

Quantum Incompatibility

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The main goal is to identify

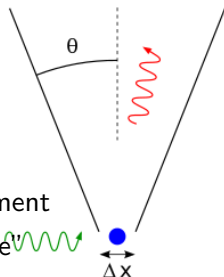
Quantumness

Quantumness = features specific to quantum theory, independent of particular physical implementations

- **Inside approach:** Study operational limitations assuming quantum theory is correct
- **Outside approach:** Characterize quantum theory within the broader framework of operational theories

Heisenberg argued that position and momentum cannot be measured simultaneously.

- A heuristic argument based on a thought experiment
- No precise definitions of “error” and “disturbance”

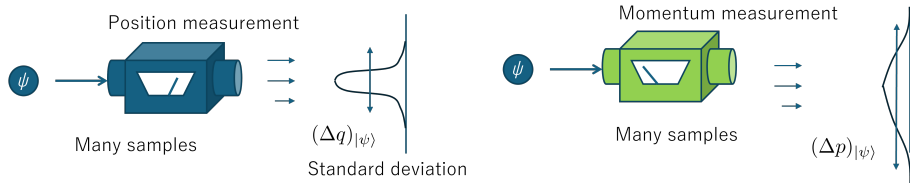


The “Proof” That Followed

One year later, E. H. Kennard showed

$$(\Delta q)_{|\psi\rangle} \cdot (\Delta p)_{|\psi\rangle} \geq \frac{\hbar}{2}$$

What do $(\Delta q)_{|\psi\rangle}$ and $(\Delta p)_{|\psi\rangle}$ mean?

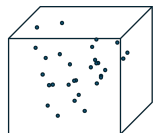


- This is not about simultaneous measurements
- It concerns two individual measurements
- Message: No quantum state yields definite outcomes for both position and momentum

Inevitable Probability

In quantum theory, measurement outcomes are probabilistic.
Is this probabilistic nature the essence of quantumness?

In statistical physics:



Partial knowledge is available

$$p(\mathbf{x}, \mathbf{p}) = \frac{e^{-\frac{1}{kT} H(\mathbf{x}, \mathbf{p})}}{Z(T)}$$

Probability distribution

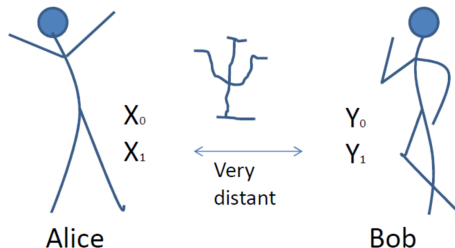
10^{23} molecules

- Statistical physics: Incomplete knowledge of underlying parameters
- Quantum physics: No access to underlying deterministic parameters

Is this the essential difference?

Bell's Inequality (CHSH)

X_0, X_1, Y_0, Y_1 : dichotomic devices (outcomes taking values ± 1)



Alice measures X_n

Bob measures Y_m : $C_{nm} = \langle \Psi | X_n \otimes Y_m | \Psi \rangle$ (correlation)

$K := C_{00} + C_{01} + C_{10} - C_{11} \leq 2$ (if governed by hidden parameters)

However, quantum theory allows $K = 2\sqrt{2} > 2$

The probability is not due to ignorance of hidden parameters

What Have We Learned?

- In quantum theory, outcomes are probabilistic
- This probability is not due to ignorance of hidden parameters
- It is dangerous to use terms like “true outcome value”.
- The probability itself is outcome.

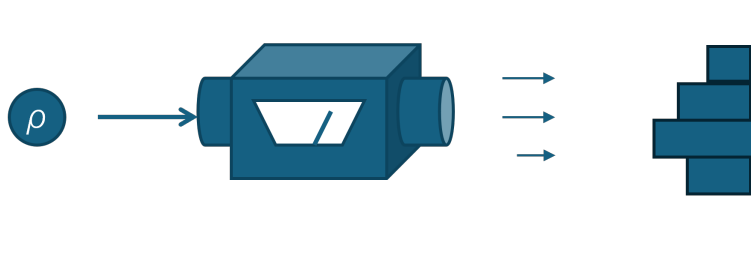
If probabilities are fundamental, what, then, is a proper definition of “joint measurement” of observables?

Observable

State: preparation procedure of a physical system

Observable: measurement procedure

Outcome: probability distribution of measurement results



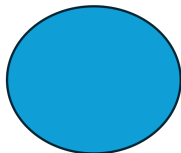
An observable is a map that assigns a probability distribution to each state.

Observable

State: preparation procedure

Observable: measurement device

Outcome: probability distribution



state space



space of probability
distributions

An observable maps states to probability distributions.

