Numerical Evaluation of Loop Integrals

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Tsukuba, April 22^{nd} 2006

In collaboration with Babis Anastasiou

Rationale

(Why do we need complicated loop amplitudes?)

Perturbative calculations play a key role in understanding and predicting phenomena in particle physics

Precise determination of parameters	*New phenomena*	*Complicated backgrounds*

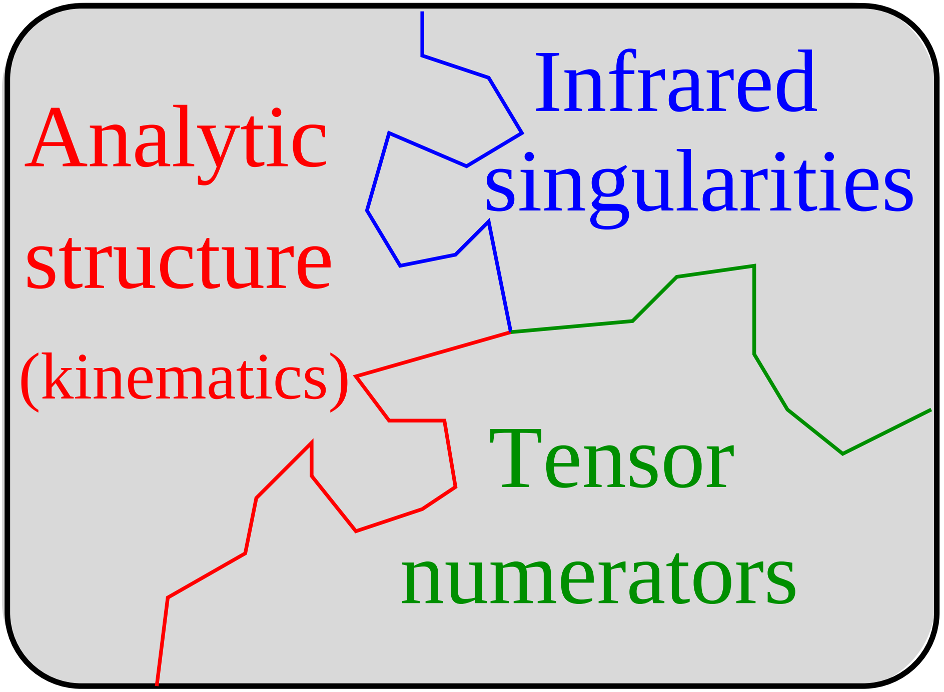
↓

Perturbative calculations for processes with higher particle multiplicities, number of loops and kinematical scales

Difficulties and challenges

Integrals must be regularized and singularities extracted

Complicated analytic continuation of results when many scales are present



Proliferate the number of terms

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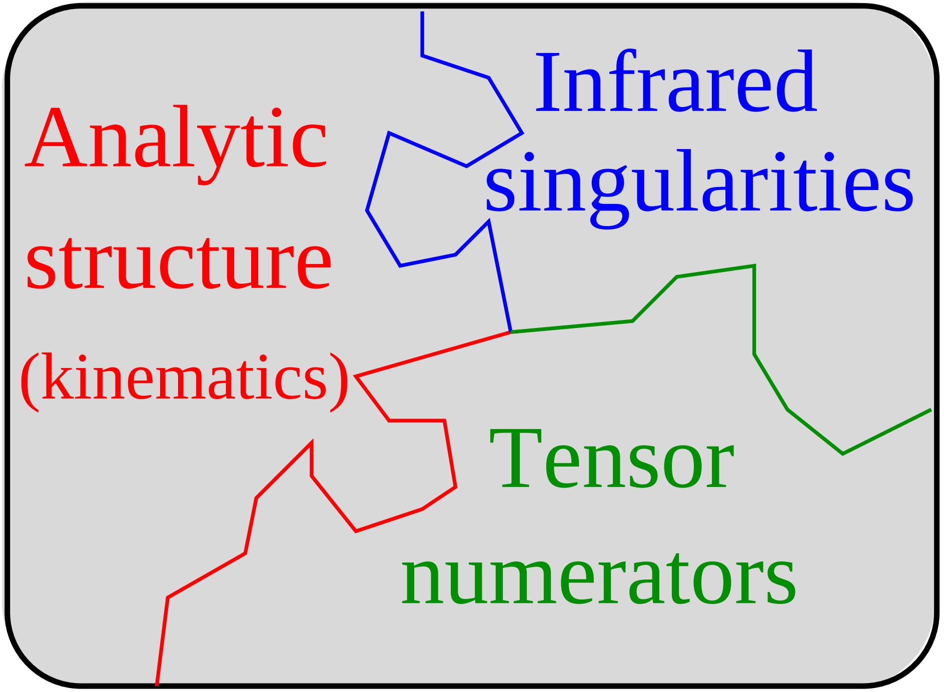


Proliferate the number of terms

...and we would like automatization

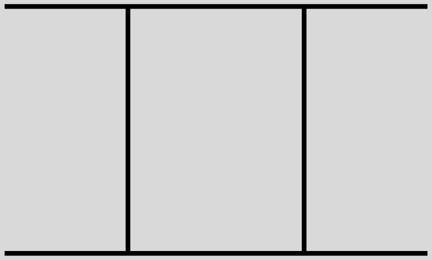
Lots of inspired work

- *Reduction to master integrals* *Laporta, Anastasiou, Lazopoulos, A.V. Smirnov, V.A. Smirnov, Tarasov, Baikov, Steinhauser,. . .*
- *Sophisticated methods to reduce one-loop integrals and handling exceptional kinematics* *Giele, Glover, R.K. Ellis, Zanderighi, Denner, Dittmaier, Binoth, Heinrich,Pilon, Schubert, Kauer, Hameren, Vollinga, Weinzerl, del Aguila, Pittau, Sopper, Nagy,. . .*
- *Sector decomposition: numerical evaluation in the euclidean region* *Binoth, Heinrich,Pilon, Schubert, Kauer, Anastasiou, Melnikov, Petriello*
- *“Twistor” developments* *Britto, Buchbinder, Cachazo, Feng, Witten, Bern, Dixon, Kosower,. . .*
- *Mellin-Barnes representations* *Czakon, V.A. Smirnov, Tausk, Veretin, Anastasiou, Tejeda-Yeomans, Heinrich, Davydychev, Ussyukina,. . .*

