Linearisation instabilities and crossed products

Julian De Vuyst

Based on [JHEP 05(2025)211] with S. Eccles, P. Höhn & J. Kirklin

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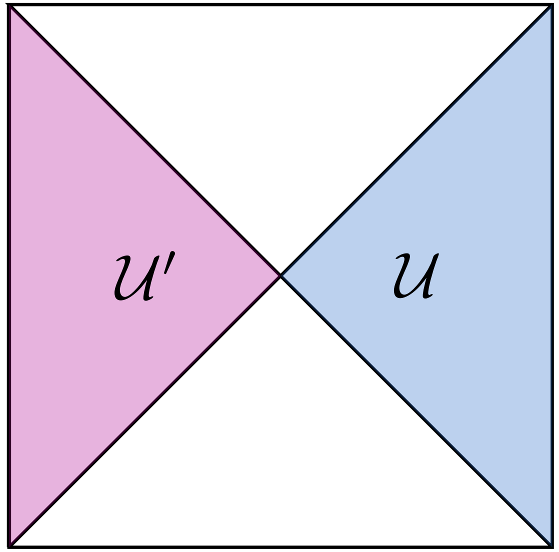
Motivation: crossed products

Generalised entropy formula for a subregion $\mathcal{U}$ of spacetime:

$$S_{\text{gen}}(\mathcal{U}) = \frac{\text{Area}}{4G_{\text{N}}} + S_{\text{QFT}}(\mathcal{U})$$

$\rightarrow$Deep connection geometry and information



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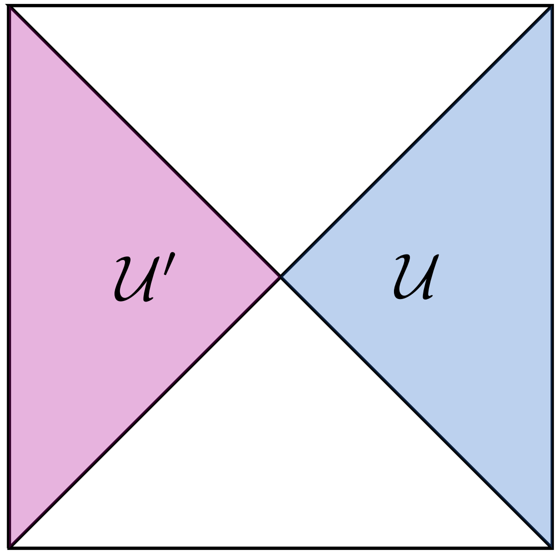


2 problems:

- $S_{\text{QFT}}(\mathcal{U})$ is ill-defined

  $\rightarrow$ von Neumann algebra ($\mathcal{A} = \mathcal{A}''$) of field operators within $\mathcal{U}$ is type III$_1$ (no trace)
  ['64 Araki, '77 Driessler, '85 Fredenhagen]

  $\rightarrow$ Either introduce UV cutoff or renormalise G_{N}

- Physical subsystem associated to gravitational subregion must be diffeomorphism invariant

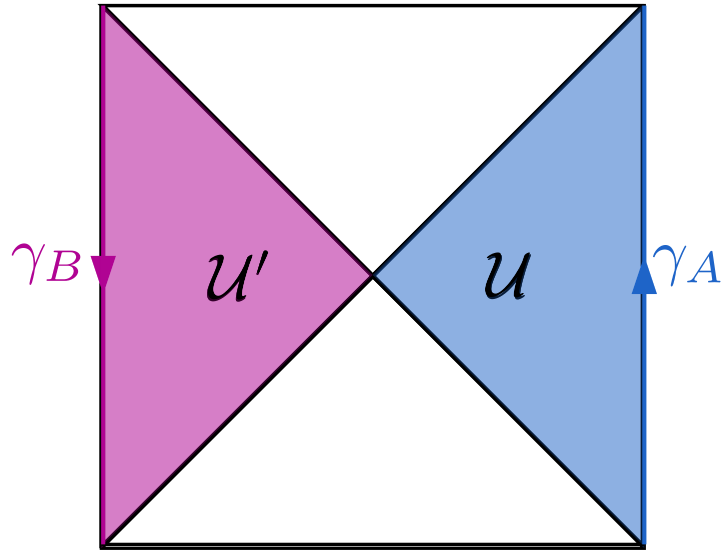
Motivation: crossed products

['22 Chandrasekaran, Longo, Penington & Witten (CLPW)] introduced observer degrees of freedom in perturbative limit $G_{\mathrm{N}} \to 0$