Observable as a Map

State space: the set of all density operators on \mathcal{H}

$$\mathcal{S}(\mathcal{H}) = \{\rho \mid \text{tr}[\rho] = 1, \rho \geq 0\}$$

An observable must respect convex mixtures:

$$\begin{aligned}\rho_1 &\mapsto \{P_{\rho_1}(x)\}_{x \in \Omega} \\ \rho_2 &\mapsto \{P_{\rho_2}(x)\}_{x \in \Omega} \\ \Rightarrow \quad p\rho_1 + (1-p)\rho_2 &\mapsto \{pP_{\rho_1}(x) + (1-p)P_{\rho_2}(x)\}_{x \in \Omega}\end{aligned}$$

Such map is represented by a positive-operator-valued measure (POVM)

$$\mathbf{A} = \{A(x)\}_{x \in \Omega}$$

- $A(x) \geq \mathbf{0}$
 - $\sum_{x \in \Omega} A(x) = \mathbf{1}$
- $$\rho \mapsto P_\rho(x) = \text{tr}[\rho A(x)].$$

Joint measurement (a possible definition)

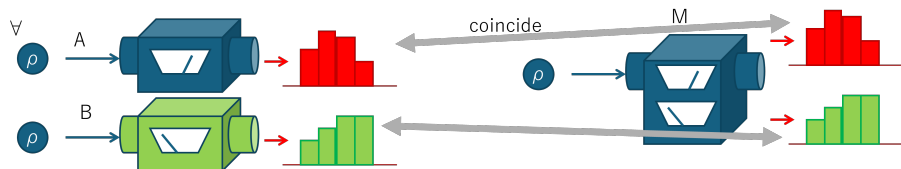
An observable is an affine map from the state space to outcome space.

Definition 1

A pair of observables $A = \{A(x)\}_{x \in \Omega_1}$ and $B = \{B(y)\}_{y \in \Omega_2}$ is jointly (simultaneously) measurable, **compatible**

\Leftrightarrow There exists an observable $\mathbf{M} = \{M(x, y)\}_{(x, y) \in \Omega_1 \times \Omega_2}$ such that for any $\rho \in \mathcal{S}(\mathcal{H})$ and $x \in \Omega_1$ and $y \in \Omega_2$,

$$\sum_{y' \in \Omega_2} \text{tr}[\rho M(x, y')] = \text{tr}[\rho A(x)], \quad \sum_{x' \in \Omega_1} \text{tr}[\rho M(x', y)] = \text{tr}[\rho B(y)].$$



Existence of compatible observables

Let A and B satisfy

$$[A(x), B(y)] = \mathbf{0} \quad \text{for all } x, y$$

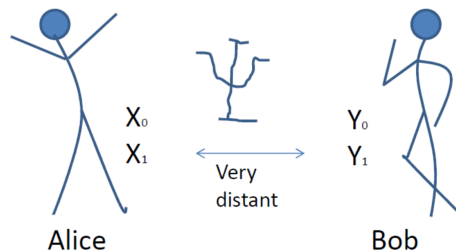
$\mathbf{M} = \{M(x, y)\}$ defined by $M(x, y) = A(x)B(y)$ is a joint observable of A and B .

(There are many other compatible pairs.)

How about incompatible pairs?

Incompatible pairs through CHSH

X_0, X_1, Y_0, Y_1 : dichotomic devices (outcomes taking values ± 1)



Alice measures X_n
Bob measures Y_m : $C_{nm} = \langle \Psi | X_n \otimes Y_m | \Psi \rangle$ (correlation)

$$K := C_{00} + C_{01} + C_{10} - C_{11} > 2$$

Incompatible pairs through CHSH

Game:

- (i) Alice randomly chooses X_0 or X_1 and performs the measurement of chosen observable.
- (ii) Bob guesses the choice of Alice (X_0 or X_1)

Bob's strategy: (assuming that Y_0 and Y_1 are compatible with a joint observable $\mathbf{M} = \{M(x, y)\}$)

- (a) Bob measures \mathbf{M} .
- (b) If the outcome (x, y) is $x = y$, Bob guesses Alice's choice as X_0 .
If the outcome (x, y) is $x \neq y$, Bob guesses Alice's choice as X_1 .

Then success probability is

$$P_{\text{success}} \geq \frac{K}{4} > \frac{1}{2}.$$

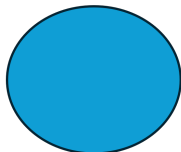
It violates causality! Y_0 and Y_1 are incompatible

Observable

State: preparation procedure

Observable: measurement device

Outcome: probability distribution



state space



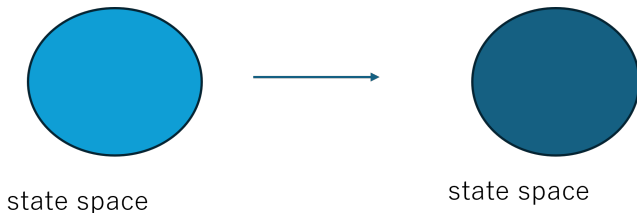
space of probability
distributions

An observable maps states to probability distributions.

