Concepts of Quantum and Spacetime, Mar.09th-12th @KEK, Tsukuba, Japan

Wigner Function Shapelets

and its application

S. Arai, arXiv: 2602.01141, PRD submitted

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Background：Shapes of astronomical bodies tells rich information

- Images of astronomical bodies contain information of fundamental physics
- Galaxy shapes are essential to probe cosmology/gravity

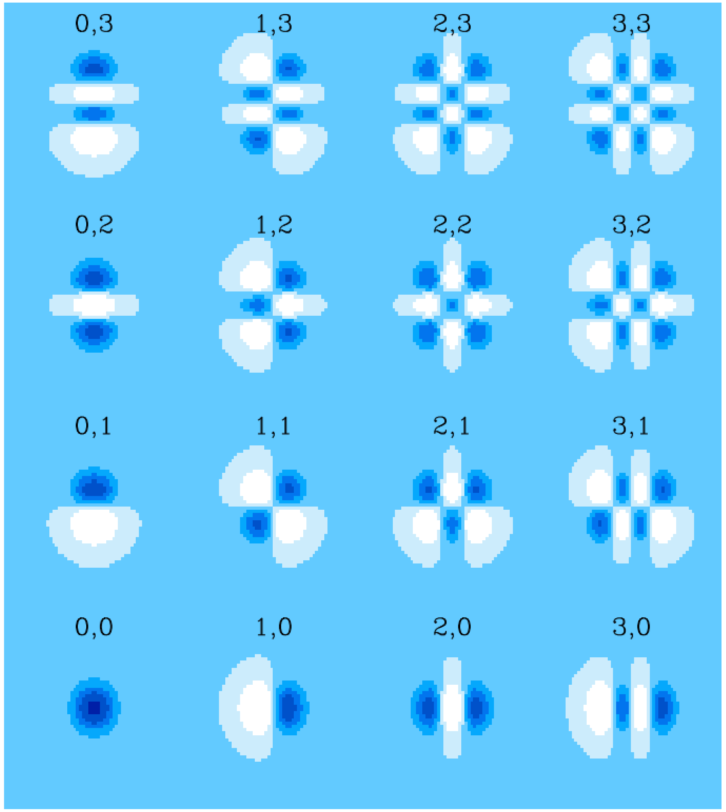
cf. S.Arai et.al 2023 +

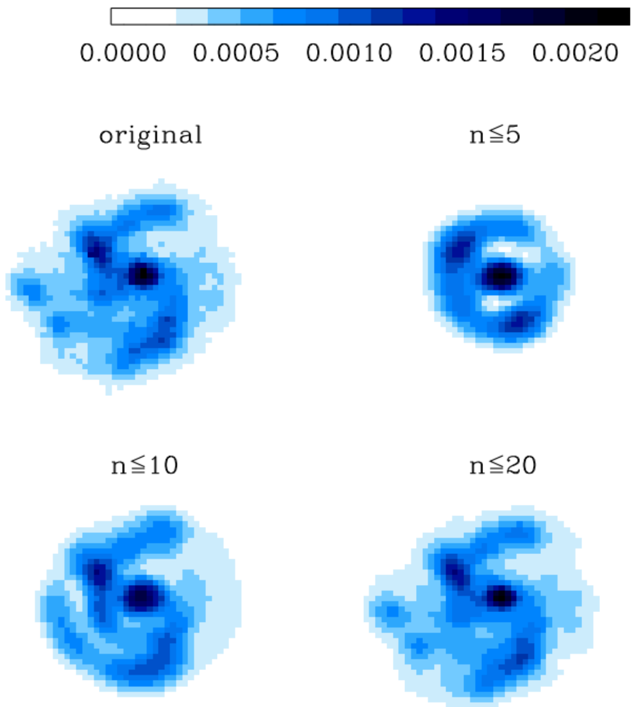
- Galaxy evolution imprinted in its shape
- Shape distortions by gravitational lensing
- Higher shape moments of galaxies can exibits 3D parity violation, higher-spin moments etc.
- Machine-learning Simulation-based inference

©Galaxy Zoo from Euclid

2D representation of images: Shapelets

A.Refregier 2003, R.Massey and A.Refregier 2004 +





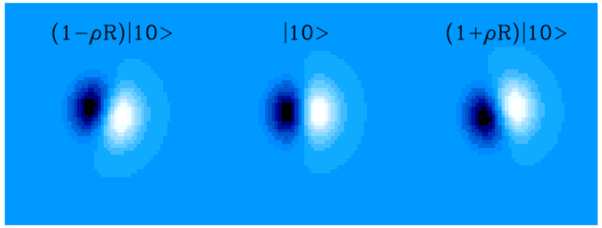
A.Refregier 2003 Fig.2, 3

$\psi(\boldsymbol{x})$ image field

$$\psi(\boldsymbol{x}) = \sum_{nm} \psi_{nm} \Psi_{nm}(\boldsymbol{x})$$

- Hermite/Laguerre-Gaussian modes are adopted

-grav. lensing follows selection rule (m→m±2)



Motivation: Why Wigner Function?

