Dynamical Black Holes and their Entropies

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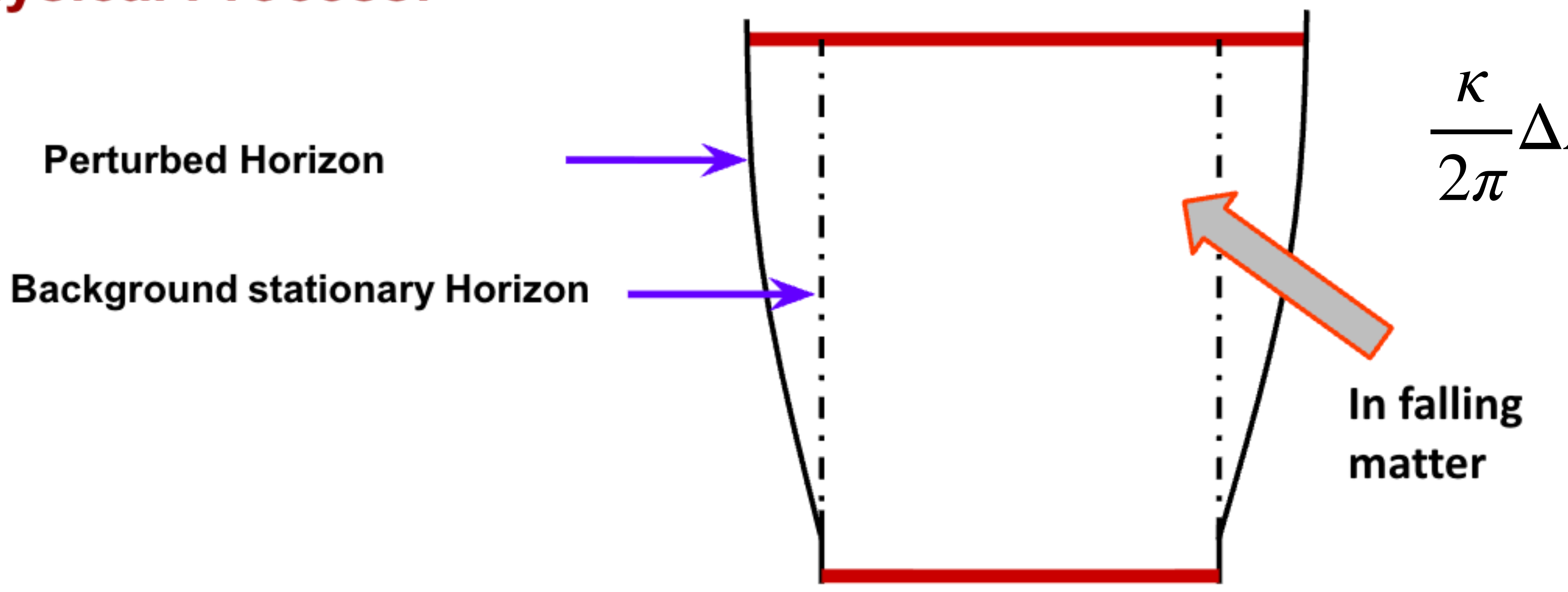
Introduction

- Black Holes are fascinating objects predicted by GR. They are simplest objects in GR usually characterised by very few parameters like Mass (M), Angular momentum (J), etc.
- Most of the intriguing features of BHs are obtained at the Event Horizon (EH). EHs of BHs obey mathematical relations amazingly similar to the laws governing thermodynamic systems
- BHs possess entropy and in GR it is proportional to the area of its EH
- BH radiates thermally when treated semi-classically- Hawking Radiation. This generated a lot of activities - finding microscopic origin of Entropy of BHs, Information loss puzzle..
- BHs are like hydrogen atoms of quantum gravity. Uncovering physics of BHs necessary to understand the fundamental structure of spacetime.

Black Hole Mechanics in Stationary setup

- Event horizon of a stationary BH is a Killing horizon -> A local notion of horizon where a Killing vector becomes null
- Killing symmetry is crucial ingredient for establishing BH mechanics
- Establishing the first and second law of BHM(T), one actually considers a dynamical phase slightly perturbed from equilibrium or from a stationary configuration admitting a Killing vector. The initial and final states must be infinitesimally separated.