Channel

State: preparation procedure

Channel: Dynamics



A quantum channel maps quantum states to quantum states.

$$\Lambda : \mathcal{S}(\mathcal{H}) \rightarrow \mathcal{S}(\mathcal{K})$$

affine and extendable to composite system (Trace-preserving CP map).

An observable is a kind of a channel such that its output states are always diagonal. (Output state space is a state space on an Abelian algebra)

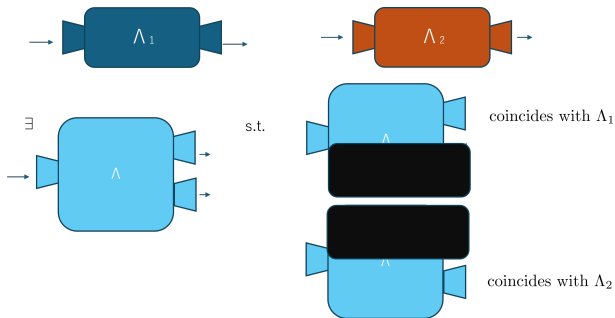
(In)compatibility

Definition 2

A pair of channels $\Lambda_1 : \mathcal{S}(\mathcal{H}) \rightarrow \mathcal{S}(\mathcal{K}_1)$ and $\Lambda_2 : \mathcal{S}(\mathcal{H}) \rightarrow \mathcal{S}(\mathcal{K}_2)$ is compatible

\Leftrightarrow There exists a channel $\Lambda : \mathcal{S}(\mathcal{H}) \rightarrow \mathcal{S}(\mathcal{K}_1) \otimes \mathcal{S}(\mathcal{K}_2)$ such that for all ρ ,

$$\text{tr}_{\mathcal{K}_2}(\Lambda(\rho)) = \Lambda_1(\rho), \quad \text{tr}_{\mathcal{K}_1}(\Lambda(\rho)) = \Lambda_2(\rho).$$

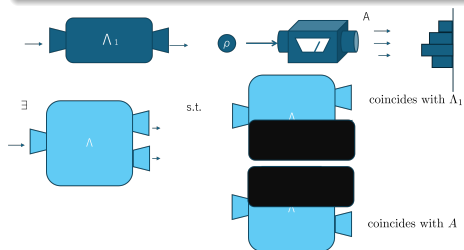


Definition 3

A channel $\Lambda_1 : \mathcal{S}(\mathcal{H}) \rightarrow \mathcal{S}(\mathcal{K}_1)$ and an observable $A = \{A(x)\}_{x \in \Omega}$ is compatible

\Leftrightarrow There exists a generalized channel (called an instrument)
 $\Lambda : \mathcal{S}(\mathcal{H}) \rightarrow \mathcal{S}(\mathcal{K}_1) \otimes P(\Omega)$ (which maps ρ to a family of subnormalized states $\{\Lambda_x(\rho)\}_{x \in \Omega}$, i.e., $\Lambda_x(\rho) \geq \mathbf{0}$ for each $x \in \Omega$) such that for all ρ ,

$$\sum_{x \in \Omega} \Lambda_x(\rho) = \Lambda_1(\rho), \quad \text{tr}_{\mathcal{K}_1}(\Lambda_x(\rho)) = \text{tr}[\rho A(x)].$$



- Now that we have defined the incompatibility.

Some natural questions arise.

- Which sets of operations (channels, observables) are (in)compatible?
 - Necessary and sufficient conditions are not known.
- How can we test the incompatibility?
 - How can we confirm that there is no joint channel?
 - Do we need all states to test compatibility?

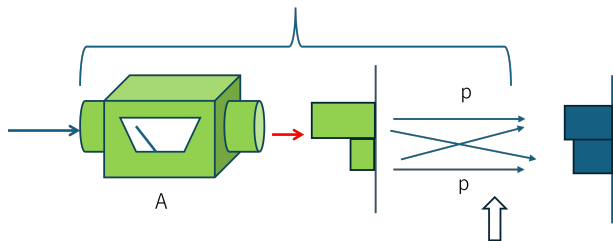
Example: Observable-Observable

A qubit:

$$A_1 = \sigma_z = \{|1\rangle\langle 1|, |-1\rangle\langle -1|\}, \quad A_2 = \sigma_x = \{|+\rangle\langle +|, |-\rangle\langle -|\}$$

A_1 and A_2 are incompatible.

noise-added (coarse grained) observable



Post processing: coarse graining
 $1-p$: noise strength

But if you add noise, they become compatible.

$$(2p_z - 1)^2 + (2p_x - 1)^2 \leq 1$$

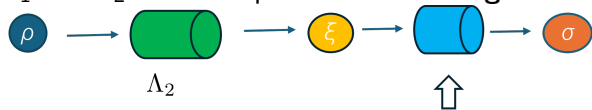
One is not very noisy (i.e., informative), then another must be very noisy.