- Shapelets in config/Fourier sp. are mathematically well-defined but…

❓ How does it relate to image propagation/processing

e.g. radiative transfer, point spreading, sensored on CCD…

What this study reveals is

"Wigner Function Shapelets" organises

✅ Wigner transport is a conterpart of transfer eq. , given a resolution limit of a telescope

✅ group-theoretic representation of observables

✅ Local signatures in phase space reflects the nature of galaxy morphology

Intro. Wigner Function

$$W(\boldsymbol{x},\boldsymbol{p})[\psi] = (2\pi\lambdabar)^{-2}\int \mathrm{d}^2\xi\,\psi(\boldsymbol{x}+\boldsymbol{\xi}/2)\psi^*(\boldsymbol{x}-\boldsymbol{\xi}/2)\exp(-i\boldsymbol{p}\cdot\boldsymbol{\xi}/\lambdabar)$$

H.Weyl 1927, E. Winger 1932

$\psi(\boldsymbol{x})$: Image field

Parity invariant

Symplectic covariant

Exp. val.

$$\langle\hat{A}\rangle = \mathrm{Tr}(\hat{\rho}\,\hat{A}) = \int \mathrm{d}^2x\mathrm{d}^2p A_W(\boldsymbol{x},\boldsymbol{p})\,W(\boldsymbol{x},\boldsymbol{p})$$

"Weyl symbol"

$$A_W(\boldsymbol{x},\boldsymbol{p}) = \int \mathrm{d}^2\xi A(\boldsymbol{x}+\boldsymbol{\xi}/2,\boldsymbol{x}-\boldsymbol{\xi}/2)\exp(-i\boldsymbol{p}\cdot\boldsymbol{\xi}/\lambdabar)$$

Normalisation

$$1 = \mathrm{Tr}(\hat{\rho}) = \int \mathrm{d}^2x\mathrm{d}^2p W(\boldsymbol{x},\boldsymbol{p})$$

"quasi"probability dist (∵ it can be negative)

Intensity

$$I(\boldsymbol{x}) = \int \mathrm{d}^2p W(\boldsymbol{x},\boldsymbol{p})$$

Power spectrum

$$F(\boldsymbol{p}/\lambdabar) = \int \mathrm{d}^2x W(\boldsymbol{x},\boldsymbol{p})$$

Intro. Cross Wigner Function

$$W(\boldsymbol{x},\boldsymbol{p})[\psi_a,\psi_b] = (2\pi\lambdabar)^{-2}\int \mathrm{d}^2\xi\, \psi_a(\boldsymbol{x}+\boldsymbol{\xi}/2)\psi_b^*(\boldsymbol{x}-\boldsymbol{\xi}/2)\exp(-i\boldsymbol{p}\cdot\boldsymbol{\xi}/\lambdabar)$$

⭐a, b: polarisation (I, Q, U, V), colours (ugriz), qunatum number

Parity inv. iff. a = b

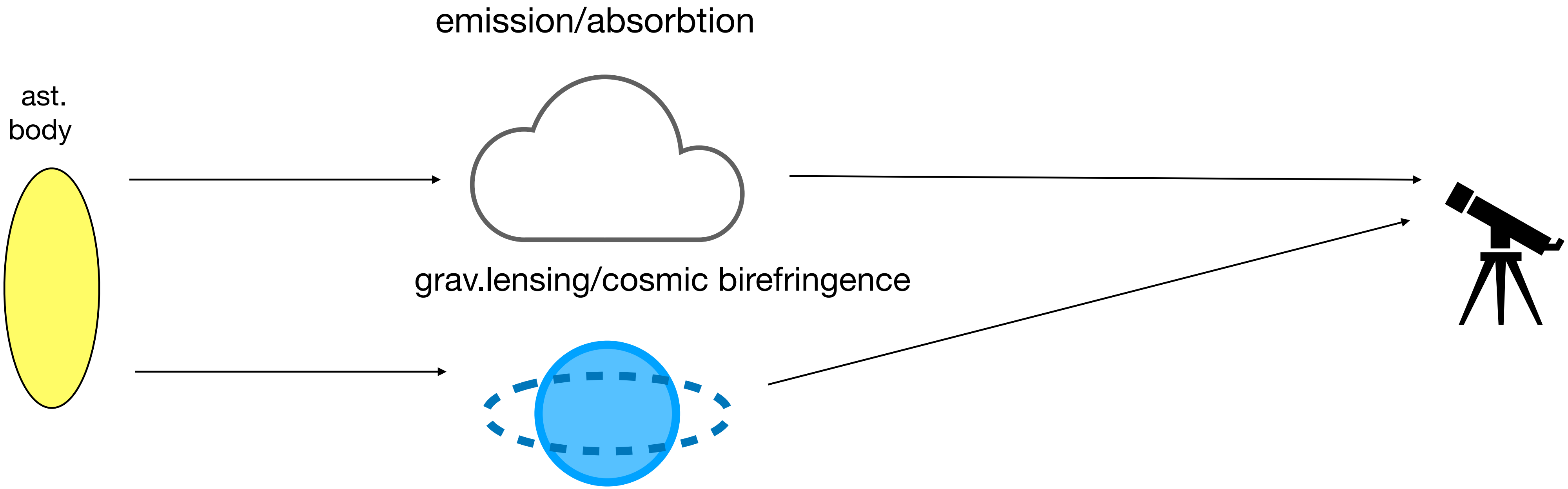
Symplectic cov.

