Heavy-Quark Fragmentation Functions in e^+e^- Collisions

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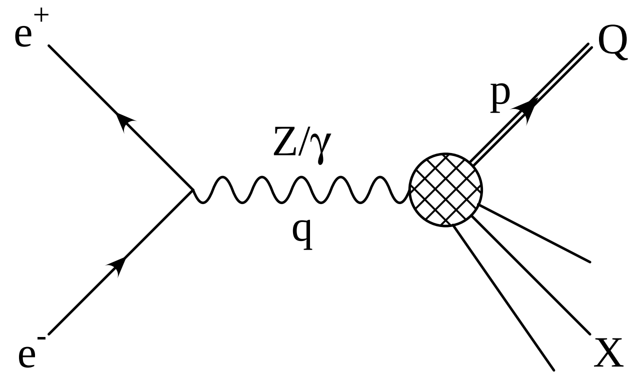
- Theoretical introduction
  - collinear logarithms
  - soft logarithms
- Non-perturbative contributions
- Results for c (and b) fragmentation functions
- Conclusions

What is a fragmentation function?



The differential cross section for the production of a massive quark Q in the process

$$e^+e^- \to Z/\gamma\ (q) \to Q\ (p) + X$$

is calculable in PQCD, because the mass acts as a cut-off for final-state collinear singularities

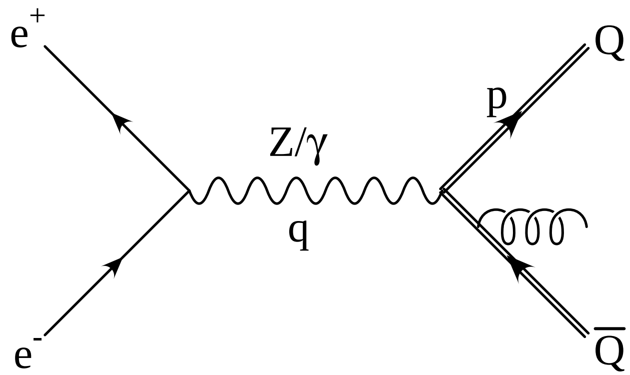
$$\frac{d\sigma}{dx}(x, q^2, m^2) = \sum_{n=0}^{\infty} a^{(n)}(x, q^2, m^2, \mu^2)\, \bar{\alpha}_s^n(\mu^2) \qquad x = \frac{2\, p \cdot q}{q^2} \qquad E = \frac{\sqrt{q^2}}{2}$$

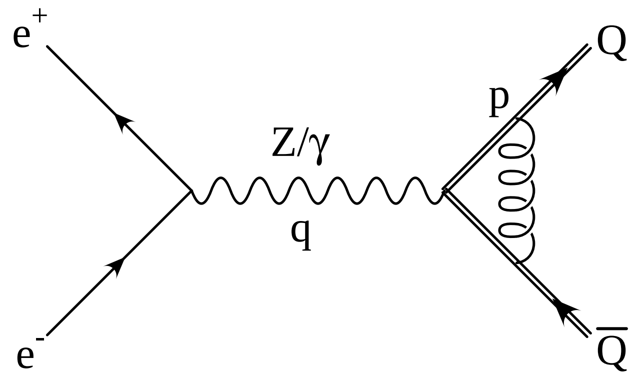
with μ = renormalization scale and $\bar{\alpha}_s \equiv \alpha_s/(2\pi)$.

The heavy-quark fragmentation function (FF) in e^+e^- annihilation is defined as

$$\hat{D}(x, q^2, m^2) \equiv \frac{1}{\sigma_{\rm tot}} \frac{d\sigma}{dx}(x, q^2, m^2)$$

Differential cross-section at order α_s





$$x = \frac{E_Q}{(E_Q)_{\max}}$$

$$\begin{aligned}\frac{1}{\sigma_0}\frac{d\sigma}{dx} = \; & \delta(1-x) + \frac{\alpha_s(q^2)}{2\pi} C_F \left\{ 1 + \ln\frac{q^2}{m^2}\left(\frac{1+x^2}{1-x}\right)_+ - \left(\frac{\ln(1-x)}{1-x}\right)_+ (1+x^2) \right. \\ & \left. + 2\frac{1+x^2}{1-x}\log x + \frac{1}{2}\left(\frac{1}{1-x}\right)_+ (x^2-6x-2) + \left(\frac{2}{3}\pi^2 - \frac{5}{2}\right)\delta(1-x) \right\} + \mathcal{O}\left(\frac{m^2}{q^2}\right)\end{aligned}$$

$$\int_0^1 \left(\frac{1}{1-x}\right)_+ f(x) \equiv \int_0^1 \left(\frac{f(x)-f(1)}{1-x}\right) \qquad \Longrightarrow \qquad \sigma = \sigma_0\left(1+\frac{\alpha_s}{\pi}\right) + \mathcal{O}\left(\frac{m^2}{q^2}\right)$$

Two main issues

- Is this a "well-behaved" perturbation expansion?
- We do NOT "see" free quarks, even if they are heavy, but bound states (mesons and baryons). How can we incorporate these non-perturbative (hadronization) effects?

