(t) = \gamma|\vec{x}\rangle\langle\vec{x}| \qquad \widehat{E_{x_D}}(\mathrm{ND}) = \hat{1} - \int_{\mathbb{R}} \mathrm{d}t\,\widehat{\Pi_{x_D}}(t) \qquad \widehat{E_{x_D}}(\mathcal{J}) := \int_{\mathcal{J}} \mathrm{d}t\,\widehat{\Pi_{x_D}}(t)$$

Gaussian detector M_{X_D,σ_D}(POVM):

$$\widehat{\Pi_{X_D,\sigma_D}}(T_D) = \gamma\int_{\mathbb{R}^d} d^d\boldsymbol{P}\frac{|\vec{X_D},\boldsymbol{P};\sigma_D\rangle\langle\vec{X_D},\boldsymbol{P};\sigma_D|}{(2\pi)^d} \qquad \widehat{E_{X_D,\sigma_D}}(ND) = \hat{1} - \int_{\mathbb{R}} dT_D\,\widehat{\Pi_{X_D,\sigma_D}}(T_D) \qquad \widehat{E_{X_D,\sigma_D}}(\mathcal{J}) = \gamma\int_{\mathcal{J}} \mathrm{d}T_D\int_{\mathbb{R}^d} \mathrm{d}^d\boldsymbol{P}\frac{|\vec{X_D},\boldsymbol{P};\sigma_D\rangle\langle\vec{X_D},\boldsymbol{P};\sigma_D|}{(2\pi)^d}$$

admissible subspace $\mathcal{H}_{\mathrm{adm}}$: restricted subspace where $\hat{E}(\mathbb{R}) \leq \hat{1}$ holds

$$\chi_{\mathbb{R}}(t) = 1,\ \chi_{\mathbb{R}}(ND) = 0\ ;\ \mathrm{id}_{\mathbb{R}}(t) = t,\ \mathrm{id}_{\mathbb{R}}(ND) = ND$$

pullback time operator:

$$\widehat{T_{x_D}} := \widehat{M_{x_D}^{\star}\mathrm{id}_{\mathbb{R}}} \qquad \widehat{I_{x_D}} := \widehat{M_{x_D}^{\star}\chi_{\mathbb{R}}} \qquad \widehat{T_{X_D,\sigma_D}} := \widehat{M_{X_D,\sigma_D}^{\star}\mathrm{id}_{\mathbb{R}}} \qquad \widehat{I_{X_D,\sigma_D}} := \widehat{M_{X_D,\sigma_D}^{\star}\chi_{\mathbb{R}}}$$

commutator between $\widehat{H_{\mathrm{free}}}$ and $\widehat{T_{X_D,\sigma_D}}$:

$$\left[\widehat{T_{X_D,\sigma_D}}, \widehat{H_{\mathrm{free}}}\right] = -i\widehat{I_{X_D,\sigma_D}} \qquad \left[\widehat{I_{X_D,\sigma_D}}, \widehat{H_{\mathrm{free}}}\right] = 0$$

↑very analogous to CCR!

$$\left[\widehat{T_{X_D,\sigma_D}}, \widehat{I_{X_D,\sigma_D}}\right] = O(\gamma^2)$$

Unlike PVM, Gaussian POVM is non-commuting with each other, so non-commuting

$$\left[\widehat{I_{X_D,\sigma_D}}^{-1/2}\widehat{T_{X_D,\sigma_D}}\widehat{I_{X_D,\sigma_D}}^{-1/2}, \widehat{H_{\mathrm{free}}}\right] = -i\hat{1}$$

Initial state: $\hat{\rho}_{\mathrm{S}} := |\vec{X_S},\boldsymbol{P}_{\mathrm{S}};\sigma_{\mathrm{S}}\rangle\langle\vec{X_S},\boldsymbol{P}_{\mathrm{S}};\sigma_{\mathrm{S}}|$

conditional expectation values of pullback time operators:
Under condition that particle is heading straight towards detector,

$$\frac{\left\langle\widehat{T_{x_D}}\right\rangle_{\hat{\rho}}}{\left\langle\widehat{I_{x_D}}\right\rangle_{\hat{\rho}}} \simeq T_S + m\frac{|\boldsymbol{x_D} - \boldsymbol{X_S}|}{|\boldsymbol{P_S}|} \qquad \frac{\left\langle\widehat{T_{\boldsymbol{X}_{\mathrm{D}},\sigma_{\mathrm{D}}}}\right\rangle_{\hat{\rho}}}{\left\langle\widehat{I_{\boldsymbol{X}_{\mathrm{D}},\sigma_{\mathrm{D}}}}\right\rangle_{\hat{\rho}}} \simeq T_{\mathrm{S}} + m\frac{|\boldsymbol{X}_{\mathrm{D}} - \boldsymbol{X}_{\mathrm{S}}|}{|\boldsymbol{P}_{\mathrm{S}}|}$$

suggesting that $\widehat{T_{x_D}}$ and $\widehat{T_{X_{\mathrm{D}},\sigma_{\mathrm{D}}}}$ are operators corresponding to time

Partial inverse of pullback $\widehat{H_{\mathrm{free}}}$ is non-existent, so Lee error diverges:

$$\tilde{\varepsilon}_{\hat{\rho}_S}\left[\widehat{H_{\mathrm{free}}}\,;M_{X_D,\sigma_D}\right] = +\infty$$

7. Prospects

- **Bell-CHSH**
- **phase operator, angular momentum and angle, and QCD axion**
- **Lorentz invariant/covariant GWB and relativistic extension**