Example: Channel-Channel

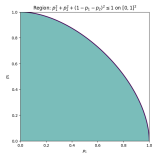
A qubit:

$$\Lambda_1(\rho) = \rho, \text{quad } \Lambda_2(\rho) = \rho$$

Λ_1 and Λ_2 are incompatible: **No-cloning theorem**



Post processing: channel
1-p: strength of noise



$$p_1^2 + p_2^2 + (1 - p_1 - p_2)^2 \leq 1$$

$$\Lambda^p(\rho) = p\rho + (1 - p)\frac{\mathbf{1}}{2}$$

One is not very noisy (i.e., informative), then another must be very noisy

Noise-Information tradeoff

Suppose that Λ_1 (channel or observable) and Λ_2 (channel or observable) are compatible.

If Λ_1 is “informative”, then Λ_2 is inevitably “noisy”

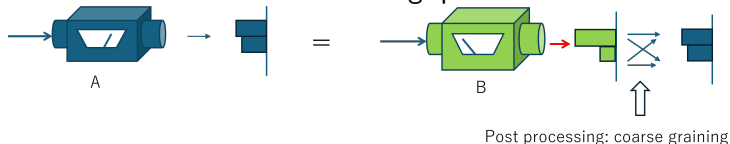
Can we understand this property without introducing any quantitative measure?

Order-relation

We consider compatibility between observables and channels.

If an observable A is “informative”, then a channel Λ compatible with A is inevitably “noisy”

“Informative” without introducing quantitative measure



B is more informative than $A \iff A \preceq B$

\preceq is a preorder on the observable space

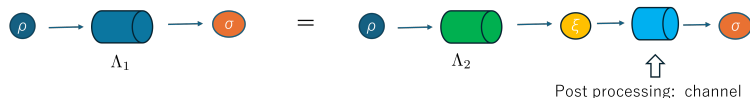
- $A \preceq A$
- $A \preceq B, B \preceq C \Rightarrow A \preceq C$

Order-relation

We consider compatibility between observables and channels.

If an observable A is “informative”, then a channel Λ compatible with A is inevitably “noisy”

The notion “noisy” without introducing quantitative measure



Λ_1 is noisier than $\Lambda_2 \iff \Lambda_1 \preceq \Lambda_2$

\preceq is a preorder on the channel space

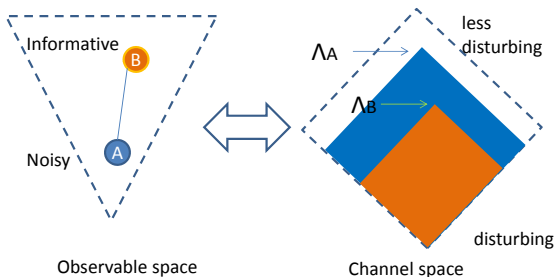
- $\Lambda \preceq \Lambda$
- $\Lambda_1 \preceq \Lambda_2, \Lambda_2 \preceq \Lambda_3 \Rightarrow \Lambda_1 \preceq \Lambda_3$

Qualitative noise-information tradeoff

Theorem

$$A \preceq B \iff \mathfrak{C}_A \supseteq \mathfrak{C}_B \iff \Lambda_B \preceq \Lambda_A$$

\mathfrak{C}_A : the set of all channels compatible with an observable A



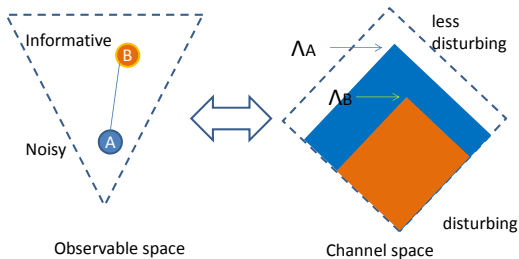
- \mathfrak{C}_A has a maximal element Λ_A .
- A more informative observable is compatible only with noisier channels
- An observable is characterized by the set of all channels compatible with it.