Collinear and soft logarithms

$$\frac{1}{\sigma_0}\frac{d\sigma}{dx} = \sum_{n=0}^{\infty} a^{(n)}(x,q^2,m^2,\mu^2)\left(\frac{\alpha_s(\mu^2)}{2\pi}\right)^n$$

$$= \delta(1-x) + \frac{\alpha_s(q^2)}{2\pi}C_F\left\{1 + \ln\frac{q^2}{m^2}\left(\frac{1+x^2}{1-x}\right)_+ - \left(\frac{\ln(1-x)}{1-x}\right)_+(1+x^2)\right.$$

$$\left. +2\frac{1+x^2}{1-x}\log x + \frac{1}{2}\left(\frac{1}{1-x}\right)_+(x^2-6x-2) + \left(\frac{2}{3}\pi^2 - \frac{5}{2}\right)\delta(1-x)\right\} + \mathcal{O}\left(\frac{m^2}{q^2}\right)$$

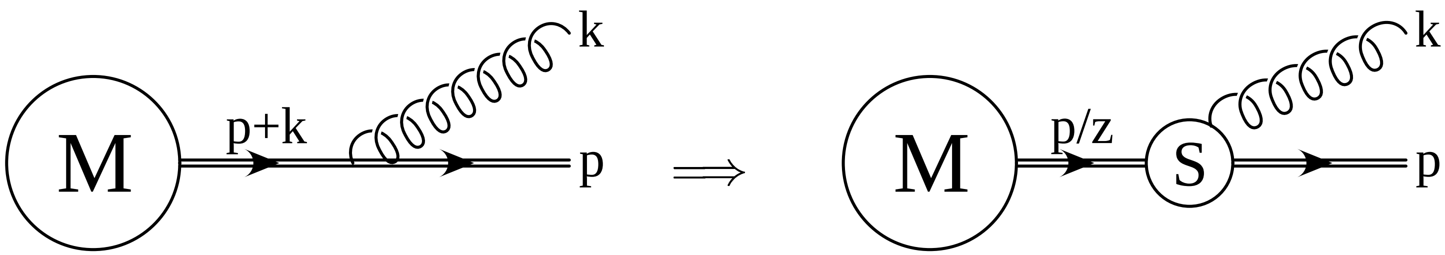
- In the limit $q^2 \gg m^2$,

$$a^{(n)} \sim \left(\log\frac{q^2}{m^2}\right)^n$$

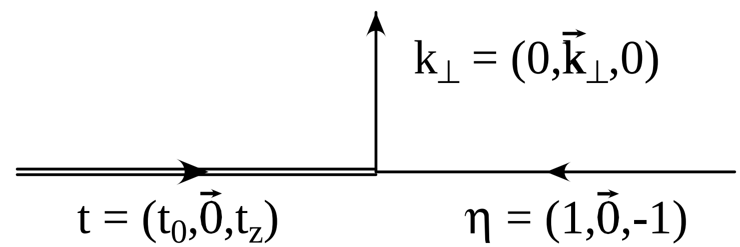
  and if $\alpha_s \log\frac{q^2}{m^2} \approx 1$ we cannot truncate the series at some fixed order, because each term in the series is of the same order as the first one $\Longrightarrow$ we have to resum these large contributions

- Same for soft logarithms, that arise when $x \to 1$.

Quasi-collinear behavior



$$(p+k)^2 - m^2 = 2p\cdot k = 2k_0\left(\sqrt{|\vec{p}|^2+m^2} - |\vec{p}|\cos\theta_{Qg}\right)$$



$$p^\nu = z t^\nu + \xi' \eta^\nu - k_\perp^\nu$$
$$k^\nu = (1-z) t^\nu + \xi'' \eta^\nu + k_\perp^\nu$$

$p^2 = t^2 = m^2 \qquad k^2 = 0 \qquad \eta^2 = 0$

$t\cdot k_\perp = 0 \qquad \eta\cdot k_\perp = 0 \qquad k_\perp^\nu k_{\perp\nu} = -\mathbf{k}_\perp^2$

Factorization theorem

- factorization of the squared amplitude

$$|\mathcal{M}(p,k;\dots)|^2 \simeq |\mathcal{M}(p/z;\dots)|^2 \, \frac{8\pi\alpha_s}{2\,p\cdot k}\, S = |\mathcal{M}(p/z;\dots)|^2\, 8\pi\alpha_s \, \frac{z(1-z)}{\mathbf{k}_\perp^2 + (1-z)^2 m^2}\, S$$

$$S\left(z, \frac{m^2}{\mathbf{k}_\perp^2}\right) = C_F\left[\frac{1+z^2}{1-z} - \frac{2z(1-z)m^2}{\mathbf{k}_\perp^2 + (1-z)^2 m^2}\right]$$

- factorization of the phase space

$$d\Phi_{n+1} = d\Phi_{n-1}\frac{d^3p}{(2\pi)^3 2p_0}\frac{d^3k}{(2\pi)^3 2k_0}\delta^4(\dots+p+k) = \underbrace{d\Phi_{n-1}\frac{d^3t}{(2\pi)^3 2t_0}\delta^4(\dots+t)}_{d\Phi_n}\,\frac{1}{16\pi^2}\,\frac{dz}{z(1-z)}d\mathbf{k}_\perp^2$$