Thursday’s talk by G. Salam

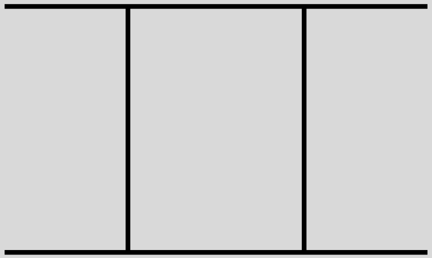
New method for the numerical evaluation of loop integrals using Mellin-Barnes representations

Loop integrals and Mellin-Barnes representations



$$\mathcal{I}_4 = \Gamma\left(4 - \frac{d}{2}\right) \int \frac{dx_1 dx_2 dx_3 dx_4}{(-s x_1 x_3 - t x_2 x_4 - i0)^{4-\frac{d}{2}}}$$

Loop integrals and Mellin-Barnes representations



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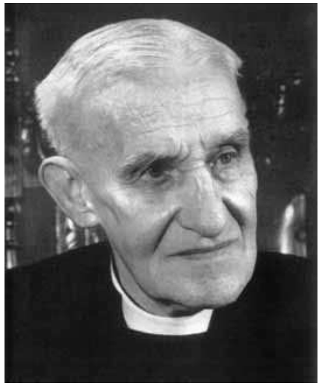


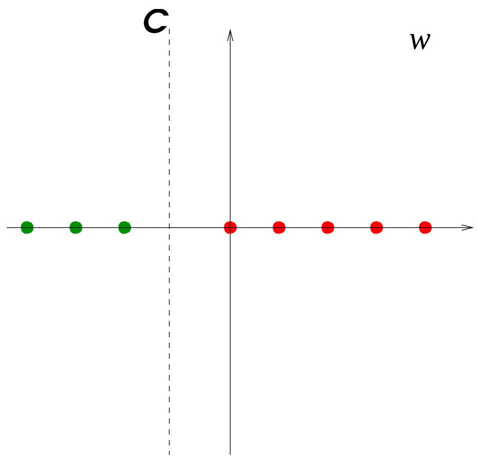
$$\frac{1}{(a+b)^\alpha} \quad = \quad b^{-\alpha} \frac{1}{2\pi i} \int_{\mathcal{C}} dw \, \left(\frac{a}{b}\right)^w \frac{\Gamma(-w)\,\Gamma(\alpha+w)}{\Gamma(\alpha)}$$

$\mathcal{C}$ separates green and red series of poles

Sum of residues reproduces the series expansion

$|\arg a - \arg b| < \pi$

All regions in s-t plane



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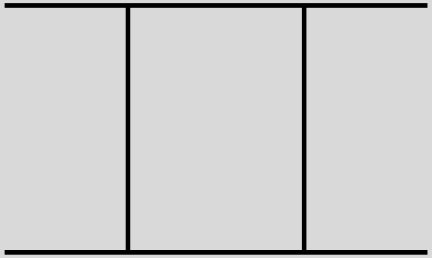

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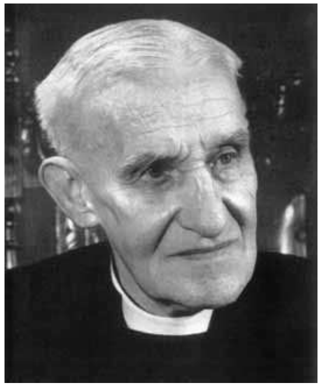

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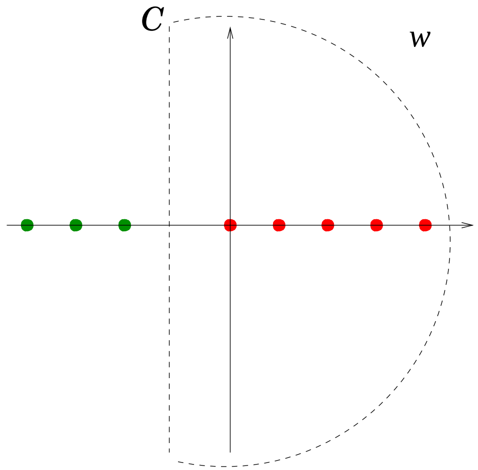

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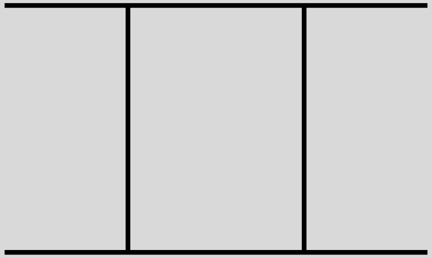

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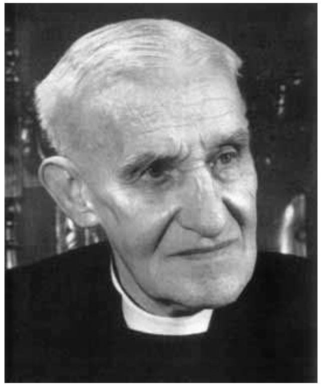

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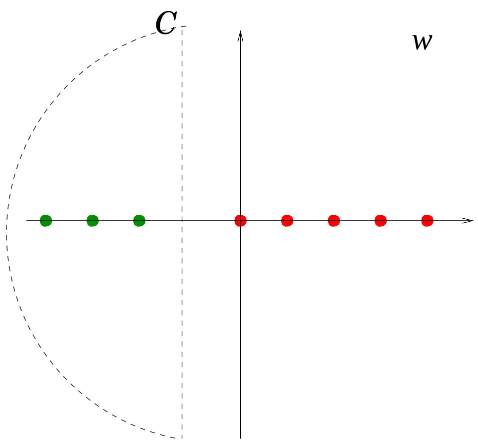

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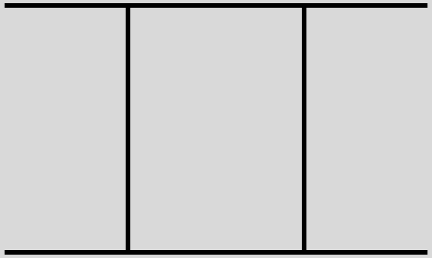