- Non-trivial physics after imposing boost constraint
- Algebra of observables dressed to observer is of type II

$\Rightarrow$ Leads to well-defined trace, density matrices, and entropy

$\Rightarrow$ Can recover generalised entropy formula in semiclassical limit



- Can be reinterpreted in terms of quantum reference frames
  ['24 **DV**, Eccles, Höhn, Kirklin; '24 Fewster, Janssen, Loveridge, Rejzner, Waldron...]

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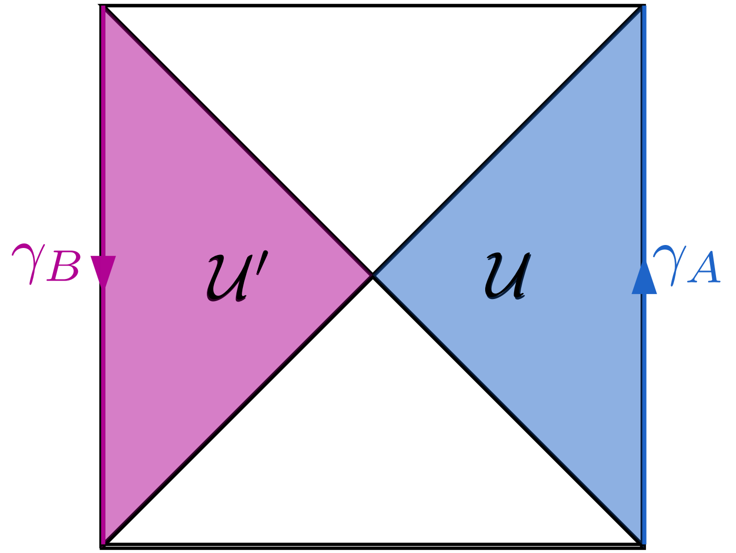

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Why impose a 2nd order constraint on a linearised theory?

$$H = H_{\mathcal{U}} + H_A - H_B$$

Linearisation instability
To linear order

Consider expansion of metric

$$g_{\mu\nu} = g^0_{\mu\nu} + \kappa h_{\mu\nu} + \tfrac{1}{2}\kappa^2 k_{\mu\nu} + \mathcal{O}(\kappa^3), \quad \kappa \propto \sqrt{G_\mathsf{N}}$$

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$$\delta_{\kappa\xi} g_{\mu\nu} = \kappa \mathcal{L}_\xi g^0_{\mu\nu} + \kappa^2 \mathcal{L}_\xi h_{\mu\nu} + O(\kappa^3) \qquad \Rightarrow \qquad \delta_{\kappa\xi} h_{\mu\nu} = \nabla^0_\mu \xi_\nu + \nabla^0_\nu \xi_\mu + \kappa \mathcal{L}_\xi h_{\mu\nu} + O(\kappa^2)$$

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ξ compactly supported and vanishing on $\partial\mathcal{M}$ ['12 Hollands, Wald; '16 Donnelly, Giddings]

Linearised gauge constraint currents $\qquad C_{\kappa\xi} = 4G^{(1)}_{\mu\nu}(h)\,\xi^\mu (\epsilon^0)^\nu + O(\kappa)$

If ξ is compactly supported Killing vector (gauge symmetry) ['80 Fischer, Marsden, Moncrief]

$$C[\kappa\xi] = \int_\Sigma C_{\kappa\xi} = 0 + O(\kappa)$$

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No matter, such as QRFs, enters the constraints!