Then we can iterate to take into account multiple quasi-collinear emissions

$$d\sigma_{n+1} = \int |\mathcal{M}(p,k;\dots)|^2 d\Phi_{n+1} = \int |\mathcal{M}(t;\dots)|^2 d\Phi_n \, \frac{\alpha_s}{2\pi}\int_0^1 dz \int_0^{\mu_f^2} d\mathbf{k}_\perp^2 \, \frac{1}{\mathbf{k}_\perp^2 + (1-z)^2 m^2}\, S$$

$$= d\sigma_n\, \frac{\alpha_s}{2\pi}\, C_F \int_0^1 dz \left\{\frac{1+z^2}{1-z}\log\left[1 + \frac{\mu_f^2}{m^2}\frac{1}{(1-z)^2}\right] + \dots\right\}$$

Factorization theorem

In the limit $m^2 \ll q^2$ we can neglect terms proportional to powers of m^2/q^2, and the single inclusive heavy-quark cross section can be written as (factorization theorem)

$$\frac{d\sigma}{dx}(x,q^2,m^2) = \sum_i \int_x^1 \frac{dz}{z} \frac{d\hat{\sigma}_i}{dz}\left(z,q^2,\mu_F^2\right) \hat{D}_i\left(\frac{x}{z},\mu_F^2,m^2\right)$$

$\frac{d\hat{\sigma}_i}{dz}$ $\overline{\rm MS}$-subtracted partonic cross section, for the production of the parton i (process dependent)

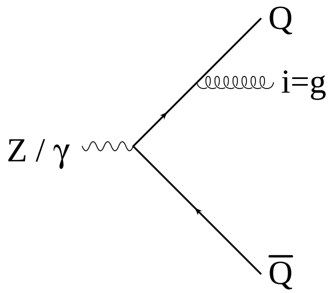
$\hat{D}_i$ $\overline{\rm MS}$ fragmentation functions for the parton i to "fragment" into the heavy quark Q (process independent)

μ_F factorization scale

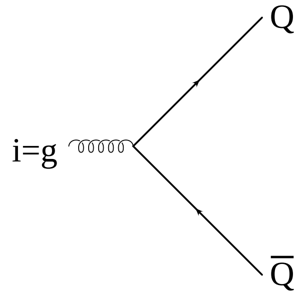
Example

$$\frac{d\sigma}{dx}(x,q^2,m^2) = \sum_i \int_x^1 \frac{dz}{z} \frac{d\hat{\sigma}_i}{dz}\left(z,q^2,\mu_F^2\right) \hat{D}_i\left(\frac{x}{z},\mu_F^2,m^2\right)$$

if $i = g$ then



$$\equiv \frac{d\hat{\sigma}_i}{dz} \otimes \hat{D}_i \equiv$$



$m = 0$!

process dependent

$m \neq 0$

process independent

Resummation of collinear logs

$$\frac{d\sigma}{dx}(x,q^2,m^2) = \sum_i \int_x^1 \frac{dz}{z} \frac{d\hat{\sigma}_i}{dz}\left(z,q^2,\mu_F^2\right) \hat{D}_i\left(\frac{x}{z},\mu_F^2,m^2\right)$$

How to choose μ_F?

If $\mu_F^2 \approx q^2$ then no large logarithms of q^2/μ_F^2 appear in $d\hat{\sigma}_i/dz$ and its perturbative expansion is reliable.

The large logarithms are moved in the fragmentation functions that obey the ($\mu_F = \mu$)

DGLAP evolution equations

$$\frac{d\hat{D}_i}{d\log\mu^2}(x,\mu^2,m^2) = \sum_j \int_x^1 \frac{dz}{z} P_{ij}\left(\frac{x}{z},\bar{\alpha}_s(\mu^2)\right) \hat{D}_j(z,\mu^2,m^2)$$

where the splitting functions P_{ij} have the expansion

$$P_{ij}\left(x,\bar{\alpha}_s(\mu)\right) = \bar{\alpha}_s(\mu)P_{ij}^{(0)}(x) + \bar{\alpha}_s^2(\mu)P_{ij}^{(1)}(x) + \bar{\alpha}_s^3(\mu)P_{ij}^{(2)}(x) + \mathcal{O}\left(\alpha_s^4\right)$$

The DGLAP equations resum correctly all the large logarithms

Leading Log, Next-to-Leading Log...

Introducing the shorthand notation $L = \log\left(q^2/m^2\right)$

✓ with an expansion for P_{ij} up to order α_s, we resum all terms of the form

$$\underbrace{\frac{d\sigma}{dx}(x, q^2, m^2)\bigg|_{\mathrm{LL}} = \sum_{n=0}^{\infty} \beta^{(n)}(x)\ (\bar{\alpha}_s L)^n}_{(\bar{\alpha}_s L)^n \Longrightarrow \text{Leading Log}}$$

✓ with an expansion for P_{ij} up to order α_s^2, we resum all terms of the form

$$\underbrace{\frac{d\sigma}{dx}(x, q^2, m^2)\bigg|_{\mathrm{NLL}} = \sum_{n=0}^{\infty} \beta^{(n)}(x)\ (\bar{\alpha}_s L)^n + \sum_{n=0}^{\infty} \gamma^{(n)}(x)\, \bar{\alpha}_s\, (\bar{\alpha}_s L)^n}_{\bar{\alpha}_s\,(\bar{\alpha}_s L)^n \Longrightarrow \text{Next-to-Leading Log}}$$

✓ ...so on, so forth.