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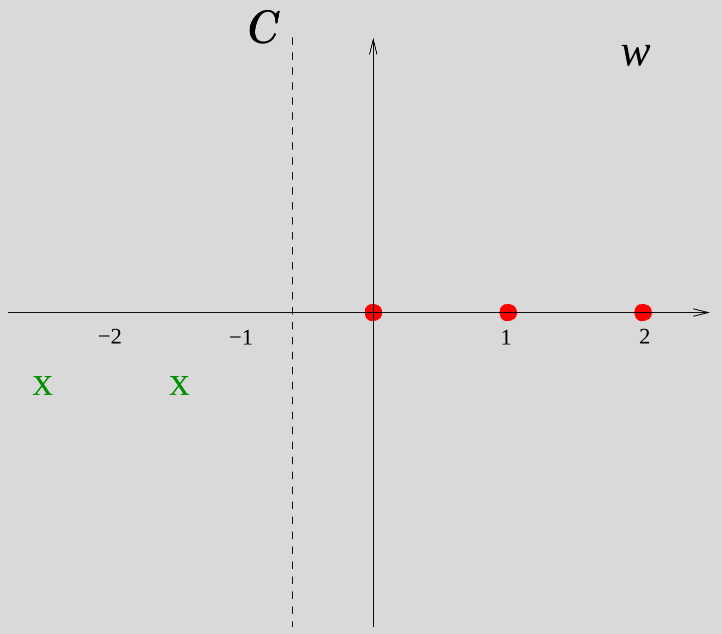
$$\mathcal{I}_4 = \int \left(\prod_{l=1}^{4} dx_l\right) \frac{1}{2\pi i} \int_{\mathcal{C}} dw \Gamma(-w)\Gamma(2+\epsilon+w)\ (-x_2 x_4 t)^w\ (-x_1 x_3 s)^{-2-\epsilon-w}$$

- contour separates poles
- Feynman parameters factorize and can be integrated out



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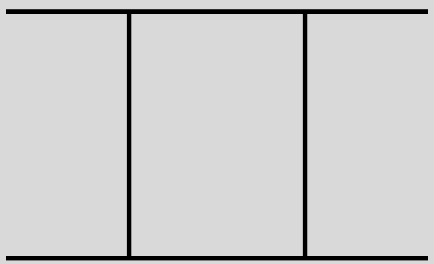

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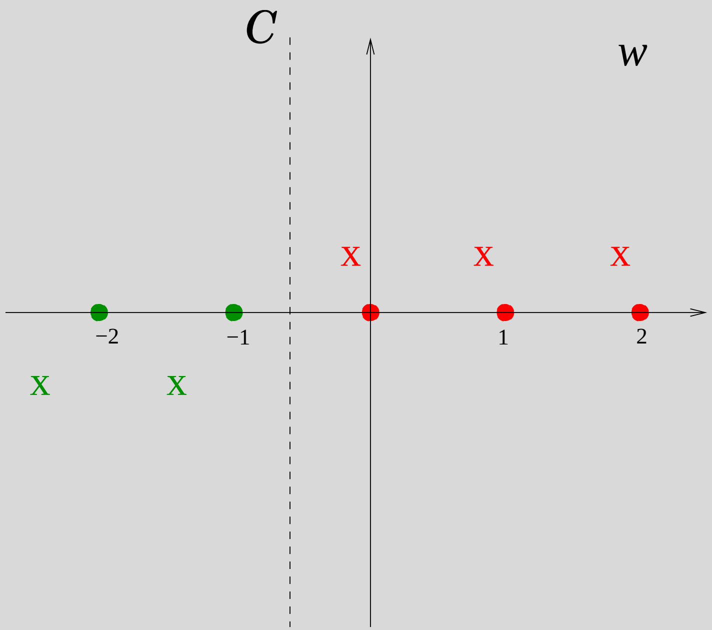
The integral only picks up the right residues if the contour separates red and green poles:

$$-1 < w < 0$$
$$-2 - w < \epsilon < -1 - w < 0$$

the representation is only valid for $\epsilon < 0$
but we need a Laurent series around $\epsilon = 0$*...*

$$\mathcal{I}_4 = \frac{C_{-2}}{\epsilon^2} + \frac{C_{-1}}{\epsilon} + C_0 + \cdots$$



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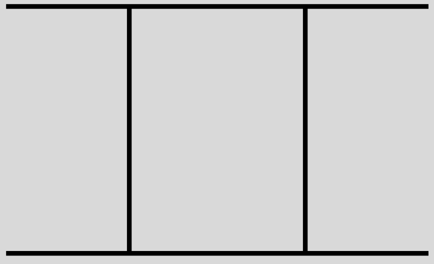

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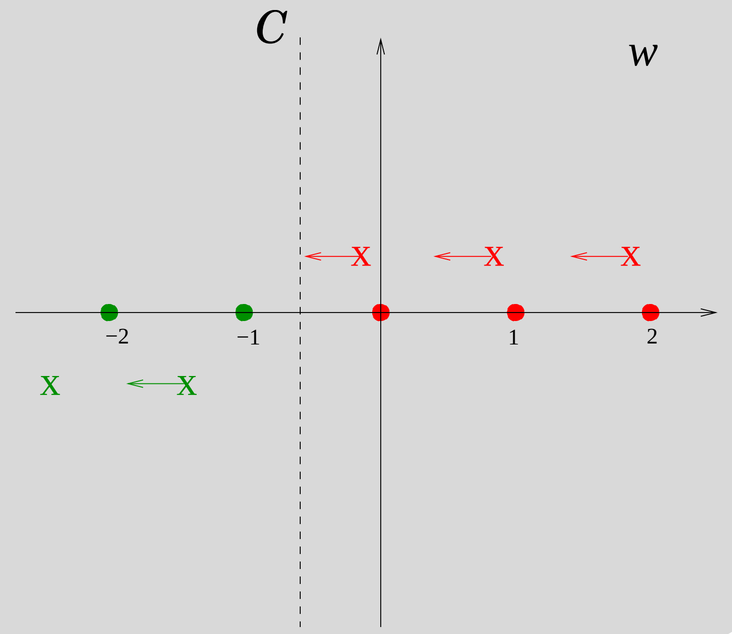

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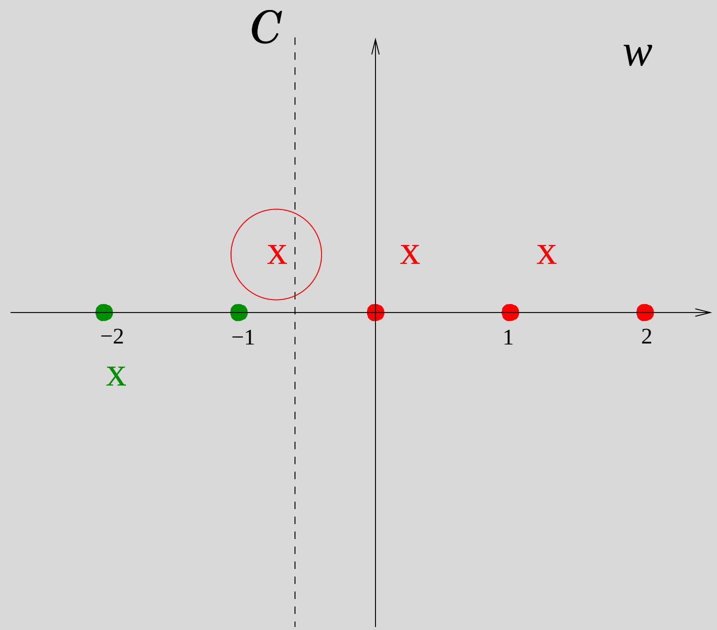