Qualitative noise-information tradeoff

Theorem

$$A \preceq B \Leftrightarrow \mathfrak{C}_A \supseteq \mathfrak{C}_B \Leftrightarrow \Lambda_B \preceq \Lambda_A$$

\mathfrak{C}_A : the set of all channels compatible with an observable A



- To obtain a quantitative trade-off relation, it suffices to examine the noisiness of Λ_A .
- The proof is based on Stinespring representation and its Radon-Nikodym theorem.

Λ is compatible with $A \Leftrightarrow \Lambda^*(X) = V^*(X \otimes \exists Y(x))V, Y = \{Y(x)\}; \text{POVM}$

- Now we have defined the incompatibility.
- We know that there exist incompatible pairs of operations (observable, channel).

Some natural questions arise.

- Which sets of operations (channels, observables) are (in)compatible?
 - Necessary and sufficient conditions are not known.
- How can we test the incompatibility?
 - How can we confirm that there is no joint channel?
 - Do we need all states to test compatibility?

Incompatibility test

- How can we test the incompatibility?
 - How can we confirm that there is no joint channel?
 - Do we need all states to test compatibility?

CHSH test:

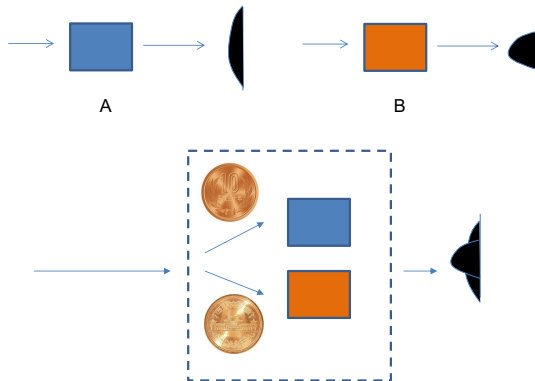
$$K = C_{00} + C_{01} + C_{10} + C_{11} > 2 \quad \Rightarrow \quad \text{incompatible}$$

There exists a test that does not need all states.

The structure behind this test is the convexity.

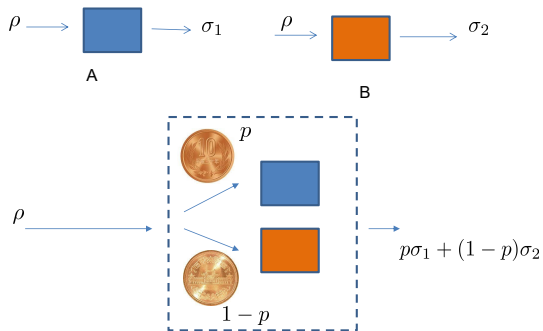
Convexity

Probabilistic mixture of two observables



This is also an observable.

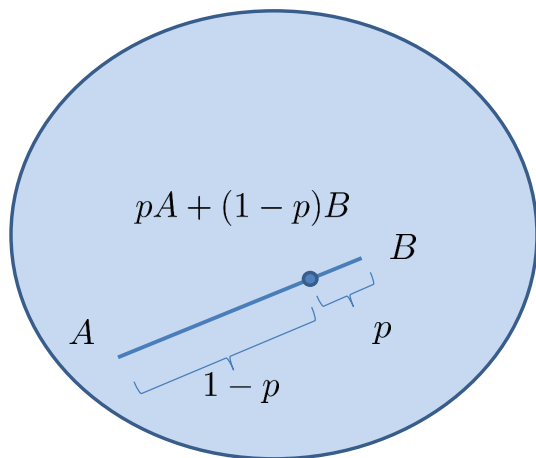
Probabilistic mixture of two channels



This is also a channel.

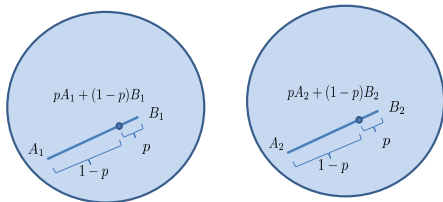
Convexity

Probabilistic mixture of two operations (observables, channels)



The set of all observables (channels) is closed under convex combinations (i.e., it forms a convex set).

Probabilistic mixture of two operations

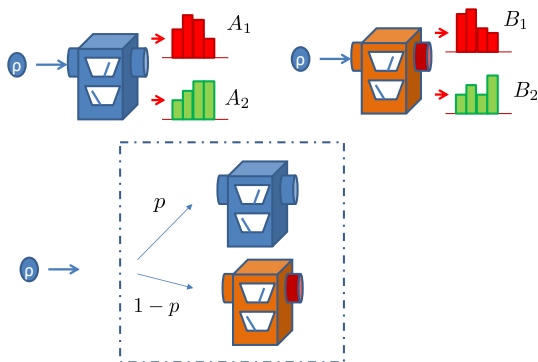


$$p(A_1, A_2) + (1-p)(B_1, B_2) = (pA_1 + (1-p)B_1, pA_2 + (1-p)B_2)$$

The set of all “pairs of observables (channels)” is also closed under convex combinations (convex set, direct sum : $\mathfrak{D} \oplus \mathfrak{D}$, $\mathfrak{C} \oplus \mathfrak{C}$)