The time-like splitting functions are known up to the third order [Mitov, Moch and Vogt, hep-ph/0604053]

Initial condition

How can we compute the initial condition for $\hat{D}_i(x,\mu^2,m^2)$?

$$\hat{D}_i(x,\mu^2,m^2) = d_i^{(0)}\delta(1-x) + \bar{\alpha}_s(\mu^2)\, d_i^{(1)}(x,\mu^2,m^2) + \mathcal{O}\left(\alpha_s^2\right) \; .$$

Use the factorization theorem

$$\frac{d\sigma}{dx}(x,q^2,m^2) = \sum_i \int_x^1 \frac{dz}{z}\,\frac{d\hat{\sigma}_i}{dz}\left(z,q^2,\mu^2\right)\,\hat{D}_i\left(\frac{x}{z},\mu^2,m^2\right)$$

and match the expansion up to order α_s of the left- and right-hand side to obtain $d_i^{(0)}$ and $d_i^{(1)}$ [Mele and Nason, *Nucl. Phys.* **B361** (91) 626]

$$\begin{aligned}
\frac{d\sigma}{dx}(x,q^2,m^2) &= a^{(0)}(x,q^2,m^2) + a^{(1)}(x,q^2,m^2,\mu^2)\,\bar{\alpha}_s(\mu^2) + \mathcal{O}\left(\alpha_s^2\right) \\
\frac{d\hat{\sigma}_i}{dx}(x,q^2,\mu^2) &= \hat{a}_i^{(0)}(x) + \hat{a}_i^{(1)}(x,q^2,\mu^2)\,\bar{\alpha}_s(\mu^2) + \mathcal{O}\left(\alpha_s^2\right)
\end{aligned}$$

The $d_i^{(2)}$ terms are known too, and have been computed by [Melnikov and Mitov, `hep-ph/0404143`; Mitov, `hep-ph/0410205`], following a different strategy.

Final recipe for collinear-log resummation

✓ start with $\hat{D}_i(x, \mu_0^2, m^2)$, with $\mu_0^2 \approx m^2$, so that no large logarithms of the ratio μ_0^2/m^2 appear in the initial conditions

✓ evolve $\hat{D}_i(x, \mu_0^2, m^2)$ from the low to the high energy scale μ with the DGLAP equation to obtain $\hat{D}_i(x, \mu^2, m^2)$

✓ use the factorization theorem to compute the resummed cross section.

Soft logarithms

In the region of the phase space of multiple soft-gluon emission ($x \to 1$), the differential cross section contains enhanced terms proportional to

$$a^{(n)} \approx c_{\rm LL}^{(n)} \left(\frac{\log^{2n-1}(1-x)}{1-x} \right)_+ + c_{\rm NLL}^{(n)} \left(\frac{\log^{2n-2}(1-x)}{1-x} \right)_+ + \ldots$$

These terms can be organized in towers of $\log N$, where we introduce the Mellin transform

$$f(N) = \int_0^1 dx\, x^{N-1} f(x) \qquad \Longrightarrow \qquad \int_0^1 dx\, x^{N-1} \left(\frac{\log^k(1-x)}{1-x} \right)_+ \approx \log^{k+1} N$$

The large-N contributions come from the regions where $x \to 1$, associated to the bremsstrahlung spectrum of soft and collinear emission.

Up to now, it is known how to resum all the Leading Log and Next-to-Leading Log [Dokshitzer, Khoze and Troyan, `hep-ph/9506425`; Cacciari and Catani, `hep-ph/0107138`]

$$\sum_{n=0}^{\infty} c_{\rm LL}^{(n)} \alpha_s^n \log^{n+1} N = \log N \, g_{\rm LL}(\alpha_s \log N) \qquad \sum_{n=0}^{\infty} c_{\rm NLL}^{(n)} \alpha_s^n \log^{n} N = g_{\rm NLL}(\alpha_s \log N)$$

Non-perturbative effects

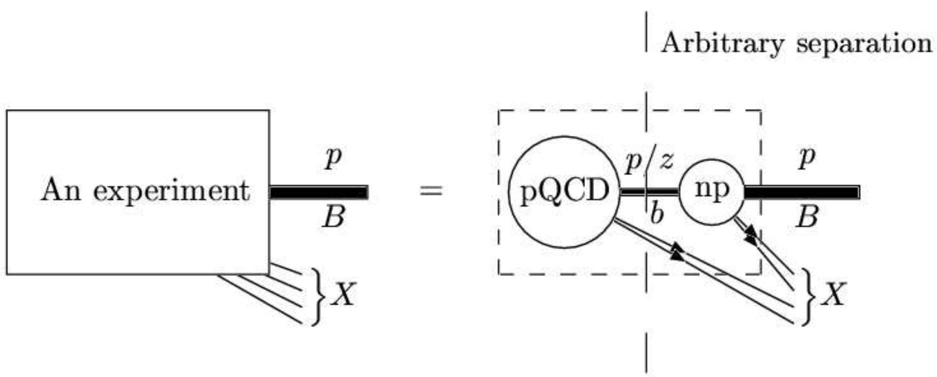
- ✗ The weak point of the factorization theorem comes from the initial condition for the evolution of the fragmentation function, which is computed as a power expansion in terms of $\alpha_s(m)$: irreducible, non-perturbative uncertainties of order $\Lambda_{\rm QCD}/m$ are present.
- ✗ The soft-gluon resummation functions g_{LL} and g_{NLL} contain singularities at large N which signal the eventual failure of perturbation theory and hence the onset of non-perturbative phenomena.
  - in the initial condition, the region $(1-x)\,m \approx \Lambda$ ($m/N \approx \Lambda$ in moment space) is sensitive to the decay of excited states of the heavy-flavoured hadrons, where Λ is a typical hadronic scale of a few hundreds MeV.
  - in the coefficient functions, when $(1-x)\,q^2 \approx \Lambda^2$, the mass of the recoil system approaches typical hadronic scales.