Physical Process:



$$\frac{\kappa}{2\pi}\Delta A(S) = \int_{\lambda_i}^{\lambda_f} \lambda T_{ab}k^a k^b \sqrt{q}\, d\lambda dA$$

$$\frac{dA(S)}{d\lambda} \geq 0$$

Matter satisfies NEC

- Physical Process First Law and local Entropy increase law are obtained under this set up

BHM Beyond GR

- BHM or BHT can also be extended for Higher curvature gravity theories
- PPFL and a local Entropy increase law have been shown to work in the regime where the dynamical perturbations have been treated perturbatively to linear order.
- The correct entropy candidate that satisfies second law is given by the Iyer-Wald entropy +JKM terms. Wall, Dong
- Verifying the second law for BHs, the perturbed non-stationary slices can be modelled as the event horizon of a dynamical black hole - the **Vaidya** solution
- $$ds^2 = -\left(1 - \frac{2m(v)}{r}\right) dv^2 + 2dvdr + r^2 d\Omega_2^2 \qquad EH : \frac{dr}{dv} = \frac{1}{2}\left(1 - \frac{2m(v)}{r}\right)$$
- Using Raychoudhuri equation one can show: $\frac{d}{dv}\left(\Theta_k(v)\right) \geq 0$ **SB,** SS**,** AW **(2015)**
- Recently an ultra local entropy increase law has been established including an entropy current term -> *both local in spatial slices of the horizon in addition to being local in the null-time evolution.*

Bhattacharya, Kundu ..(2021); Wald, Zhang, Hollands(2024)

BHM(T) beyond stationarity

- Realistic BHs are not always isolated, they are dynamic due to their interactions with surroundings
- EH is a global concept, teleological in nature and not suitable to study dynamical situations
- In dynamical spacetime we don't have the advantage of Killing symmetry
- Establishing BHM needs other ingredients
- Can the laws of BHM (T) be generalized for horizons that are not event horizons or Killing horizons?
- What is a satisfactory definition of temperature?
- What happens to the derivation of Hawking Radiation for dynamical horizons, or other quasi-local horizons?