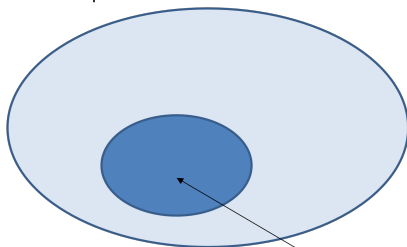
Convexity

If (A_1, A_2) is compatible and (B_1, B_2) is also compatible
 $\Rightarrow p(A_1, A_2) + (1 - p)(B_1, B_2) = (pA_1 + (1 - p)B_1, pA_2 + (1 - p)B_2)$ is also compatible



The set of compatible pairs of operations forms a convex subset.

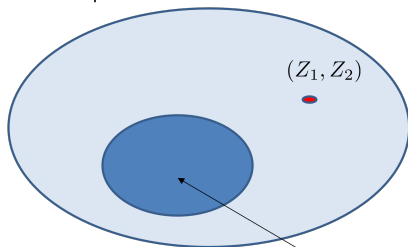
The set of all pairs



Compatible pairs

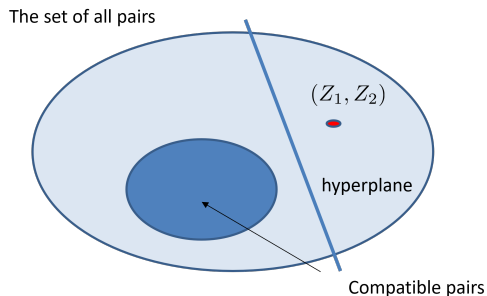
$Z = (Z_1, Z_2)$: an incompatible pair of operations

The set of all pairs



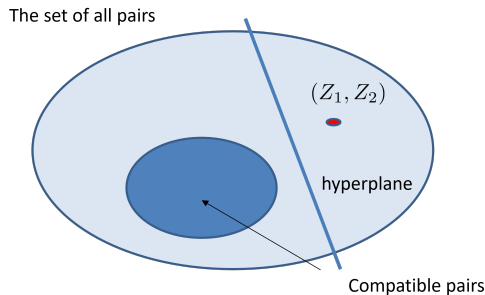
Compatible pairs

$Z = (Z_1, Z_2)$: an incompatible pair of operations



There exists a hyperplane separating Z from the set of compatible pairs.

$Z = (Z_1, Z_2)$: an incompatible pair of operations



$\exists \xi : \mathfrak{C} \oplus \mathfrak{C} \rightarrow \mathbf{R}$ Hyperplane = $\{X \mid \xi(X) = 0\}$.

$\xi(pX + (1-p)Y) = p\xi(X) + (1-p)\xi(Y)$

$\xi(X) \leq 0$ for compatible X , $\xi(Z) > 0$.

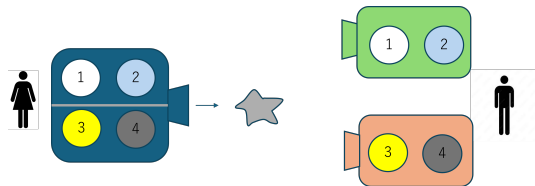
Witness is associated with a game

Operational realization of $\xi : \mathfrak{C} \oplus \mathfrak{C} \rightarrow \mathbf{R}$

We treat observable-observable case.

Prior information game: $\Omega = \Omega_1 \cup \Omega_2$, where $\Omega_1 = \{(x, 1) \mid x \in X_1\}$, $\Omega_2 = \{(y, 2) \mid y \in X_2\}$. Bob has $A_1 = \{A_1(x)\}_{x \in X_1}$ and $A_2 = \{A_2(x)\}_{x \in X_2}$.

- (i) Alice chooses $z = (x, i) \in \Omega$ with probability $p(z) = p(x, i)$ and sends $\sigma_{x,i}$ to Bob. (Denote $\rho_{x,i} = p(x, i)\sigma_{x,i}$.)
- (ii) Alice informs Bob of i , which indicates from which subset the element was chosen.
- (iii) Bob measures the observable A_i corresponding to i and guesses x .