Moyal/Plancherel identity

$$\langle W_{cd}|W_{ab}\rangle_{\mathrm{HS}} = \frac{1}{(2\pi\lambdabar)^2}\langle\psi_a\,|\,\psi_c\rangle\,\langle\psi_b\,|\,\psi_d\rangle$$

$\langle\psi_a\,|\,\psi_b\rangle \propto \delta_{ab}$ → Hilbert-Schmibt product becomes orthogonal

Intro. Wigner transport



Wigner func. $W = W(\boldsymbol{x}, \boldsymbol{p}, s)$ $\qquad \partial_s W = \{H_W, W\}_{\text{Moyal Product}}$

$\bar{\lambda} \to 0 \quad \boldsymbol{k} \equiv \boldsymbol{p}/\bar{\lambda}$

Specific intensity $L = L(\boldsymbol{x}, \boldsymbol{k}, s)$ $\qquad \partial_s L = \{H, L\}_{\text{Poisson Bracket}}$

✅ Wigner transport naturally corresponds to radiative transf.

Wigner func. and Van-Cittert Zernike (VCZ) theorem

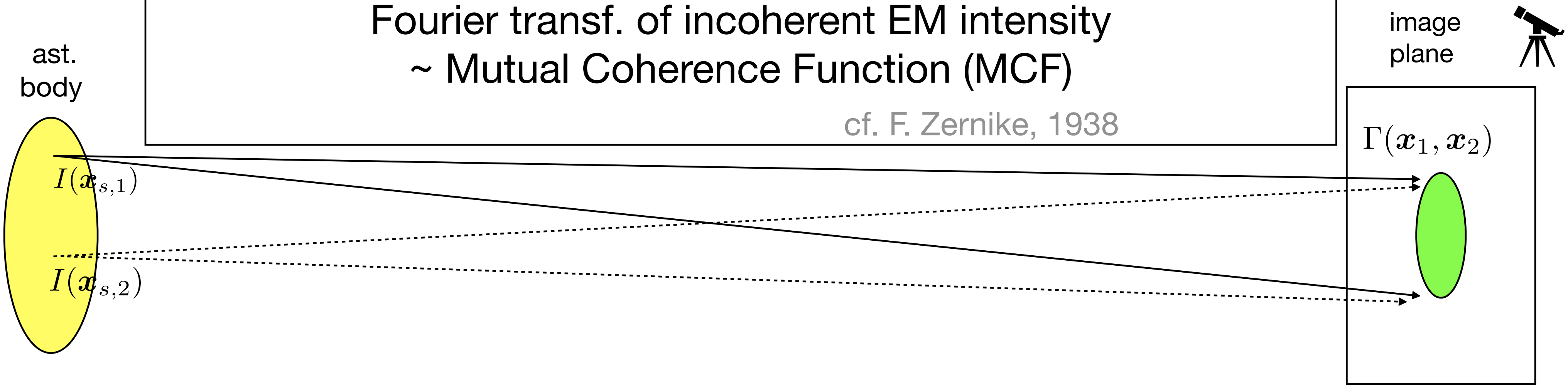
VCZ theorem

Given a distant observer,

Fourier transf. of incoherent EM intensity

~ Mutual Coherence Function (MCF)