Linearisation instability

To 2nd order

For each Killing vector ξ ['80 Fischer, Marsden, Moncrief; '82 Arms, Marsden, Moncrief]

Taub charges $$\int_\Sigma \left(-2G^{(2)}_{\mu\nu}(h,h) + T^{(0)}_{\mu\nu}\right)\xi^\mu(\epsilon^0)^\nu \approx 2\int_{\partial\Sigma}(\epsilon^0)^{\mu\nu}F_{\mu\nu}(k) + O(\kappa)$$

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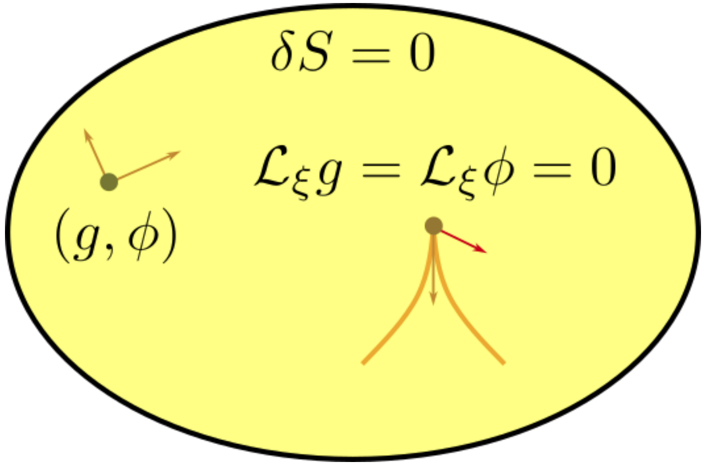

For symmetric background with closed Cauchy clices

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2nd-order nontrivial equation for 1st-order metric perturbation $h_{\mu\nu}$

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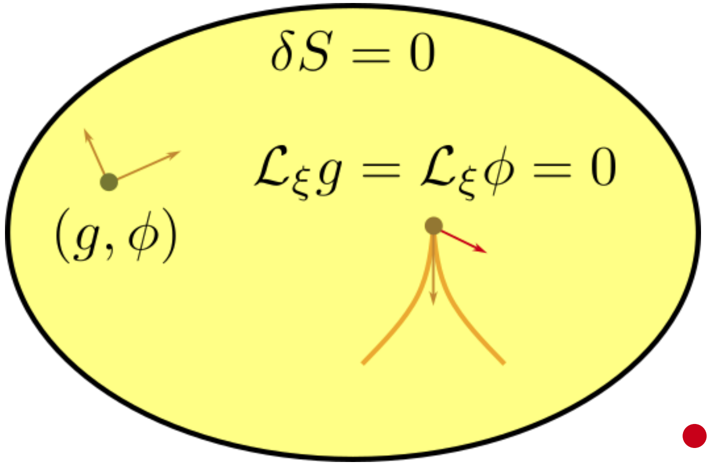

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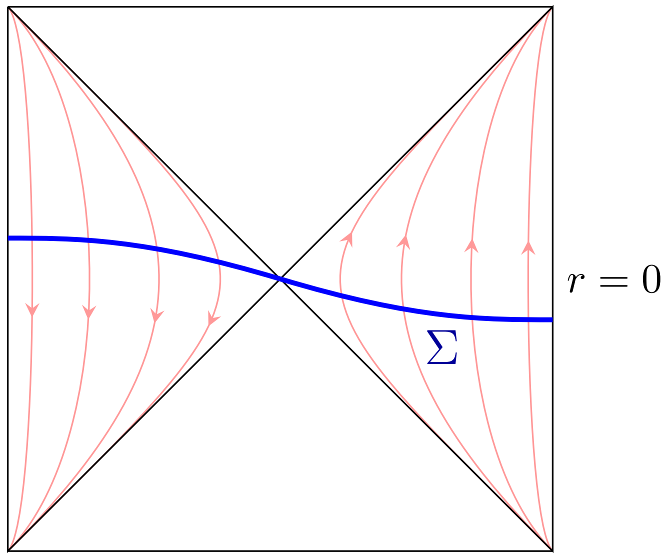
Linearisation instabilities