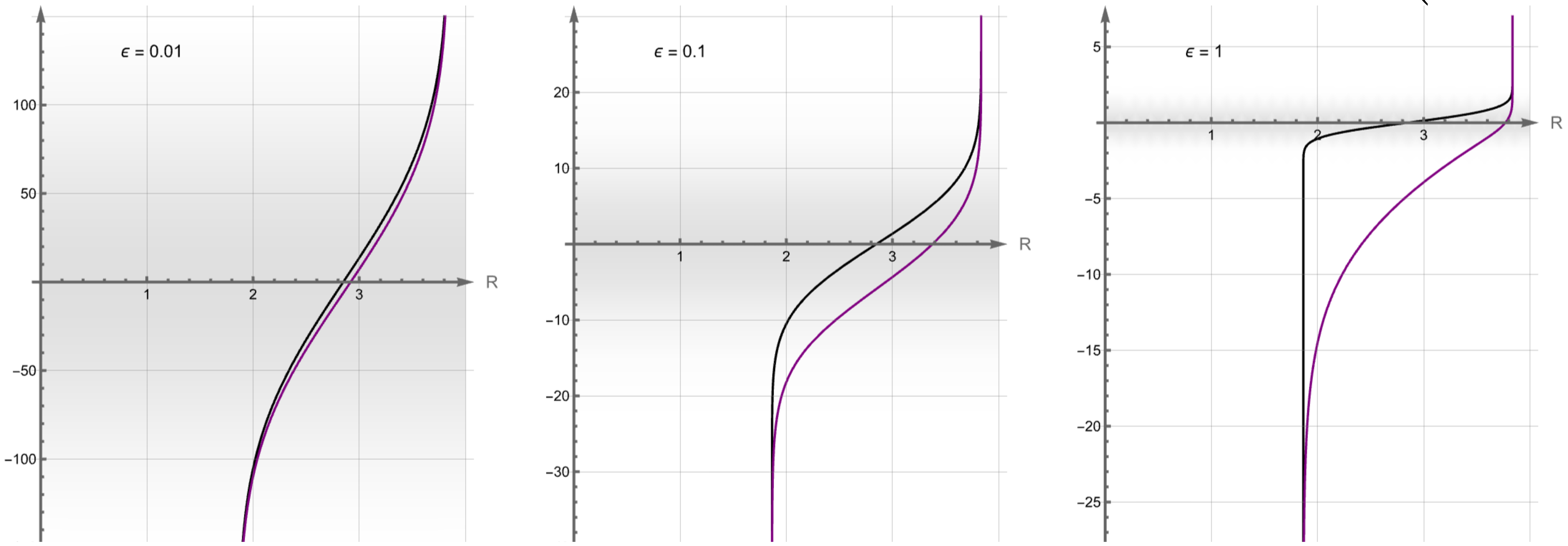
BHM for Slowly Evolving horizons

- For Dynamical BHs local horizons that are characterised by expansions of ingoing (θ_l) and outgoing null (θ_n) geodesic congruences
- Black hole mechanics have been studied on quasi local horizons (boundary of trapped surfaces): eg. Isolated horizons, Trapping horizons, Slowly Evolving horizons etc. Ashtekar-Krishnan-Lewandowski-Beetles; Hayward; Booth-Fairhurst
- Future Outer Trapping Horizon (FOTH): $\theta_l = 0; \theta_n < 0; \mathscr{L}_n \theta_l < 0$
- Slowness conditions for SENS: $\frac{1}{d-1}\Theta^2_{(l)} \ll \left(\sigma^{(l)}_{AB}\sigma^{AB}_{(l)} + \mathscr{R}_{ab}l^a l^b\right), \quad \mathscr{L}_l \Theta_{(l)} \ll \kappa_{(l)}\Theta_{(l)}$
- Dynamical surface gravity: $\kappa_{(l)} = -\, l^a n_b \nabla_a l^b$
- SENS quite well captures the near equilibrium scenario. BHM can be shown to be obeyed for general spacetimes including higher curvature terms. The horizon leaves need not to be infinitesimally separated.