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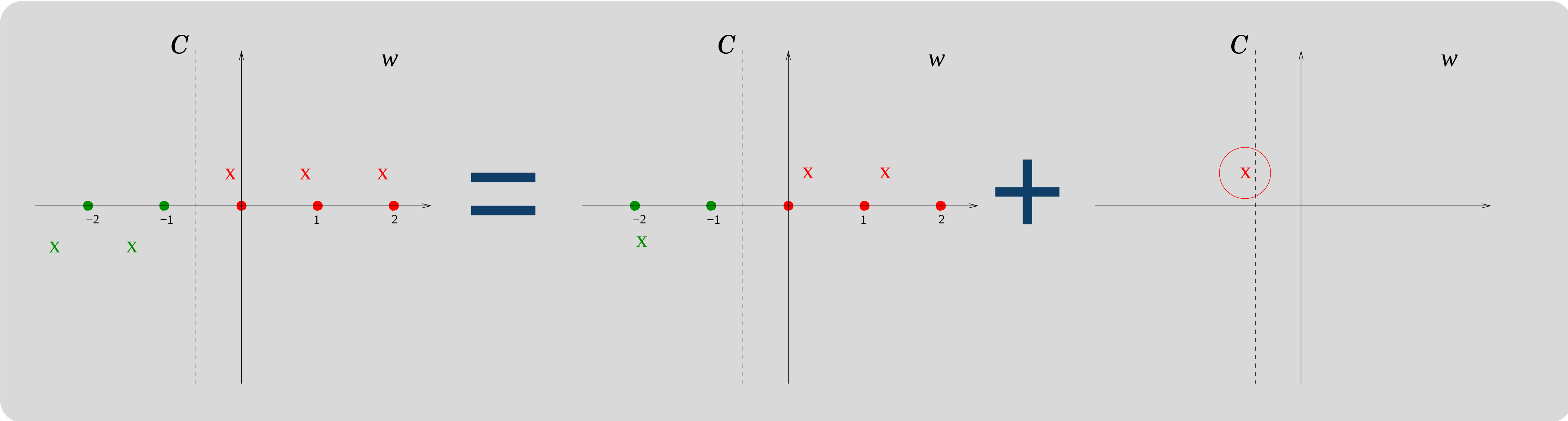

Original representation
$-3/2 < \epsilon < -1/2$

Similar to the original
$-1/2 < \epsilon < 1/2$

Single residue
$-1/2 < \epsilon < 1/2$
Simpler than original integral

Alejandro Daleo

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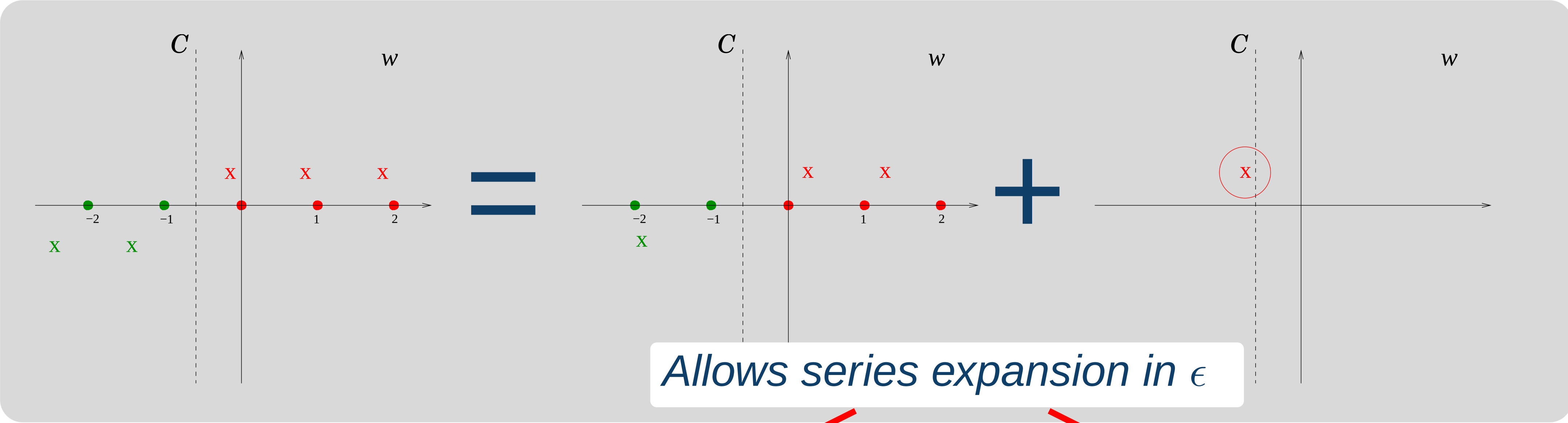

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$$\mathcal{I}_4 = -\frac{\epsilon}{\pi i}\int_{\mathcal{C}} dw\Gamma(-w)\Gamma(2+w)\Gamma^2(1+w)\Gamma^2(-1-w)\,(-t)^w\,(-s)^{-2-w} + \mathcal{O}(\epsilon^2)$$

$$+ \frac{(-t)^{-\epsilon}}{s\,t}\frac{\Gamma(-\epsilon)^2\Gamma(1+\epsilon)}{\Gamma(-2\epsilon)}\{\gamma_E - \log(-s) + \log(-t) + 2\psi(-e) - \psi(1+e)\}$$

Loop integrals and Mellin-Barnes representations

- *Iterative procedure, until reaching* $\epsilon = 0$.
- *Poles in* ϵ *appear explicitly when taking residues.*
- *Remaining integrals can be expanded in Laurent series.*
- *Straightforward to extend to more complicated loop integrals.*
- *Calculation of remaining integrals:*
  $\star$ *analytically, series resummation*
  $\star$ *very-well suited for numerical integration*
- *Whole procedure implemented in* MATHEMATICA *and* MAPLE

$$\mathcal{I}_4 = -\frac{\epsilon}{\pi i}\int_{\mathcal{C}} dw \Gamma(-w)\Gamma(2+w)\Gamma^2(1+w)\Gamma^2(-1-w)\,(-t)^{w}\,(-s)^{-2-w} + \mathcal{O}(\epsilon^2)$$
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Evaluation of integrals