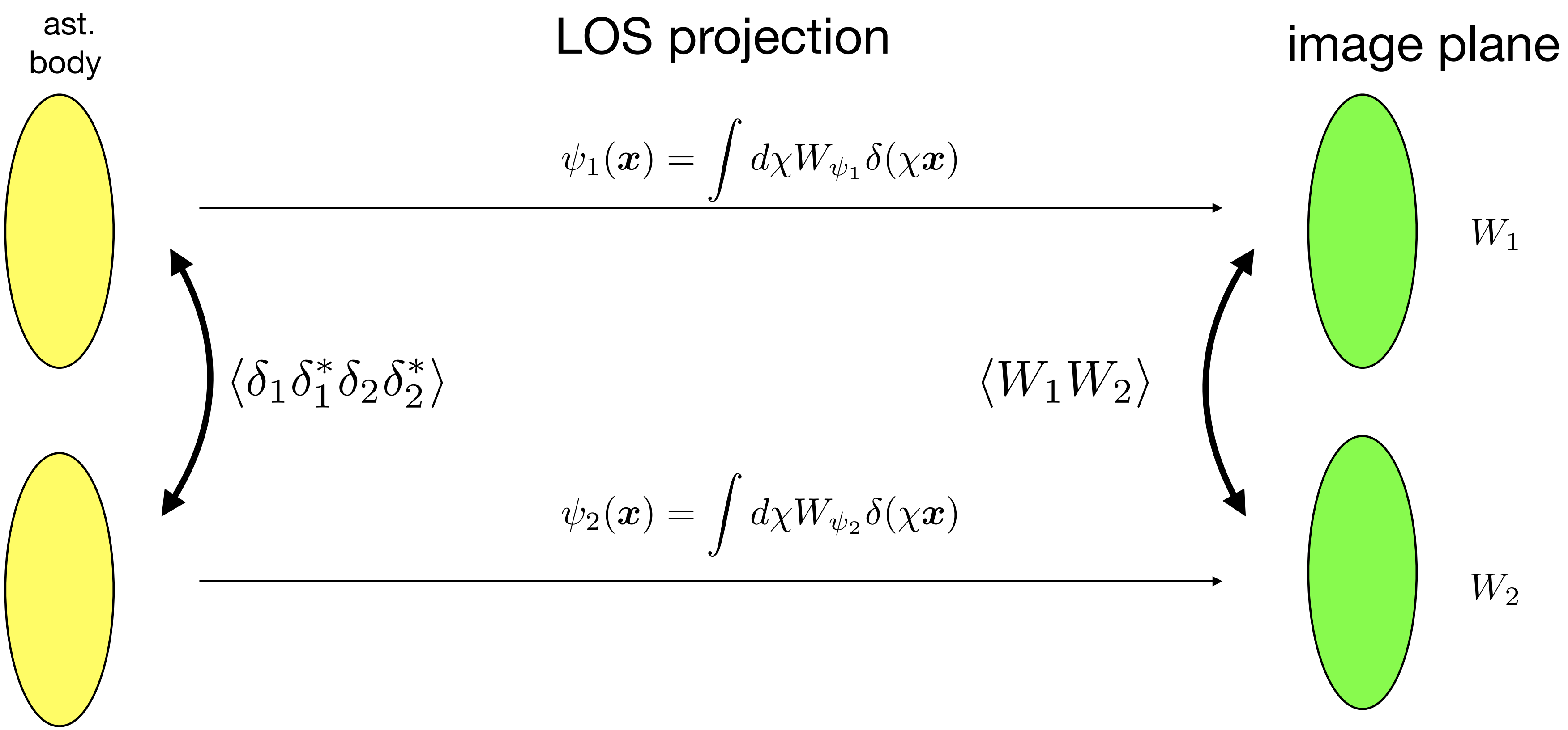
cf. F. Zernike, 1938



$$W(\boldsymbol{x}, \boldsymbol{p}) = \int \mathrm{d}^2\xi \, \Gamma(\boldsymbol{x} + \boldsymbol{\xi}/2, \boldsymbol{x} - \boldsymbol{\xi}/2) \exp(-i\boldsymbol{p} \cdot \boldsymbol{\xi}/\lambda\!\!\!^{-})$$

✅ Wigner function is Fourier inverse transf. of MCF

e.g. Wigner func. and cosmo. 4pt func.



✅ Wigner func. at different sky position ~ 4pt (collapsed limit)

Wigner Function Shapelets

R.Simons and G.Agarwal 2004 +

Point：Cross Wigner func. for Laguerre-Gaussian modes preserves SU(2)

$$W^{\mathrm{LG}}_{jm}(Q_0, Q_2) \propto (-1)^{2j} e^{-4Q_0} L_{j+m}(4(Q_0+Q_2)) L_{j-m}(4(Q_0-Q_2))$$

$$Q_0 = \frac{1}{2}\left[\frac{|\boldsymbol{x}|^2}{w^2} + \frac{w^2|\boldsymbol{p}|^2}{4\lambda\!\!\!^-{}^2}\right] \qquad Q_2 = \frac{1}{2\lambda\!\!\!^-}{}^T\boldsymbol{x}J\boldsymbol{p} \qquad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Cross Wigner func. is analytically diagonalised by Swinger bosonic oscillator

$$\begin{aligned} &W[\Psi^{\mathrm{LG}}_{js}, \Psi^{\mathrm{LG}}_{j's'}](Q_0, Q_2; \phi_u, \phi_v) \\ &= \frac{(-1)^{n_{u<}+n_{v<}}}{\pi^2 \lambda\!\!\!^-{}^2} \sqrt{\frac{n_{u<}!\, n_{v<}!}{n_{u>}!\, n_{v>}!}} \, (\sqrt{2}\,|u|)^{|\Delta_u|} \, (\sqrt{2}\,|v|)^{|\Delta_v|} \\ &\quad \times e^{-2(|u|^2+|v|^2)} \, e^{i(\Delta_u\phi_u + \Delta_v\phi_v)} \\ &\quad \times L^{(|\Delta_u|)}_{n_{u<}}(4|u|^2) \, L^{(|\Delta_v|)}_{n_{v<}}(4|v|^2), \end{aligned}$$