The matching of perturbative results with non-perturbative physics is a delicate problem, which rests, first of all, on a proper definition of the perturbative series.

Non-perturbative fragmentation function

We assume that all these effects are described by a non-perturbative fragmentation function $D^H_{\rm NP}$, that takes into account all low-energy effects, including the process of the heavy quark turning into a heavy-flavoured hadron. The full resummed cross section, including non-perturbative corrections, is then written as

$$\frac{d\sigma^H}{dx}(x,q^2) \;=\; \sum_i \frac{d\hat{\sigma}_i}{dx}\left(x,q^2,\mu_F^2\right) \otimes \hat{D}_i\left(x,\mu_F^2,m^2\right) \otimes D^H_{\rm NP}(x)$$



The non-perturbative part $D^H_{\rm NP}$ is what is missing to go from the partonic cross section to the hadronic one $\Longrightarrow$ very sensitive to the perturbative part

It is expected to be universal $\Longrightarrow$ exctract it from e^+e^- data and use it in hadronic heavy-quark production.

Non-perturbative fragmentation function

The Mellin transform of the full resummed cross section, including non-perturbative corrections, is

$$\sigma_H(N,q^2) = \sigma_Q(N,q^2,m^2)\; D_{\rm NP}(N)$$

We [Cacciari, Nason and C.O., hep-ph/0510032] have fitted CLEO and BELLE D^* data using the two-component form for $D_{\rm NP}$

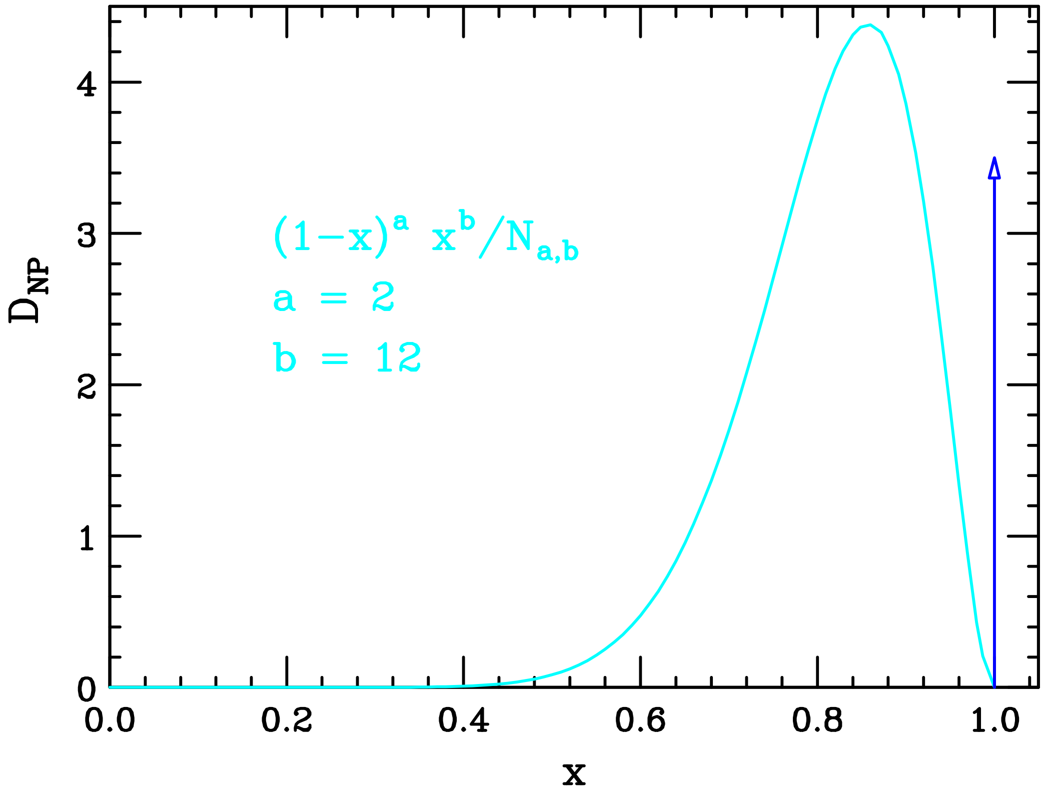
$$D_{\rm NP}(x) = {\rm Norm.} \times \left[\delta(1-x) + c\,\frac{(1-x)^a x^b}{N_{a,b}}\right], \qquad N_{a,b} = \int_0^1 (1-x)^a x^b$$