- *Analytic evaluation implies the infi nite sum over residues*
  - *extremely diffi cult for multidimensional MB integrals*
  - *involved analytic continuation in kinematical variables*
- *Numerical integration on the other hand:*
  - *rapid damping of integrand along the contour when Im$(w) \to \pm\infty$ thanks to extra gamma functions coming from Feynman par. integration*
  - *trivial analytic continuation in kinematical variables: only powers and logs*

$$\mathcal{I}_4 = -\frac{\epsilon}{\pi i}\int_{\mathcal{C}} dw \Gamma(-w)\Gamma(2+w)\Gamma^2(1+w)\Gamma^2(-1-w)(-t)^w (-s)^{-2-w} + \mathcal{O}(\epsilon^2)$$

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Tensor integrals

$$I_{n,m} = \int \frac{d^dk}{i\,\pi^{\frac{d}{2}}} \frac{k^{\mu_1}\dots k^{\mu_m}}{(k+q_1)^2(k+q_2)^2\cdots(k+q_n)^2}$$

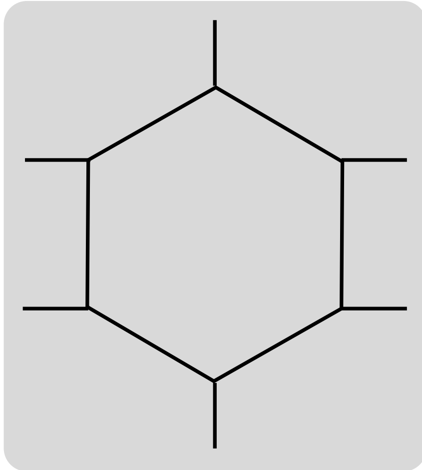
$$= \sum_{r\leq m} \int \left(\prod_i dw_i\right) \Gamma^{(scalar)}_{d\to d+r}(\vec{w})\, h^{(m,r)}(\vec{w})$$

$\Gamma^{(scalar)}_{d\to d+r}(\vec{w})$: *analogous to the representation of the scalar case with shifted dimension* $d \to d+r$

$h^{(m,r)}(\vec{w})$: *polynomial in the MB variables with tensor coefficients (external momenta). Does not affect the analytic continuation in* ϵ

-This decomposition allows to perform a unique analytic continuation in ϵ *for each topology, using a general polynomial.*
-Whole diagrams can be evaluated at the same time.

1 loop massless hexagon



- *Many important processes with six external legs at the LHC*
- *Recent calculation of 6 gluons amplitude* *[Ellis, Giele, Zanderighi]*
- *Our calculation:*
  8 dimensional MB representation
  analytic continuation for arbitrary tensors
  sampled several points in the physical region $2 \to 4$
  check in the euclidean region with sector decomposition
  check some phase space points with explicit reductions using IBP identities
- *Evaluation of tensors up to rank 6*
- *Ready to evaluate whole diagrams*

A rank 6 tensor in the physical region

$$q_2 \cdot k\, q_2 \cdot k\, q_3 \cdot k\, q_4 \cdot k\, q_5 \cdot k\, q_6 \cdot k$$

Point	c_2	c_1	c_0
P_3	$-\ 47.453378\ (0)$ $+\ i \cdot 0.000000\ (0)$	$-\ 149.099\ (6)$ $-\ i \cdot 145.128\ (4)$	$-\ 79.\ (2.)$ $-\ i \cdot 485.\ (2.)$
P_4	$-\ 3876.804240\ (0)$ $+\ i \cdot 0.000000\ (0)$	$-\ 23083.200\ (2)$ $-\ i \cdot 12161.074\ (2)$	$-\ 53835.\ (6.)$ $-\ i \cdot 72507.\ (5.)$
P_5	$-\ 52.452711\ (0)$ $+\ i \cdot 0.000000\ (0)$	$-\ 214.000\ (2)$ $-\ i \cdot 158.220\ (2)$	$-\ 214.\ (2.)$ $-\ i \cdot 663.\ (1.)$
P_7	$-\ 54.532983\ (0)$ $+\ i \cdot 0.000000\ (0)$	$-\ 172.177\ (4)$ $-\ i \cdot 161.007\ (7)$	$-\ 26.\ (2.)$ $-\ i \cdot 531.\ (2.)$

2 loop boxes

- *planar double box with one off-shell leg*

  test of analytic continuation in ϵ

  test of numerical integration feasibility

  effective 3-fold integral

  V.A. Smirnov (MB), Gehrmann and Remiddi (diff. eq.)
  non-trivial check of analytic continuation in kin. var.

- *planar double box with two adjacent off-shell legs*

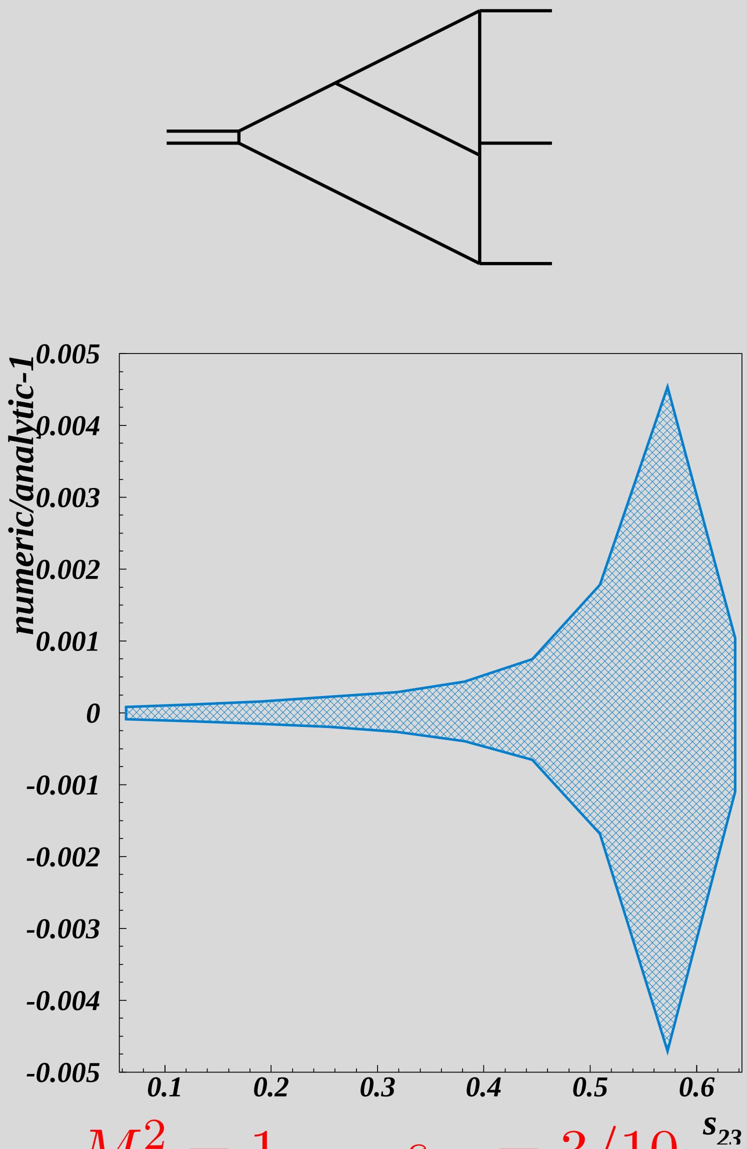
  fi rst calculation in the physical region