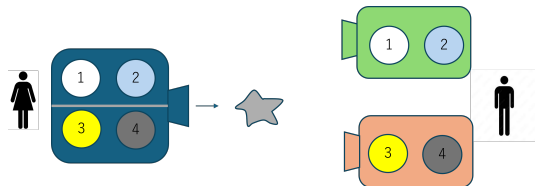


Witness is associated with a game

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- (ii) Alice informs Bob of i , which indicates from which subset the element was chosen.
- (iii) Bob measures the observable A_i corresponding to i and guesses x .

$$P_{\text{success}}(\{\rho_{x,i}\}, \mathbf{A}) = \sum_i \sum_x \text{tr}[\rho_{x,i} A_i(x)]$$

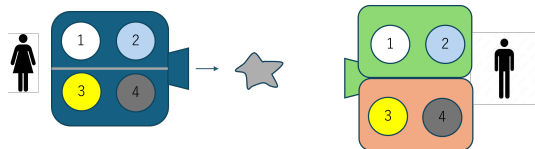


Witness is associated with a game

Post information game:

- (i) Alice chooses $z = (x, i) \in \Omega$ with probability $p(z) = p(x, i)$ and sends $\sigma_{x,i}$ to Bob.
- (ii') Bob measures a single observable $G = \{G(x, y)\}$ which is a joint measurement of compatible pair $\mathbf{B} = \{B_1, B_2\}$.
- (iii') Alice informs Bob of i , which indicates from which subset the element was chosen.
- (iv') From the measurement outcome (x, y) in (ii') and the value of i , Bob guesses (x, i) .

$$P_{\text{success}}(\{\rho_{x,i}\}, \mathbf{B}) = \sum_i \sum_x \text{tr}[\rho_{x,i} B_i(x)], \quad \mathbf{B} \in \mathcal{C} \quad \text{compatible}$$



Theorem

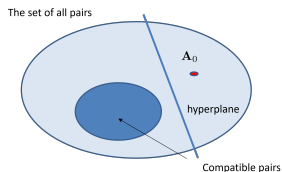
Any witness ξ can be written as

$$\xi(\mathbf{A}) = \alpha(P_{\text{success}}(\{\rho_{x,i}\}, \mathbf{A}) - \beta) + \delta,$$

where

- $\{\rho_{x,i}\}$ is a family of states.
- $P_{\text{success}}(\{\rho_{x,i}\}, \mathbf{A})$ is the guessing probability in a prior-information game using \mathbf{A} .
- $\alpha > 0$, β , and δ are constants.
- β is the maximal guessing probability in the posterior-information game:
$$\beta = \max_{\mathbf{B}: \text{compatible}} P_{\text{success}}(\{\rho_{x,i}\}, \mathbf{B}).$$

- For any incompatible pair of observables A_0 , there exists a witness (hyperplane) ξ that takes $\xi(A_0) > 0$ and $\xi(B) \leq 0$ for all compatible B .
- Any witness can be realized by a prior-information game.

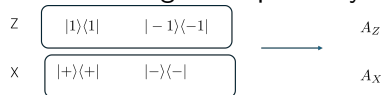


- For any incompatible pair A_0 , there exists a test detecting A_0 that needs only finite numbers of states.
- For every incompatible pair of operations, there exists a game in which one gains an advantage precisely because of that incompatibility.

Witness as a game

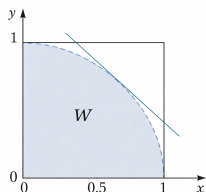
A qubit

Test detecting incompatibility between noisy σ_z and σ_x



Every state is sent with probability $1/4$.

$$\xi(A_x, A_z) = \frac{1}{4} (\langle 1|A_z(1)|1\rangle + \langle -1|A_z(-1)|-1\rangle + \langle +|A_x(+)|+\rangle + \langle -|A_x(-)|-\rangle) - \frac{1}{2} \left(1 + \frac{1}{\sqrt{2}} \right)$$



$$x = p_z - 1, \quad y = p_x - 1$$

For any incompatible \mathbf{A} , there exists a game specified by $\{\rho_{x,i}\}$ such that

$$P_{success}(\{\rho_{x,i}\}, \mathbf{A}) > \max_{\mathbf{B} \in \mathcal{C}} P_{success}(\{\rho_{x,i}\}, \mathbf{B})$$

For every incompatible pair of operations, there exists a game in which one gains an advantage precisely because of that incompatibility.

To what extent the prior information strategy is better

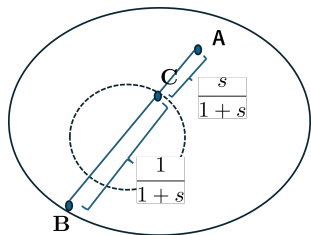
Paul Skrzypczyk, Ivan Supic, and Daniel Cavalcanti (2019)

For any pair of (incompatible) observables $\mathbf{A} = \{A_1, A_2\}$,

$$\max_{\{\rho_{x,i}\}} \frac{P_{\text{success}}(\mathbf{A}, \{\rho_{x,i}\})}{\max_{\mathbf{C}:\text{compatible}} P_{\text{success}}(\mathbf{C}, \{\rho_{x,i}\})} = 1 + s(\mathbf{A}),$$

where $s(\mathbf{A})$ is incompatibility robustness.