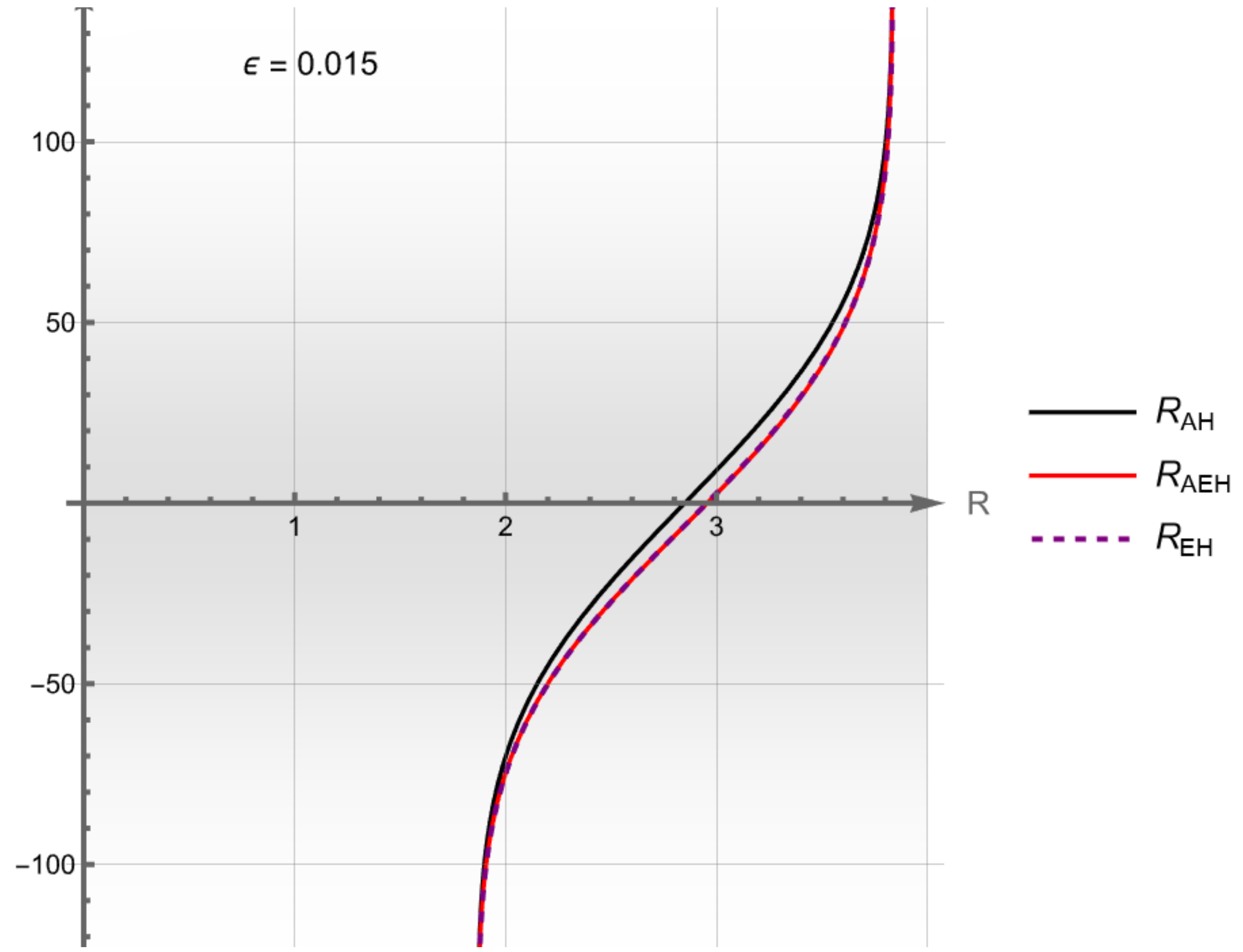
SENS and BHM

$$FOTH : r_{\pm}(v) = m(v) \pm \sqrt{m(v)^2 - q(v)^2}$$

$$ds^2 = -\left(1 - \frac{2m(v)}{r} + \frac{q(v)^2}{r^2}\right) dv^2 + 2dvdr + q_{AB}dx^A dx^B,$$



FOTH and Event Horizon



SENS and EH almost coincide in the slow evolution regime

$$\kappa_{(l)}\Theta_{(l)} \approx \left(\|\sigma_{(l)}\|^2 + \mathcal{R}_{ab}l^a l^b\right) \rightarrow Clasius \quad relation : FL \ for \ SENS$$

AT, **SB** CQG (2022)

Second law can be obtained by using Raychaudhuri eqn. $\kappa_{(l)}\Theta_{(l)} - \mathcal{L}_l\Theta_{(l)} = \frac{1}{d-1}\Theta_{(l)}^2 + \|\sigma_{(l)}\|^2 + \mathcal{R}_{ab}l^a l^b$

Works for HD gravity theories as well

Conformal Killing Horizon

- Can anything be extracted for such horizons that are not slowly evolving?
- Vaidya Black Hole provides another class of horizons known as Conformal Killing Horizons for linear mass function: $m(v) = \mu v$
- Conformal Killing horizon is a null hyper surface where a Conformal Killing vector becomes null

$$\mathcal{L}_\xi g_{ab} = \nabla_a \xi_b + \nabla_b \xi_a = 2\lambda g_{ab} = \frac{2}{D}(\nabla \cdot \xi) g_{ab}$$

$$\nabla_a \left(\xi^b \xi_b\right) = -2\kappa_1 \xi_a; \quad \xi^b \nabla_b \xi^a = \kappa_2 \xi^a; \quad -\frac{1}{2}\left(\nabla^a \xi^b\right)\left(\nabla_a \xi_b\right) = (\kappa_3)^2$$

- For Killing vectors all definitions coincide on KH. For CKV only the first def. remains the same under conformal transformation.
- Zeroth law can be established on CKH as $\mathcal{L}_\xi(\kappa_2 - 2\lambda) = 0 = \mathcal{L}_\xi \kappa_1$

Jacobson-Kang, CQG (1993); Sultana-Dyer, JMP (2004)