  relevant for heavy boson pair production at colliders

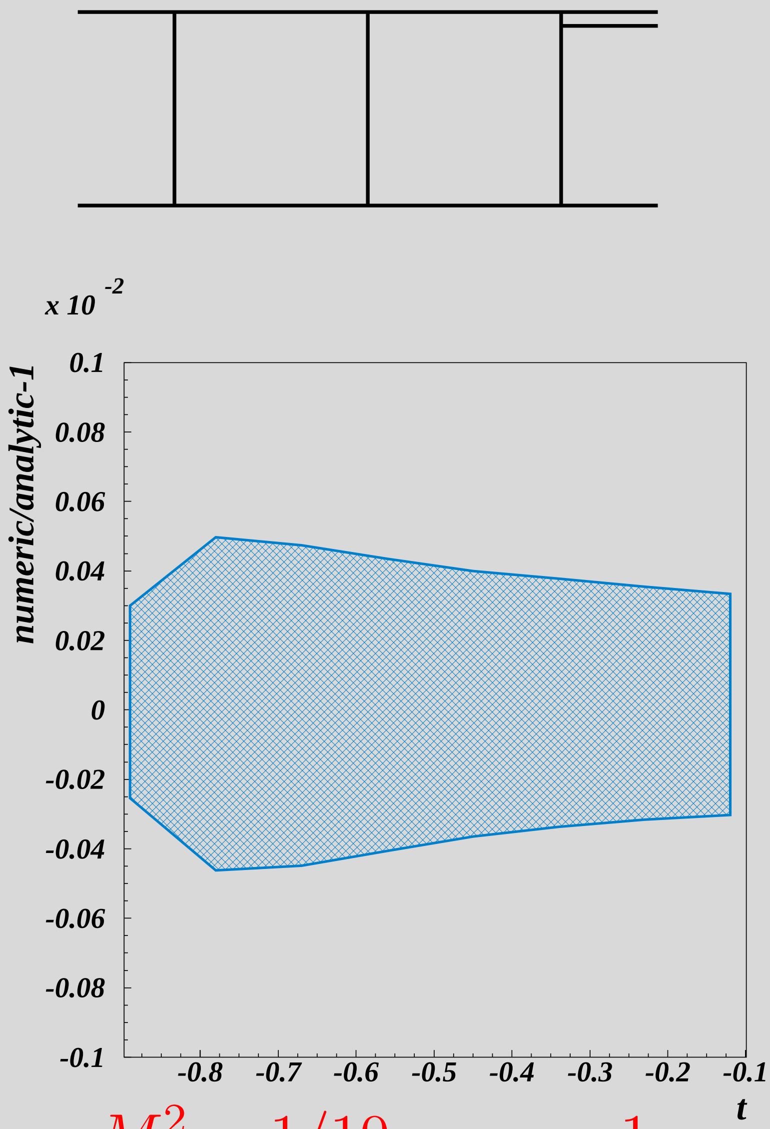
  euclidean points by Binoth and Heinrich with sec. dec.