Simple phenomenological interpretation

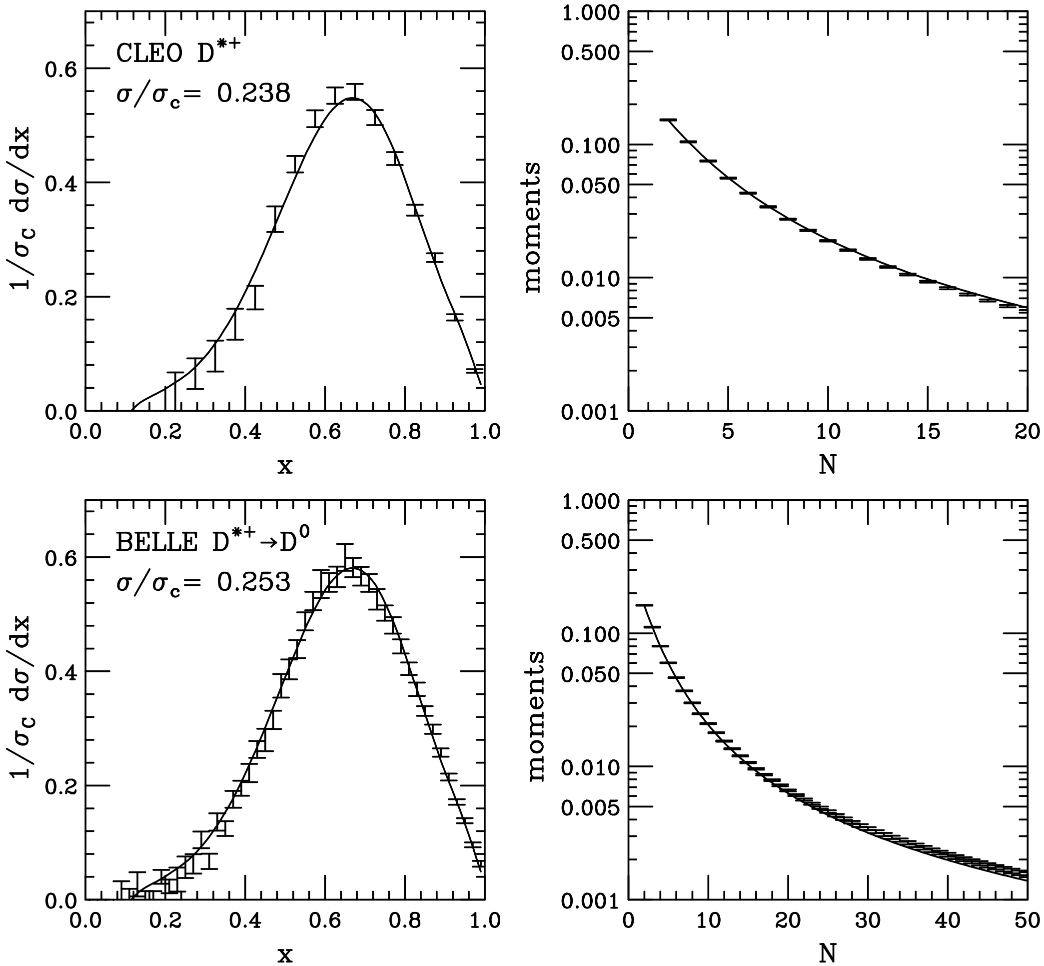
- the hard term (the delta function) corresponds, in some sense, to the direct exclusive production of the D^*
- the rest accounts for D^*'s produced in the decay chain of higher resonances.

Non-perturbative fragmentation function

$$D_{\rm NP}(x) = {\rm Norm.} \times \left[\delta(1-x) + c\,\frac{(1-x)^a x^b}{N_{a,b}} \right], \qquad N_{a,b} = \int_0^1 (1-x)^a x^b$$