⭐ Hirbert-Schmibt space w. orthogonal and complete basis

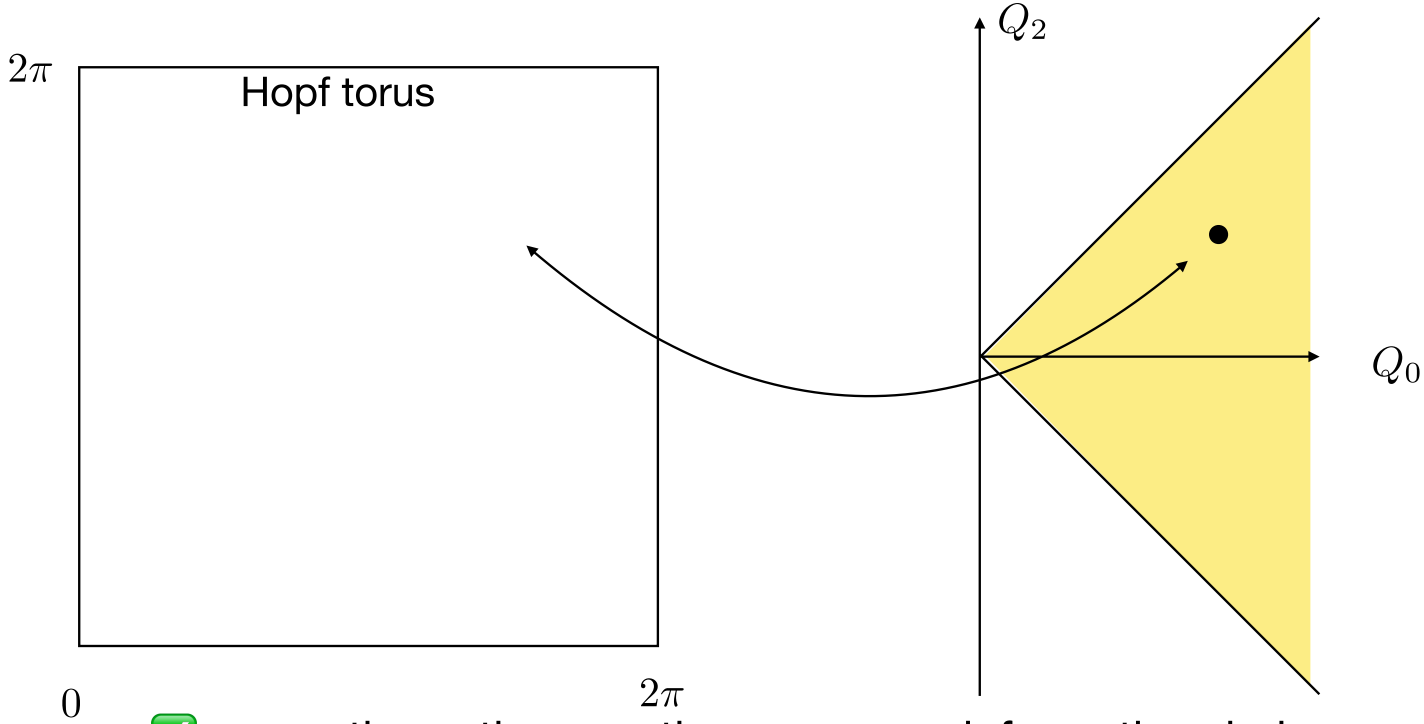
Compression of Wigner Function Shapelets to 2D plane

WFS is complex in 4D phase space that prevents me from intuitive analyses

⭐ WFS is further characterised by "Hopf Fibration" H. Hopf 1931+

$$\mathrm{SU}(2) \supset U(1) \times U(1)$$

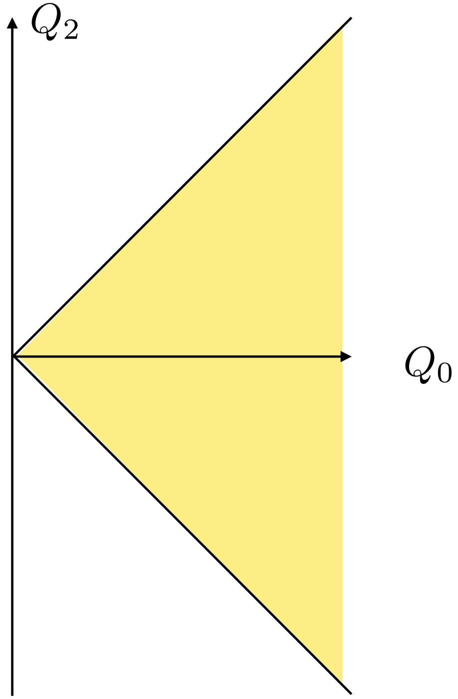
$$u = \sqrt{Q_0 + Q_2}\boldsymbol{e}(\phi_u)\,, v = \sqrt{Q_0 - Q_2}\boldsymbol{e}(\phi_v) \qquad \boldsymbol{e}(\phi) = {}^T(\cos(\phi), \sin(\phi))$$