  effective 3-fold integral
  good numerical convergence

2-Box: 1 off-shell leg

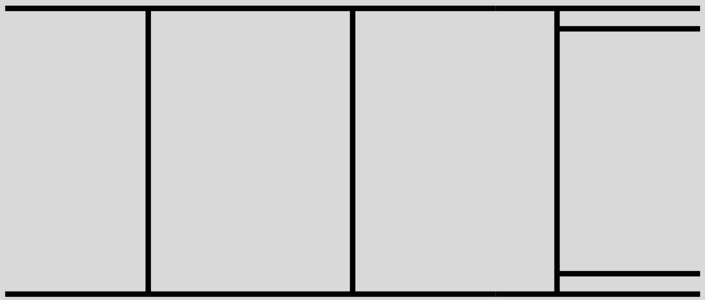


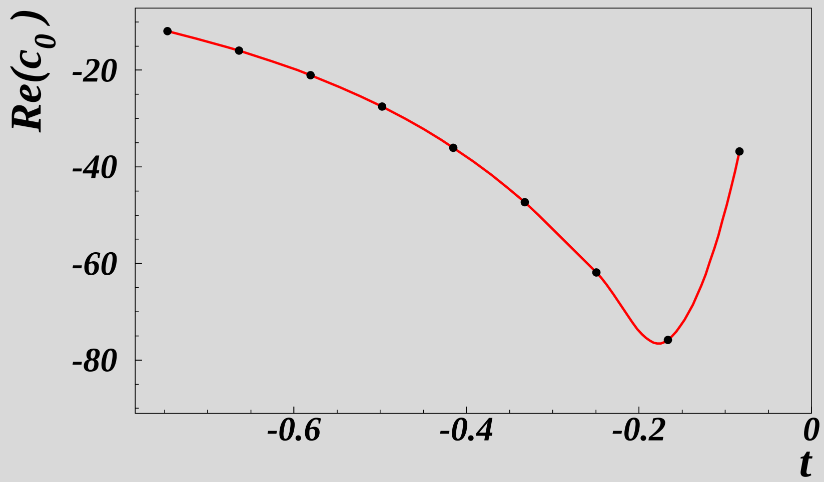
$M^2 = 1 \qquad s_{13} = 3/10$

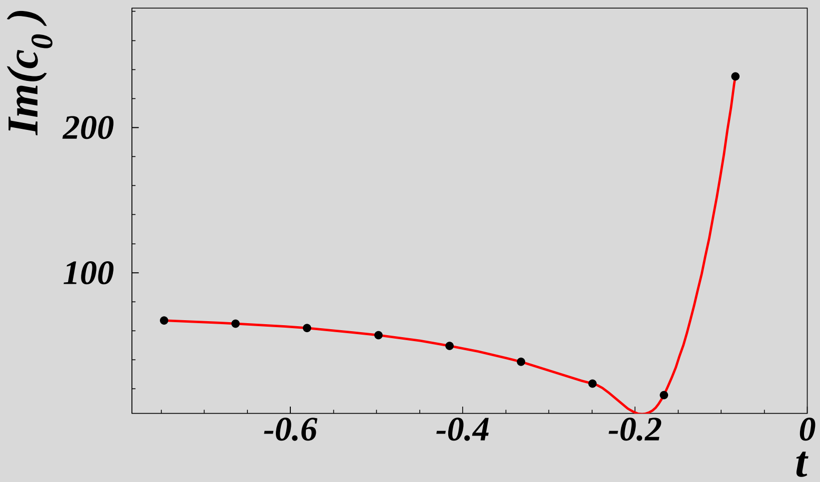


$M^2 = 1/10 \qquad s = 1$

2-Box: 2 off-shell legs





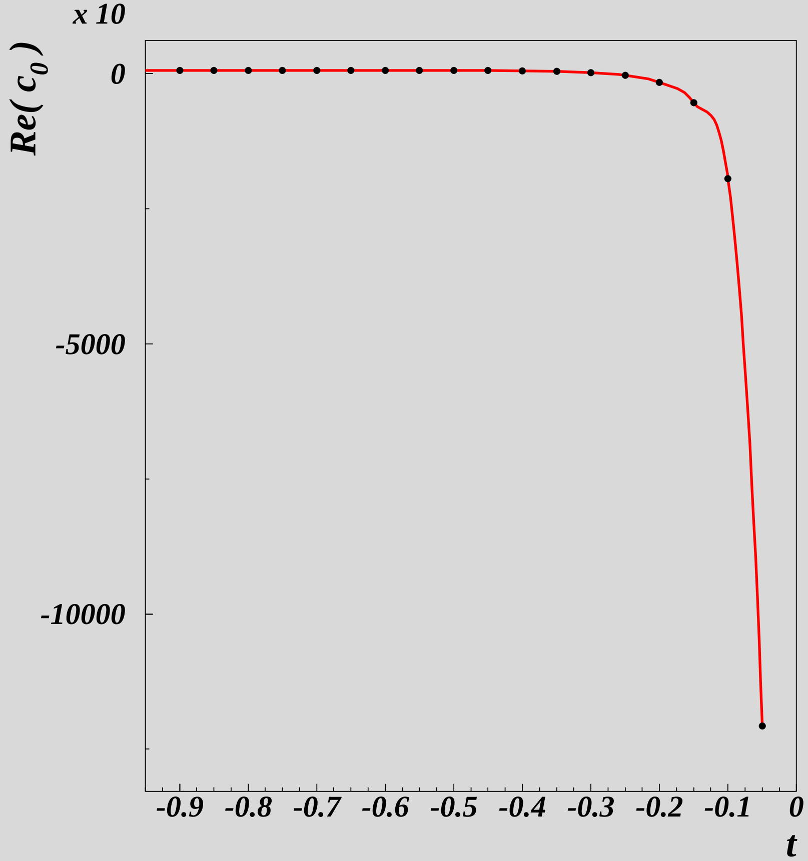


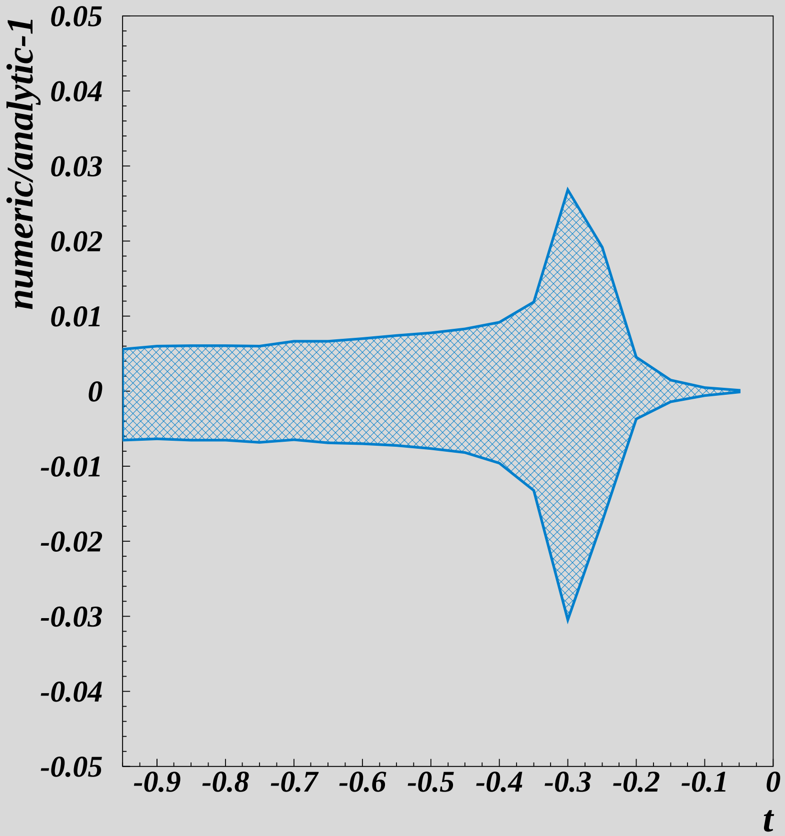
$s = 1 \qquad M_1^2 = 1/20 \qquad M_2^2 = 1/2$