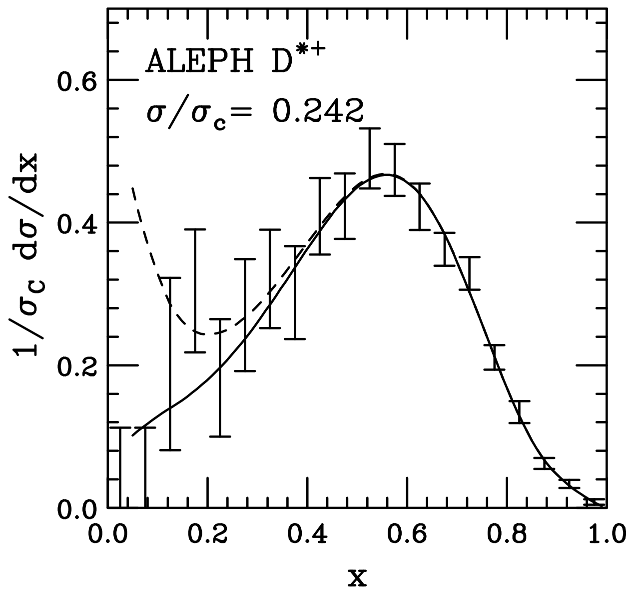


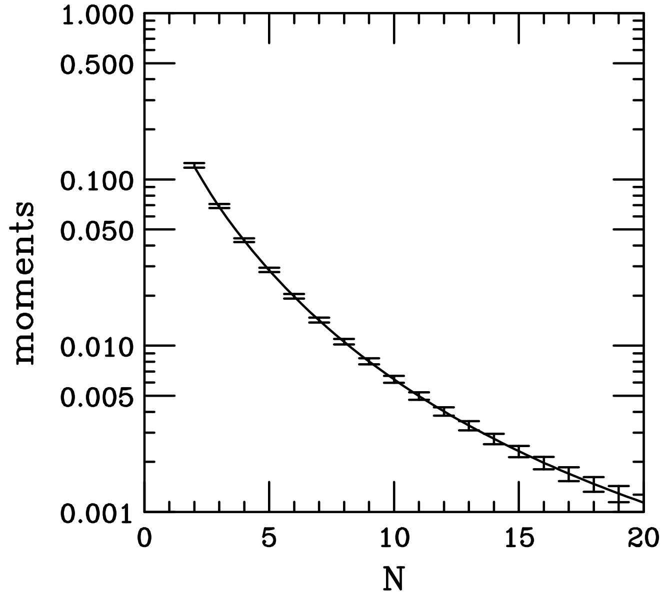
CLEO and BELLE D^* fits



- ✓ D mesons data fits near the $\Upsilon(4S)$ mass (10.6 GeV)
- ✓ Very good fit in the whole x range
- ✓ More fits in [Cacciari, Nason and C.O.], where $D^* \to D\,X$ have been modeled from decay chains and branching ratios
- ✓ B mesons data fits near the Z^0 mass (91.2 GeV)

ALEPH D^* fits at LEP





✗ very few useful points

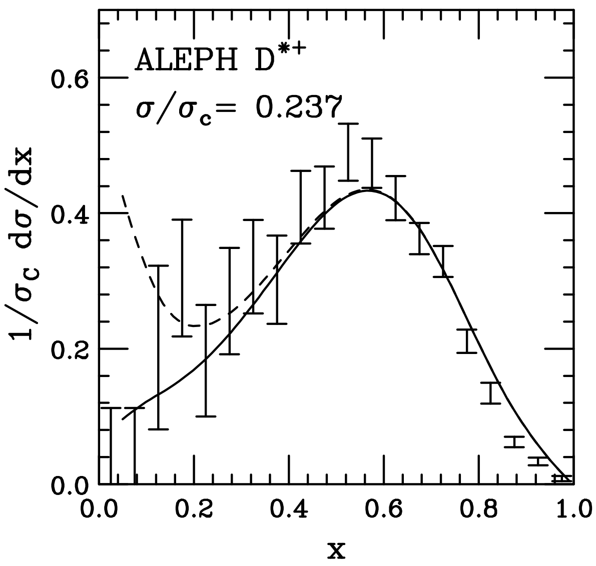
✗ large error bars

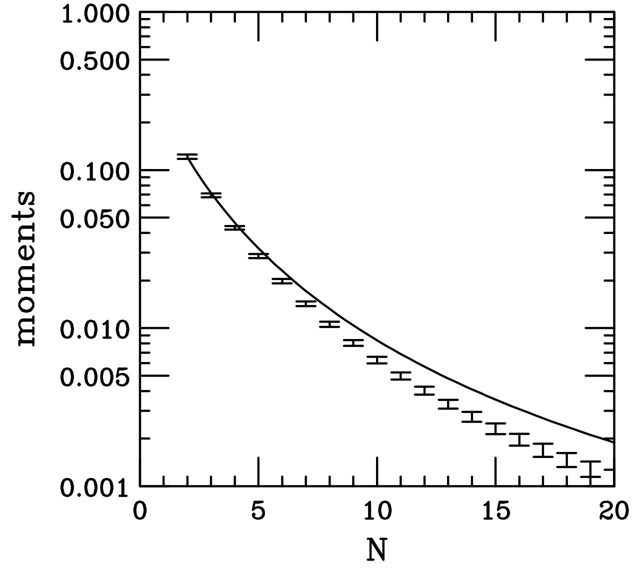
$$D_{\rm NP}(x) = {\rm Norm.} \times \left[\delta(1-x) + c\frac{(1-x)^a x^b}{N_{a,b}} \right]$$

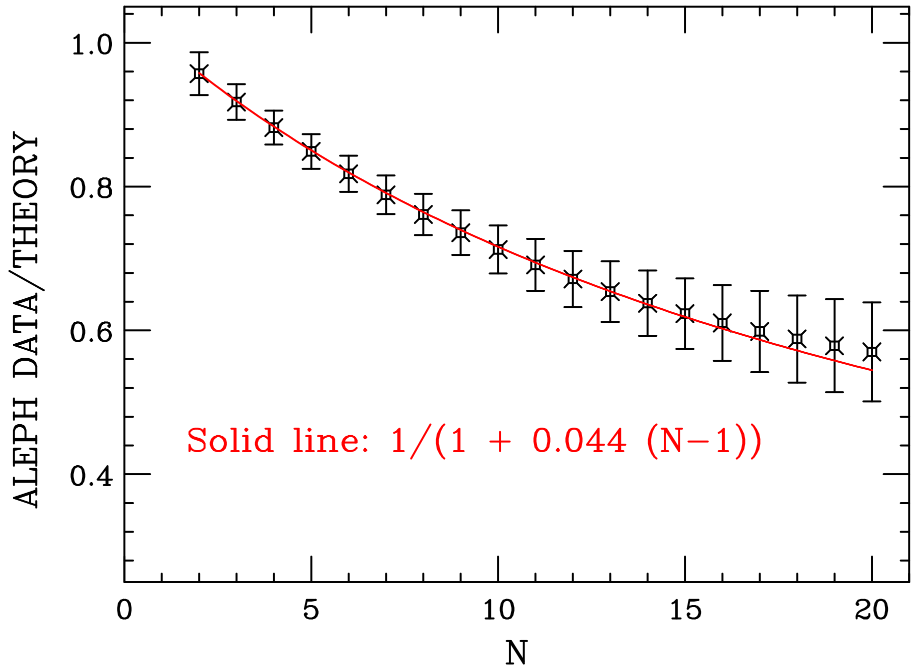
ALEPH	$a = 2.4 \pm 1.2\,,$	$b = 13.9 \pm 5.7$	$c = 5.9 \pm 1.7$
CLEO/BELLE	$a = 1.8 \pm 0.2\,,$	$b = 11.3 \pm 0.6$	$c = 2.46 \pm 0.07$

What about the supposed universality of the non-perturbative fragmentation function?