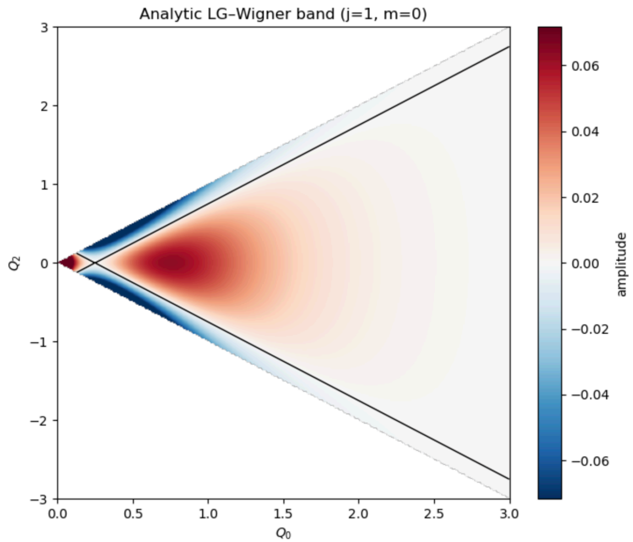


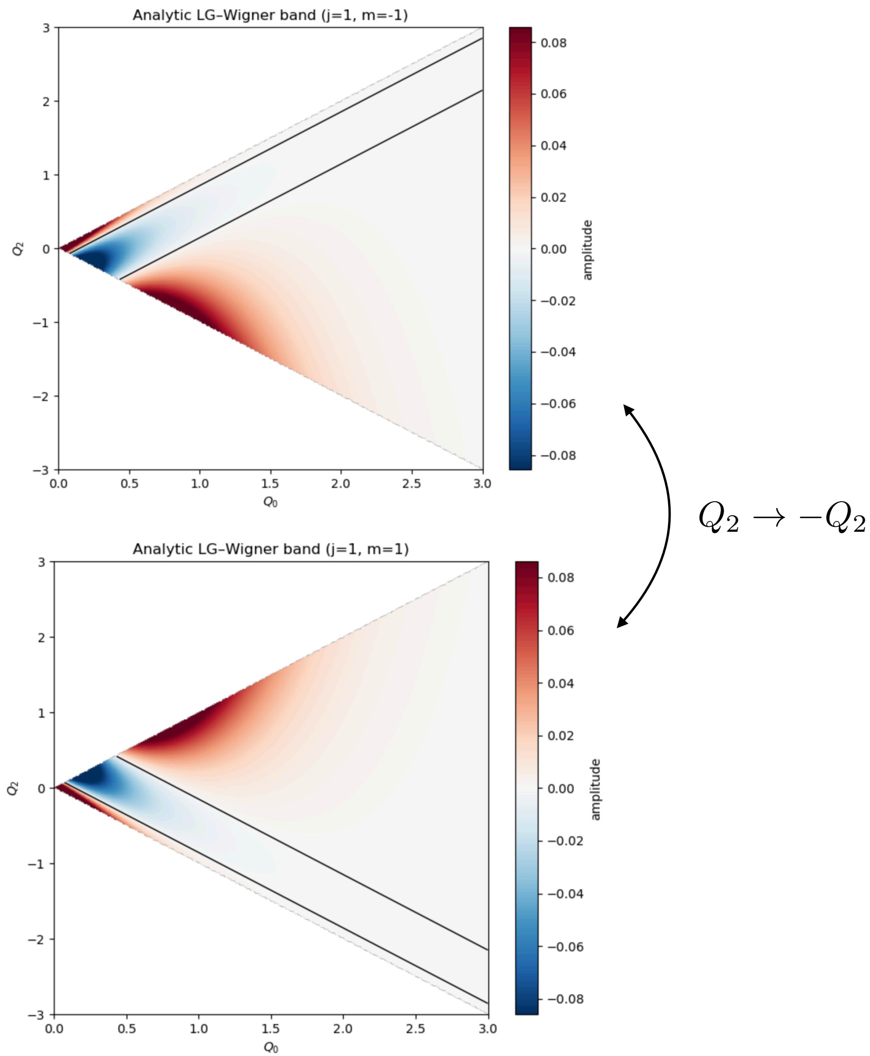
✅ group-theoretic operation preserves information during compression

Result 1: Wigner Function Shapelets ”band structure ”

$$W_{jm}^{\mathrm{LG}}(Q_0,Q_2)\propto(-1)^{2j}e^{-4Q_0}L_{j+m}(4(Q_0+Q_2))L_{j-m}(4(Q_0-Q_2))$$