Proof of existence of CKV for General Vaidya-like spacetimes

Work in progress with Ritwika Ghoshal and Nilay Kundu of IIT Kanpur

- Spherical case: $ds^2 = -f(r,v)dv^2 + 2dvdr + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2$

$$f(r,v) = \left(1 - \frac{2m(v)}{r} + \frac{(\alpha(v))^{2k}}{r^{2k}}\right)$$ -> Hussain type metric

- CKE with $\xi^a = \{\xi^v(v,r), \xi^r(v,r), 0, \chi\}$
- The CKE indicates for CVS: $m(v) = M + \mu(v - v_0)$; $q(v) = Q + \nu(v - v_0)$
- $\xi^v = \dfrac{m(v)}{M}, \quad \xi^r = \dfrac{\mu r}{M}$
- There is no solution if only any one of $m(v)$ *and* $\alpha(v)$ is kept constant!
- For Kerr-Vaidya BH also similar proof goes through and the same features obtained

More on CKV

- All these Vaidya spacetimes admit Homothetic Killing Vectors: $\nabla_c \xi^c = const$.
- In fact a converse proof exists: $ds^2 = -f(r,v)dv^2 + 2dvdr + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2$
- If there exists a HKV for the above metric then using the Einstein equation one can show that they must be of the following form: $\xi^v = cv + g(r); \ \xi^r = cr$
- This is true even for f(v, r) having a cosmological constant
- For Hussain type metrics EE without CC (conformally transformed) implies: $\mathcal{L}_\xi T_{ab} = 0$
- For linear mass, charge etc. use of vv, vr etc. components show $g(r) = 0$
- This can also be shown for Kerr-Vaidya metric

Advantages of existing a CKV/HKV

- Existence of a HKV in these spacetimes enables us to map each dynamical spacetime to a static or stationary spacetime
- Therefore we can always obtain a $\tilde{g}_{ab} = \Omega^2(x) g_{ab}$ for some non-trivial Ω where $\tilde{g}$ is static/stationary
- In this static/stationary spacetime the HKV becomes a KV
- For the choice a Kerr-Vaidya spacetime is mapped to a stationary metric $\Omega^2 = \dfrac{m(v)r}{2M^2}$
- Surface gravity: $\kappa_1 = \dfrac{a_1\left(-16\mu + \sqrt{1-16\mu} + 1\right)}{8a_0\mu}$
- This surface gravity reaches the correct limit (1/8M) for $\mu \to 0$.

Temperature of HKH

- On the HKH one can calculate the Entropy (eg. CPS methods)
- The conformally static/stationary metrics are not asymptotically flat. They are solutions of conformally transformed EEs
- Surface gravity for HKH for dynamical BH is a constant. This is remarkable and surprising! An outcome of Homothety
- Nielsen (PRD, 2016) for Vaidya black hole proposed an alternative definition compatible with the geometric surface gravity defined in terms of an asymptotic Killing observer
- $\bar{\xi}'^{a} = \frac{1}{\sqrt{-\bar{\xi}^a \bar{\xi}_a}} \bar{\xi}^a = \frac{Mr}{\sqrt{m(v)^2 r^2 - 2m(v)^3 r + 2m(v)^2 q(v)^2 - 2\mu m(v) r^3}} \bar{\xi}^a; \quad \bar{\kappa}_1 \simeq \frac{\mu}{M} + \frac{4\mu^2}{M}$
- **Temperature of a CKH needs to be calculated directly to see if it matches with the definitions**
- **A hint can be obtained by estimating the Hawking temperature**