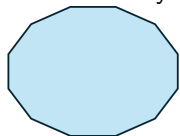
$s(\mathbf{A}) > 0 \Leftrightarrow \mathbf{A}$ is incompatible $s(\mathbf{A}) = 0 \Leftrightarrow \mathbf{A}$ is compatible



The stronger the incompatibility \mathbf{A} is, the better the prior information strategy is.

A glance at Outside approach

we consider a broad framework that includes both quantum theory and classical theory as special cases.



state space
= arbitrary convex set



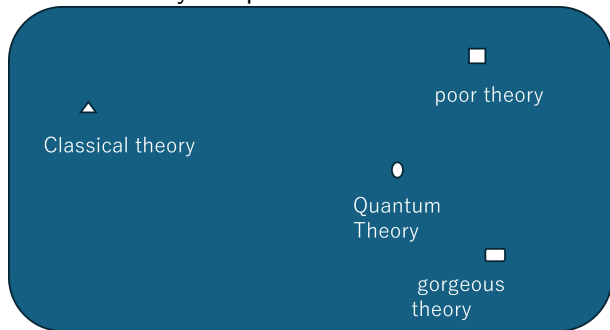
space of probability
distributions

General Probabilistic Theory

- Classical theory: State space = probability distributions (=simplex)
- Quantum theory: State space = $\mathcal{S}(\mathcal{H})$
- Hypothetical theory: State space = any other convex set

A glance at Outside approach

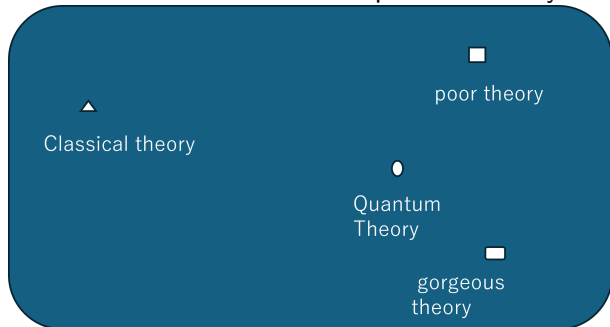
we consider a broad framework that includes both quantum theory and classical theory as special cases.



- every theory except classical theory contains incompatible pairs of operations.
- Incompatibility is not specific to the quantum theory.

State discrimination game and the quantum bound

How can we characterize the quantum theory?

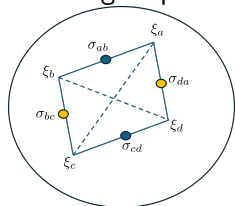


- Quantum theory has incompatible pairs
- The quantum incompatibility is not maximally strong
- $P_{success}^{prior}(A)$ is relevant with the strength

Do we have any quantum-specific bound on $P_{success}(\{\rho_{X,i}\}, A)$?

State discrimination game and the quantum bound

We consider prior-information game in which states $\{\rho_{x,i}\}$ are chosen following a specific rule.



state space

$i = 1$: Discriminate σ_{ab} and σ_{cd} using A_1

$i = 2$: Discriminate σ_{bc} and σ_{da} using A_2

$i = 1, 2$ is chosen randomly

$$\begin{aligned} P_{\text{success}}(\{\rho_{x,i}\}, A) &= \frac{1}{2} (\text{tr}[\sigma_{ab}A_1(ab)] + \text{tr}[\sigma_{cd}A_1(cd)] + \text{tr}[\sigma_{bc}A_2(bc)] + \text{tr}[\sigma_{da}A_2(da)]) \\ &\leq \frac{1}{2} + \frac{1}{4\lambda(A_1, A_2)} \leq \frac{1}{2} + \frac{1}{2\sqrt{2}} \quad \text{device-independent bound} \end{aligned}$$

This inequality connects $P_{\text{success}}(\{\rho_{x,i}\}, A)$ with $\lambda(A_1, A_2)$ the strength of incompatibility of $A = \{A_1, A_2\}$.

The inequality is violated by super-quantum theory.

Why incompatibility matters?

- Uncertainty relation
- No-cloning theorem
- Quantum cryptography (not mentioned)
- Broad framework (beyond quantum theory)

In this talk:

- Definition of incompatibility (operations as maps)
- Noise-information relation
- Witness, how to detect incompatibility

Future dream

- Natural characterization (derivation) of quantum theory based on incompatibility
- I would be very happy if there could be fruitful interaction with other fields of physics!