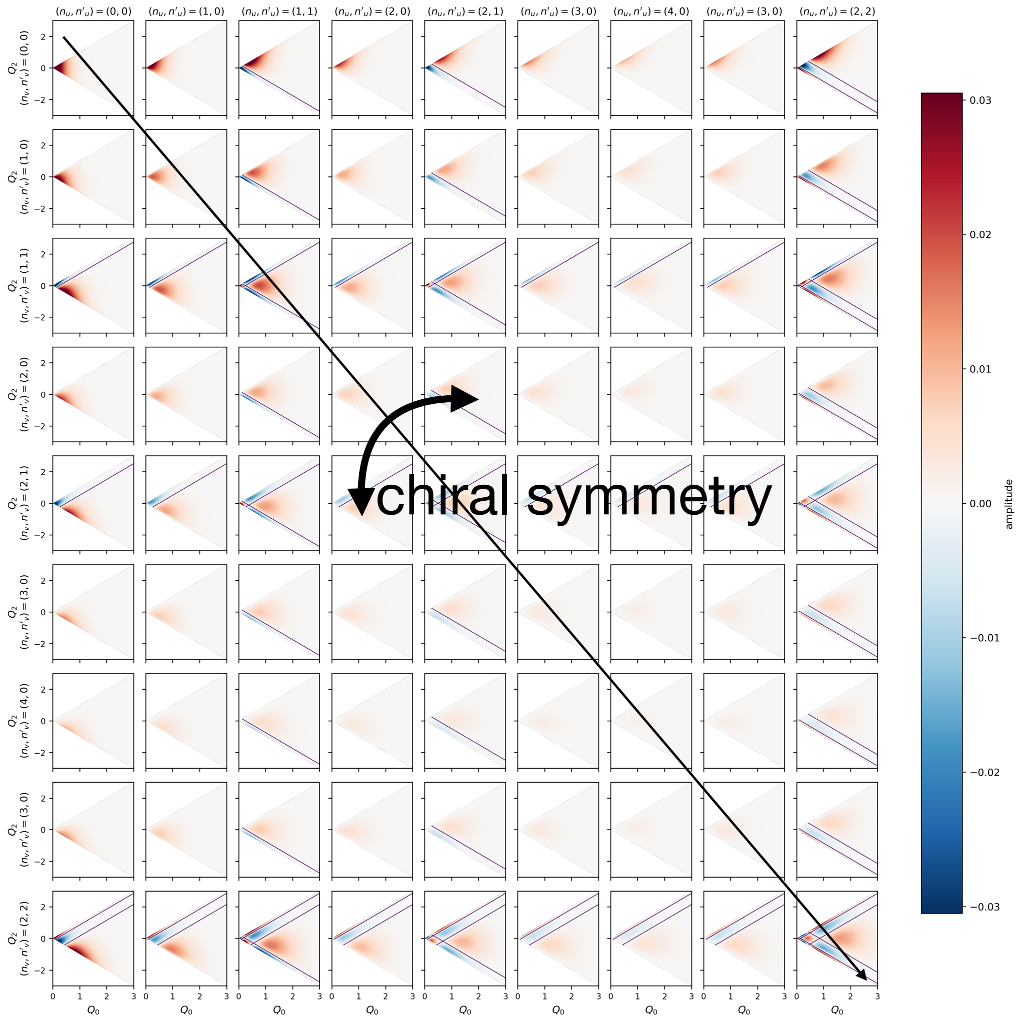
⭐WFS exhibits +/-/0 in geometry i.e. “band structure”

⭐Wigner is everywhere positive iff. it is a Gaussian state

* single image

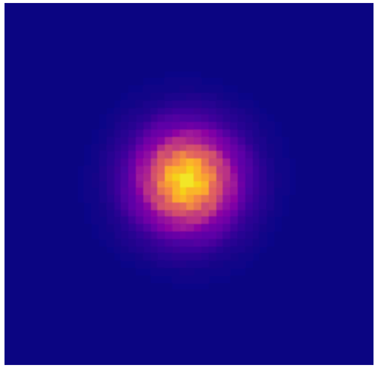
WFS rep.

$$\mathcal{W}_{k\ell}(Q_0, Q_2) = \int \frac{\mathrm{d}\phi_u}{2\pi}\frac{\mathrm{d}\phi_v}{2\pi} W \exp(-ik\phi_u - i\ell\phi_v)$$



Q. How to use WFS?

config. sp.

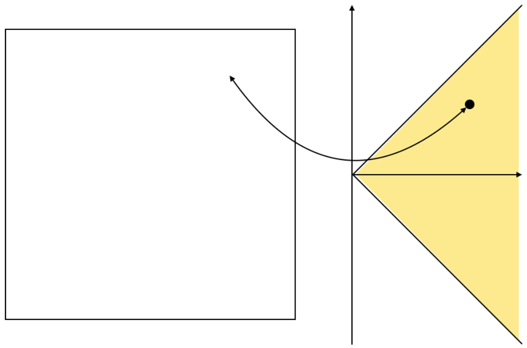


$\psi \to W[\psi]$

$I(\boldsymbol{x}) = \int \mathrm{d}^2 p W(\boldsymbol{x}, \boldsymbol{p})$

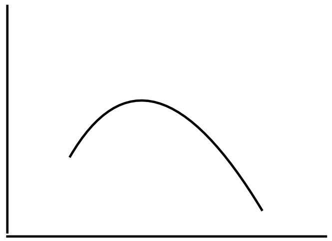
$\psi \to \tilde{\psi}(\boldsymbol{k})$

phase sp. $\mathrm{Sp}(4, \mathbb{R}) \simeq \mathrm{SU}(2)$ level sets



$F(\boldsymbol{p}/\lambda) = \int \mathrm{d}^2 x W(\boldsymbol{x}, \boldsymbol{p})$