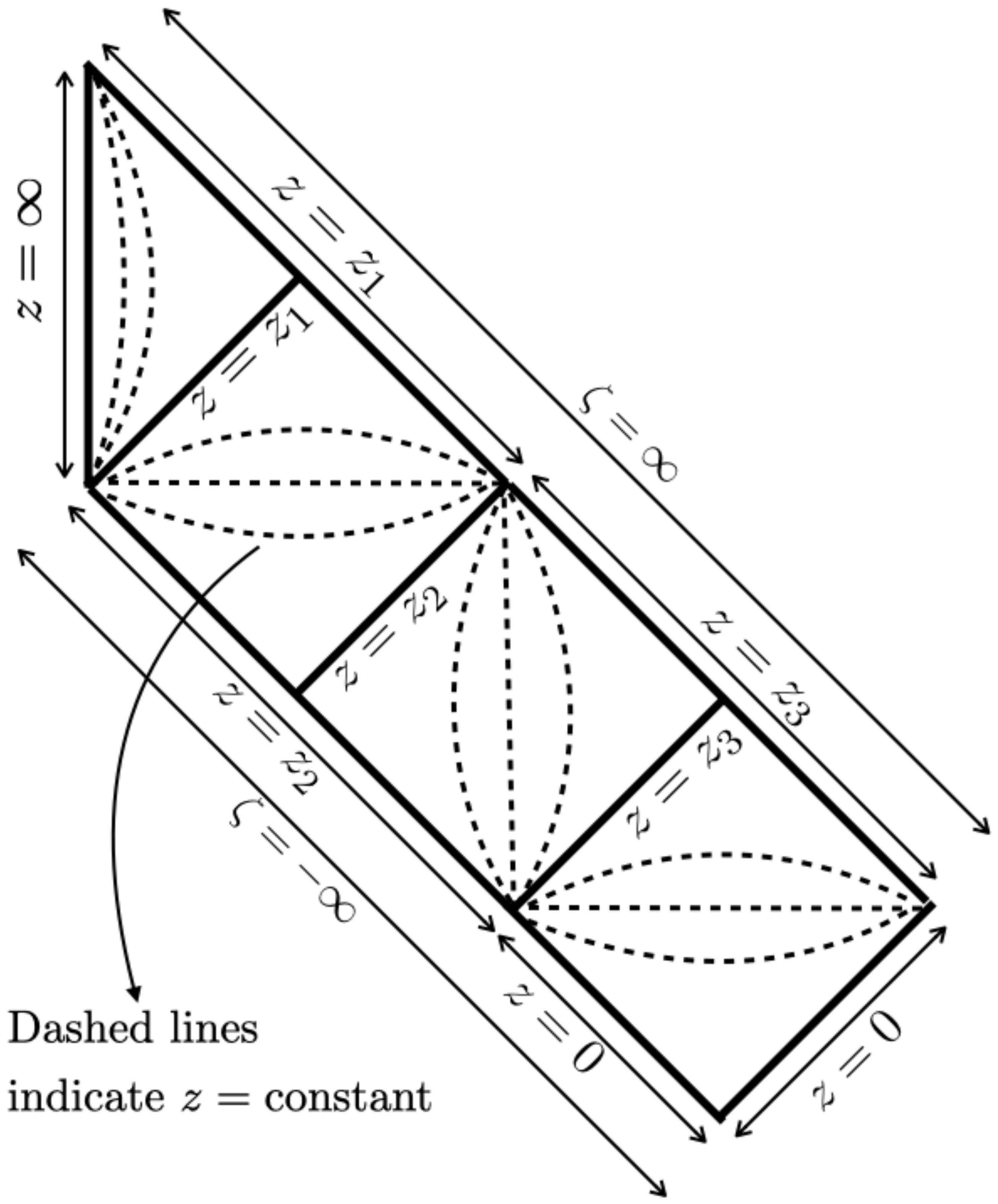
Hawking effect in self-similar Vaidya spacetimes

