NUMERICAL EVALUATION OF LOOP INTEGRALS

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We report on a new method for the numerical evaluation of loop integrals in dimensional regularization developed in hep-ph/0511176 [1].

1. Introduction

Perturbative calculations play a crucial role in our current understanding of particle interactions. At the TeV energy frontier, many processes with high final-state multiplicity, number of loops and kinematic scales are important for precision studies and searches of physics beyond the Standard Model. We present a method aimed to compute loop amplitudes for such complicated processes.

The calculation of loop integrals gets involved due to the appearance of infrared singularities that must be regulated and made explicit. In multiscale processes, the analytic structure in the kinematical parameters also posses formidable difficulties both for the calculation and for the extension of the results to the regions of physical interest. Additionally, gauge theories give rise to integrals with tensor numerators, that in traditional approaches are reduced, proliferating the number of terms.

We present a method for the calculation of loop integrals based on Mellin-Barnes representations. Such representations have been already employed successfully in several complicated calculations [2,3,4,5,6,7,8,11]. In [2,3], it has been shown how to extract infrared singularities exploiting the analytic properties of Mellin-Barnes contour integrals on the complex plane.

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Following the guidelines of [3] we developed an algorithm that automatizes the procedure of extraction of infrared singularities from a given Mellin-Barnes representation. The resulting contour integrals are then expanded in power series of ϵ and evaluated by direct numerical integration on the complex plane [1]. A similar implementation has been presented in [12].

2. Outline of the method

Our starting point is the Feynman parameterization of a regulated loop integral. By repeated use of the formula

$$\frac{1}{(A_1+A_2)^{\alpha}} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} dw A_1^w A_2^{-\alpha-w}\frac{\Gamma(-w)\Gamma(\alpha+w)}{\Gamma(\alpha)},$$

the Feynman parameters can be integrated out in terms of Gamma functions, giving the Mellin-Barnes representation of the integral. As seen from the formula above, this representation will involve integrals over paths in the complex plane. The contours must be chosen such that, for each Gamma function in the integrand, all its poles lie on the same side of the contours. For divergent integrals this can only be satisfied for values of ϵ different from 0. When taking the limit $\epsilon \to 0$, some of the poles of the Gamma functions cross the contours, changing the value of the integral and thus invalidating the representation.

This feature can be used to extract the singularities in ϵ by simply accounting for the residues on the poles that crossed the contour. The final result consists in the original integral, now valid in the region $\epsilon \simeq 0$, plus the sum of the residues. These last terms involve less Mellin-Barnes integrals than the original representation. All the pieces can be safely expanded in powers of ϵ, the poles appearing in the residues, in the form of factorized Gamma functions. In [1], this procedure has been fully implemented into a set of MATHEMATICA and MAPLE routines, allowing for a fast analytic continuation.

The case of loop integrals with tensor numerators can also be efficiently handled within this approach. The representations for these integrals coincide with the one for the scalar case, evaluated with shifted space-time dimension, times a polynomial in the Mellin-Barnes variables. As the only additional factor is an analytic function, the analytic continuations has to be performed only once, keeping a general polynomial. Whole diagrams can be evaluated at the same time without resourcing to any reduction procedure.

We found that the contour integrals obtained after the analytic continuation are very well suited for direct numerical integration. When there are no masses in the internal lines, the integrands vanish fast when moving away from the real line. Gamma functions arising when integrating over the Feynman parameters are crucial for this damping. The integrals can, then, be reliably evaluated in all kinematic regions by direct integration. The only analytic continuation in the kinematical variables needed is for simple logs and powers, no polylogarithms involved.

Integrals with massive internal lines are more delicate. Internal masses generate terms linear in the Feynman variables. After integration, these produce a deficit of Gamma functions in the numerator of the integrand and additional Gamma functions in the denominator, spoiling the damping away from the real line. However, we have found that several cases of physical interest, not involving thresholds, are perfectly suited for numerical integration.

We have implemented routines that completely automatize the steps for handling tensor integrals and producing FORTRAN code to perform the numerical evaluation of the contour integrals.

3. Applications

In order to show the power of the method outlined above, we have applied it to calculate a set of one, two and three loop integrals. The studied integrals give rise to Mellin-Barnes representations of high dimensionality and, in many cases the final expressions, after the analytic continuation, involve up to hundreds of terms. This stresses the advantage of the automatic algorithm we described, since the book-keeping is done automatically and our routines perform the ϵ expansion in fractions of a minute.

To test the method, we applied it to one loop hexagon tensors. As mentioned, the method does not involve any reduction. Analytic continuation of Mellin-Barnes representations, followed by direct numerical integration proved to be perfectly suited for the evaluation of tensors of up to rank six.

We also performed several comparisons to existing results for box integrals with two and three loops, these included the on-shell, massless planar double box computed in [2], the crossed double box [3], the double box with an off-shell leg [13,6,14], and the on-shell massless triple box calculated analytically in [7]. In all these cases, our method provided fast evaluations in different kinematical regions, with errors in the numerical integration at the per mill level or better.

In [1], we presented the first calculation of the double box with two adjacent massive legs in the physical region. Results in the euclidean region agreed with the ones of [15]. Again, our codes proved to be efficient at handling this difficult integral providing results with errors typically under the 1% level for the constant pieces in ϵ.

At the three loops level, we also took a further step and performed the first calculation for a triple box with one leg off-shell. Our results are valid in all physical regions with errors in the per cent range or better.

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