- There is a finite number of them and they are sufficient (no higher-order) (vacuum GR, GR + Maxwell, GR + YM) ['69 Taub; '86 Arms, Anderson]
- With boundary: same situation if $k_{\mu\nu} = O(r^{-x}),\ x > d-3$
- Gauge invariant (linearised small diffeos)

Back to crossed products
Spatially closed

Work in $\kappa \to 0$ limit of $\mathcal{H}_{\text{QFT}}$ (gravitons + other), assume already invariant under linearised diffeos



Boost Hamiltonian for Killing horizon of boost Killing vector ξ

$$H_\xi = \int_\Sigma (\epsilon^0)^\mu \xi^\nu T_{\mu\nu} = C$$

Linearised constraint vanishes identically, so must be Taub charge!

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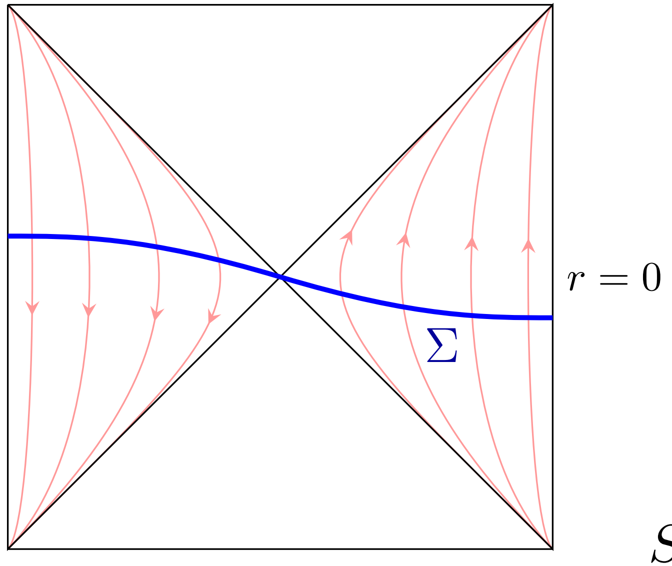

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Add clock actions at the same order of other matter

['23 Witten; '24 Kolchmeyer, Liu]

$$S_i = \int_{\gamma_i} \mathrm{d}s \left(p_i \dot{t} - \sqrt{-g^0_{ss}} (H_i(t_i, p_i)) \right)$$

γ_i along Killing flow lines

Clock contributes to boost Hamiltonian $H_i = \int_\Sigma (\epsilon^0)^\mu \xi^\nu T^{(0)}_{i,\mu\nu} \quad \Rightarrow \quad C = H_\xi + \sum_i H_i \approx 0 + O(\kappa)$

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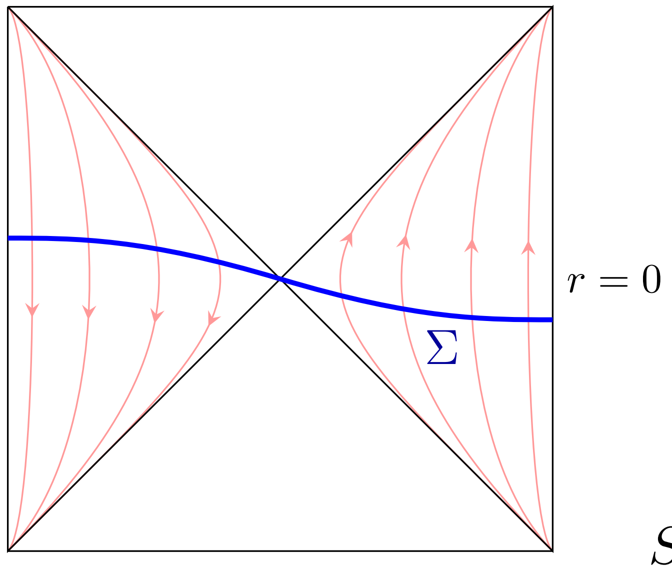

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- Remaining Taub charges (finitely many) don't affect the type transition

['22 CLPW, '24 Ahmad et al.]