There are curvature singularities for $z=z_2$ and z_3 is a Cauchy horizon

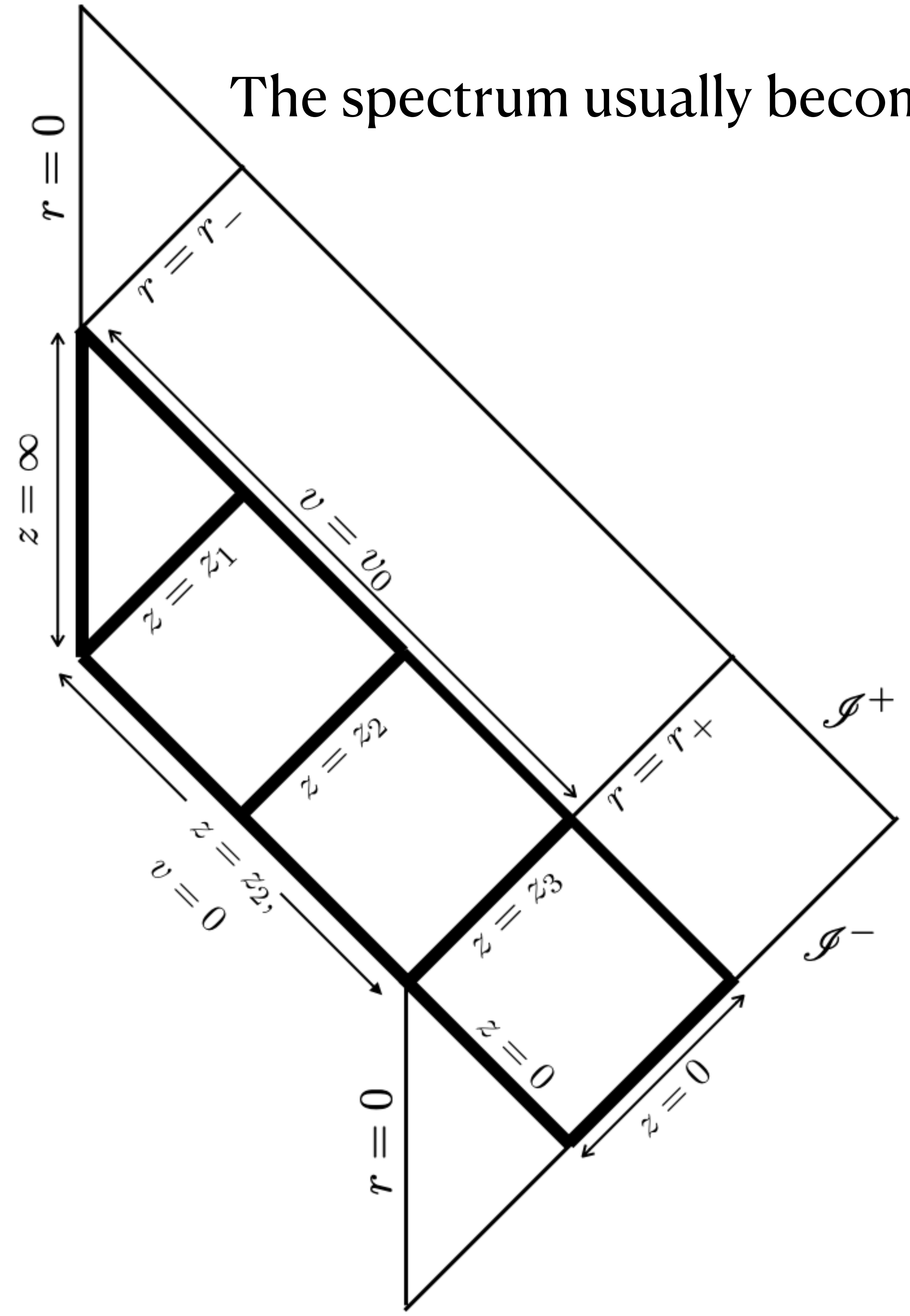
Study of particle creation is possible as we have null infinities with HKV

The spectrum usually becomes quasi-thermal

Hiscock et al. (1982)



Maximum analytic extension of self-similar VRN spacetime



VRN sptm with null infinities obtained by attaching a RN spacetime along the CH

Summary & Outlook

- BHM can be established for dynamical BHs in different setups
- A large class of Vaidya solutions contain Homothetic Killing horizons and they are useful to study the properties of dynamical spacetimes
- The HKH are nor trapped neither marginally trapped. It is difficult to directly use many standard setups to establish BHM at CKH (eg. Kodama vector and Misner-sharp energy)
- To study the role of such symmetries, particle creation may be studied. It may shed light on the meaning of temperature of such systems
- Insights from non-equilibrium statistical mechanics may be useful. Entropy production etc. may be incorporated
- ‘Quantum’ approaches need to be explored (canonical quantizations techniques by Friedel and others). Use of Quantum Thermodynamics for studying BH evaporation may provide interesting insights on dynamical BHs.

Thank You!