ALEPH with CLEO/BELLE parameters







- ✗ Discrepancy in the large-x (large-N) region
- ✗ The ratio data/theory well modeled by

$$\frac{1}{1+0.044\,(N-1)}$$

Possible explanation

We can only speculate about the possible origin of the discrepancy.
If this were due to a non-perturbative correction to the coefficient function of the form

- $1+\frac{C(N-1)}{q^2}$ this would lead to the extra factor

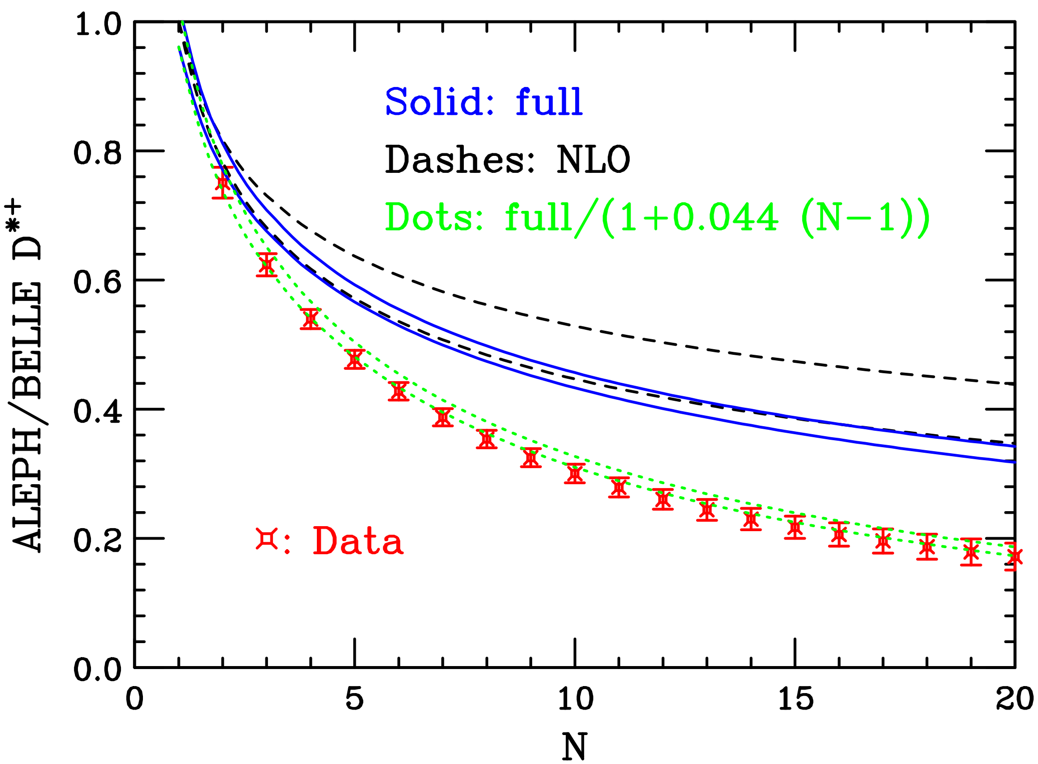
$$\frac{1+\frac{C(N-1)}{M_Z^2}}{1+\frac{C(N-1)}{M_\Upsilon^2}} \Longrightarrow \sim \frac{1}{1+0.044\,(N-1)} \quad \text{if} \quad C \sim 5\,\mathrm{GeV}^2 \qquad \text{large value!!}$$

- $1+\frac{C(N-1)}{E}$ with $E=\sqrt{q^2}/2$ then $C \sim 0.52$ GeV, a much more acceptable value.

Demonstrating the absence (or the existence) of $1/E$ corrections in fragmentation functions would be a very interesting result, since it would help to validate or disprove renormalon-based predictions.

Conclusions

$$\frac{\sigma_Q(N, M_Z^2, m^2)}{\sigma_Q(N, M_\Upsilon^2, m^2)} = \frac{\bar{a}_q(N, M_Z^2, \mu_Z^2)}{1 + \alpha_s(\mu_Z^2)/\pi} E(N, \mu_Z^2, \mu_\Upsilon^2) \frac{1 + \alpha_s(\mu_\Upsilon^2)/\pi}{\bar{a}_q(N, M_\Upsilon^2, \mu_\Upsilon^2)}$$



- low-scale effects, both at the heavy-quark mass scale and at the non-perturbative level, cancel completely
- its prediction is then entirely perturbative (the evolution function E is totally perturbative)
- scale-variation effects do not explain the discrepancy
- mass effects in charm production on the $\Upsilon(4S)$ where computed at order α_s^2, and found to be small [Nason and C.O., hep-ph/9903541]

Unfortunately, the low precision of the available data does not allow, at the moment, to resolve the issue of the absence (or the existence) of $1/E$ corrections in fragmentation functions.