Back to crossed products
Black holes and boundaries

Let there be an asymptotic boundary, then

$$C^{(2)}[\kappa\xi] = \int_\Sigma \left(2G^{(2)}_{\mu\nu}(h,h) - T^{(0)}_{\mu\nu} + 2hG^{(1)}_{\mu\nu}(h)\right)\xi^\mu(\epsilon^0)^\nu + 2\int_{\partial\Sigma}(\epsilon^0)^{\mu\nu}F_{\mu\nu}(k)$$

Which on-shell of linearised Einstein equation means

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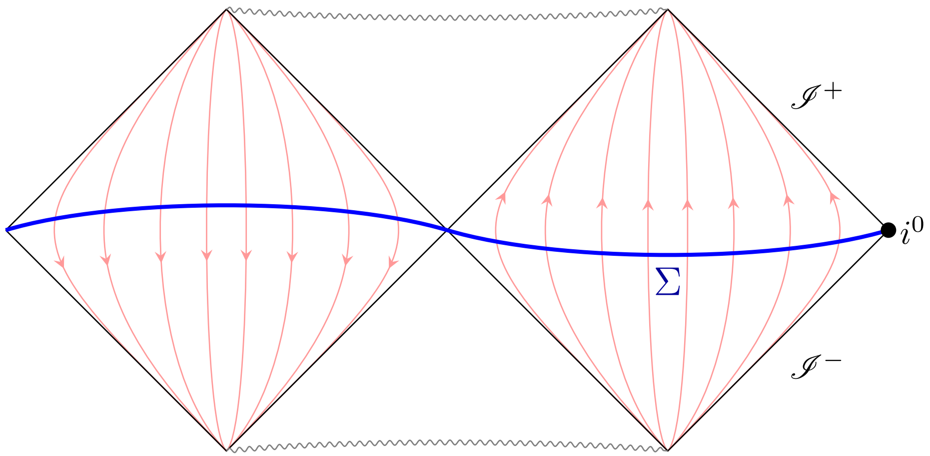

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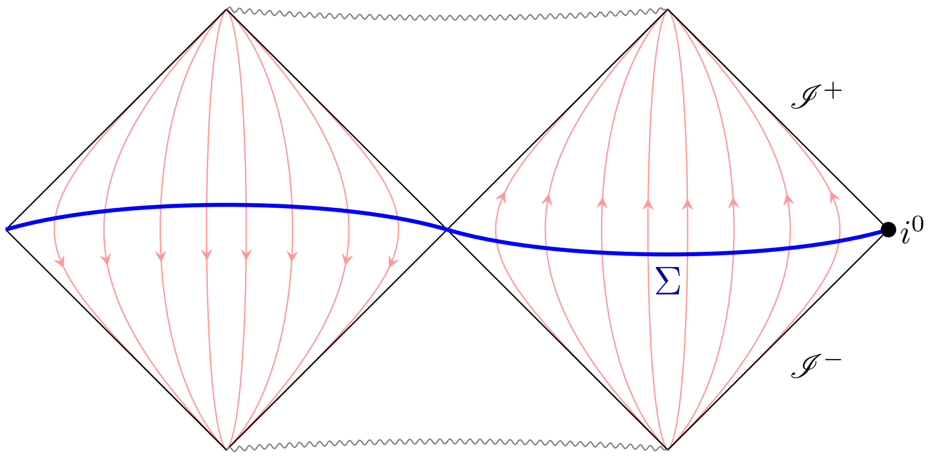

No additional clock needed, dress to R after

$$H^{(2)}_{\mathrm{ADM}} = H_R - H_L$$

- No instability, so reason to impose constraint must come from somewhere else (Impose when interested in 2nd-order variables such as area fluctuations)

Conclusion

Unambiguous justification for imposing 2nd order constraints on linearised theory in spatially closed spacetimes

Thank you!