3 loop boxes

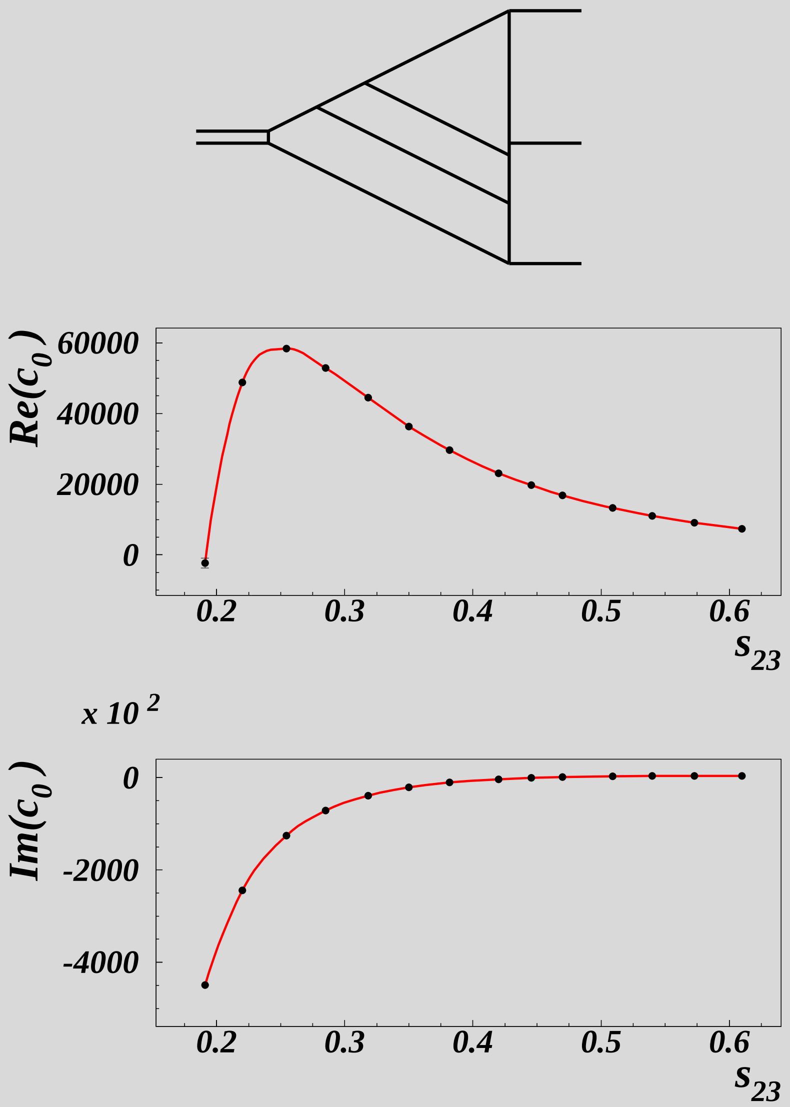
- *on-shell triple box*
  - *analytic result by V.A. Smirnov with MB technique*
  - *all physical regions*
  - *poles up to e^{-6}*
  - *effective 5-fold integral*
- *triple box with one off-shell leg*
  - *fi rst evaluation of a 3 loop box with 3 scales*
  - *evaluation in different physical regions*
  - *production of a heavy boson in association with a jet*
  - *effective 6-fold integral*

3-Box: on-shell





3-Box: 1 off-shell leg



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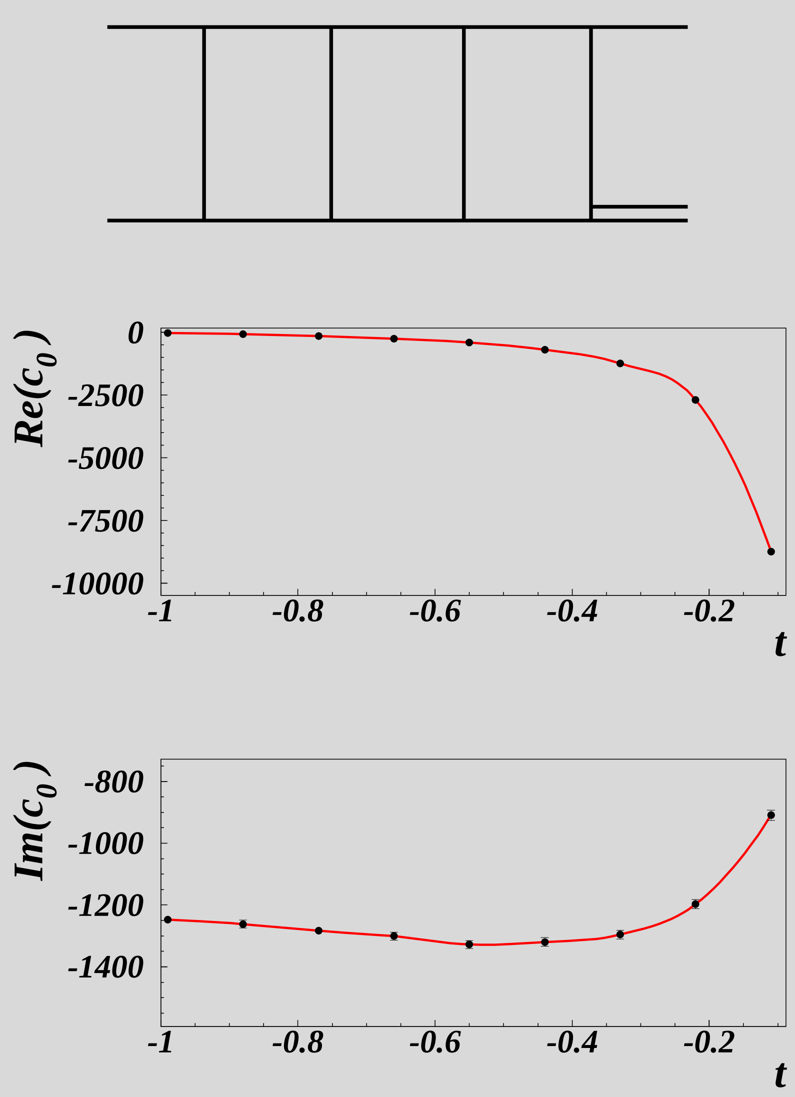

$M^2 = 1/10 \qquad s = 1$

Summary

- *Framework for the numerical evaluation of loop integrals using Mellin-Barnes representations*

  algorithmic extraction of infrared singularities
  very well suited for multi-loop multi-scale problems
  all phase space regions without cumbersome analytic continuations
  generalization for effi cient evaluation of tensor integrals
  direct numeric integration of contour integrals

- *Full automatization of the whole procedure*

- *Numerical integration:*

  possible to evaluate numerically with good accuracy
  fast and compact code

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- *Full automatization of the whole procedure*

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  fast and compact code

- *Novel results:*

  evaluation of high rank tensors for the 1 loop hexagon
  fi rst evaluation of two loop box with two adjacent massive legs
  fi rst evaluation of three loop box with one massive leg