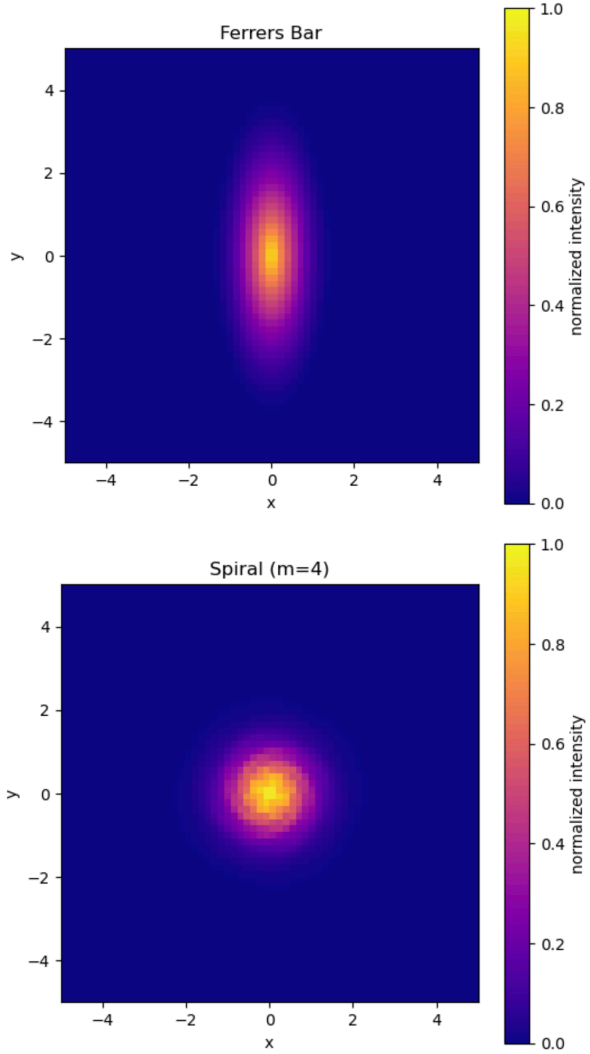
Fourier sp.

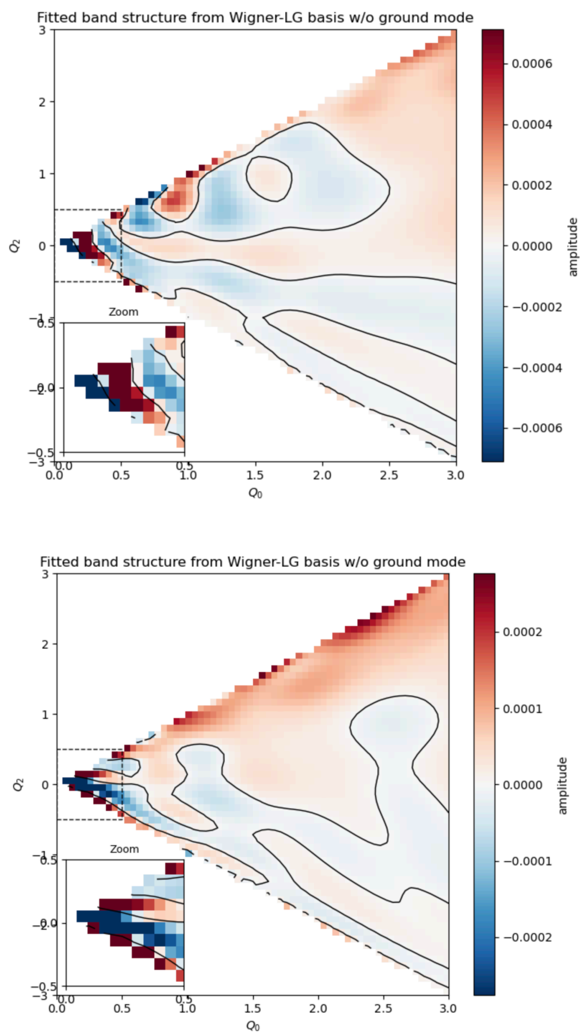


2D observables
w. band structure

Result 2：Galaxy morphology response

$$W = \sum_{j=0,\frac{1}{2},1,\cdots} \sum_{m=-j}^{j} \mathcal{W}_{jm} W^{\mathrm{LG}}_{jm} + \text{cross Wigner}$$





Takeaways

- We extend the conventional shapelets, employing Wigner function

What this study reveals is

“Wigner Function Shapelets” organises

✅ Wigner transport is a conterpart of transfer eq. , given a resolution limit of a telescope

✅ group-theoretic representation of observables

✅ Local signatures in phase space reflects the nature of galaxy morphology

prospects :

- Mock data challenges (Point spread function, model bias, deblending…)
- app. for extended fields (CMB, LSS, 21cm BT etc.) by Wigner function

“The great advances in science usually result from new tools rather than from new doctolines…”
F. Dyson, NYBR essay 1995