Lattice QCD study of the heavy quark potential

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1. Introduction

▶ **Heavy quarkonia:**
bound states of heavy quark and heavy antiquark
⇒ η_c , J/Ψ , χ_{cJ} etc. for $c\bar{c}$
⇒ η_b , Υ , χ_{bJ} etc. for $b\bar{b}$

▶ **Phenomenological approach**
⇒ assume "Coulomb + linear (confining) potential"
between heavy quarks and solve the Schrödinger equation
(with relativistic corrections, such as spin-dependent ones)
[e.g. Godfrey&Isgur('85), there are many works along this line]
⇒ compute mass spectra, wave functions, decay widths, transition amplitudes, etc.

▶ **Can QCD support phenomenological approaches ?**
⇒ use "static potential" from lattice QCD ?
Coulomb + linear

▶ **Problems are ...**
⇒ QCD does not tell how to use the potential
(remember that QCD is not quantum mechanics)
⇒ masses of heavy quarks are not infinitely heavy
(flavor dependence ? fine & hyper-fine splitting ?)
⇒ multiscale hierarchy $m_q \gg m_q v \gg m_q v^2$ and $m_q \gg \Lambda_{\rm QCD}$

▶ **A promising approach:**
use an effective field theory "potential NRQCD"
with lattice QCD inputs

▶ **potential NRQCD**
⇒ related to QCD
⇒ potential picture of heavy quarkonium
⇒ but need nonperturbative inputs from QCD
(contain "unknown" functions corresponding to
static potential and corrections classified in powers $1/m_q$)

Determine "unknown" functions from lattice QCD !

2. Potential NRQCD

▶ **Effective field theory framework**
⇒ assume hierarchy of energy scales in QCD
$m_q \gg \Lambda_{\rm QCD}$ and $v \ll 1$ (quark velocity in the CM frame)
$m_q \gg m_q v \gg m_q v^2$
⇒ integrate out the scales above m_q and $m_q v$
[Brambilla,Pineda,Soto&Vairo('99ff), hep-ph/0410047]
⇒ **potential NRQCD**
static potential plus systematic relativistic corrections
with $1/m_q$ expansion

▶ **Effective Hamiltonian up to $O(1/m_q^2)$** [Pineda&Vairo('01)]

$$H = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} + V^{(0)}(r) + \frac{1}{m_1}V^{(1)}(r) + \frac{1}{m_2}V^{(1)}(r) + \frac{1}{m_1^2}V^{(2,0)}(r) + \frac{1}{m_2^2}V^{(0,2)}(r) + \frac{1}{m_1 m_2}V^{(1,1)}(r) + O(1/m^3)$$

(m_1 and m_2 are the masses of quark and antiquark, can be different)

- $V^{(0)}(r)$: static potential
  ⇒ well known, Coulomb plus linear
- $V^{(1)}(r)$: $O(1/m_q)$ correction
  ⇒ original in QCD due to 3 gluon vertex (absent in QED)
- $V^{(2,0)}(r)$, $V^{(0,2)}(r)$, $V^{(1,1)}(r)$: $O(1/m_q^2)$ corrections
  ⇒ including spin and momentum dependent corrections

3. Heavy quark potential

▶ **Static potential**

$$V^{(0)}(r) = -\lim_{t\to\infty}\frac{1}{t}\ln\langle W(r,t)\rangle$$

$$W(r,t) = {\rm Tr}[\prod_C U_\mu(s)] , \quad U_\mu(s) \equiv e^{igaA_\mu(s)}$$

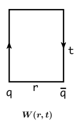
⇒ use transfer matrix theory

$$\hat{\mathbb{T}}\,|n(r)\rangle = e^{-aE_n^{(0)}}|n(r)\rangle$$

$|n(r)\rangle$ is a static q-$\bar{q}$ state

$$\Rightarrow \langle W(r,t)\rangle \simeq c_0(r)\, e^{-(E_0^{(0)}(r)-E_0)t}$$

for $T \gg t \gg a$



$W(r,t)$

▶ **$O(1/m_q)$ correction** [Brambilla,Pineda,Soto&Vairo('01)]

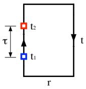
$$V^{(1)}(r) = -\frac{\delta_{ij}}{2}\int_0^\infty d\tau\, \tau \frac{\langle \mathcal{E}_i(0,t_1)\ \mathcal{E}_j(0,t_2)\rangle_W^c}{\langle W(r,t)\rangle}$$

(sum over all possible leading fluctuation of quark propagation)

$$= -\frac{\delta_{ij}}{2}\sum_{n=1}^{\infty}\frac{\langle 0(r)|\mathcal{E}_i|n(r)\rangle\langle n(r)|\mathcal{E}_j|0(r)\rangle}{(\Delta E_{n0}(r))^2}$$

(spectral representation)

where $\Delta E_{n0}(r) = E_n^{(0)}(r) - V^{(0)}(r)$



▶ **$O(1/m_q^2)$ spin-dependent corrections**
[Eichten&Feinberg('79), Pineda&Vairo('01)]
⇒ relevant to fine and hyperfine splitting

$$V_{\rm SD}(r) = \left(\frac{\vec{s}_1\cdot\vec{l}_1}{2m_1^2} - \frac{\vec{s}_2\cdot\vec{l}_2}{2m_2^2}\right)\left(\frac{V^{(0)\prime}(r)}{r} + 2\frac{V_1'(r)}{r}\right) + \left(\frac{\vec{s}_2\cdot\vec{l}_1}{2m_1m_2} - \frac{\vec{s}_1\cdot\vec{l}_2}{2m_1m_2}\right)\frac{V_2'(r)}{r} + \frac{1}{m_1m_2}\left(\frac{(\vec{s}_1\cdot\vec{r})(\vec{s}_2\cdot\vec{r})}{r^2} - \frac{\vec{s}_1\cdot\vec{s}_2}{3}\right)V_3(r) + \frac{\vec{s}_1\cdot\vec{s}_2}{3m_1m_2}V_4(r)$$

where

$$V_1'(r) = 2\int_0^\infty d\tau\, \tau \frac{\langle \mathcal{B}_y(0,t_1)\ \mathcal{E}_z(0,t_2)\rangle_W}{\langle W(r,t)\rangle}$$

$$V_2'(r) = 2\int_0^\infty d\tau\, \tau \frac{\langle \mathcal{B}_y(0,t_1)\ \mathcal{E}_z(r,t_2)\rangle_W}{\langle W(r,t)\rangle} \quad ......$$

▶ **$O(1/m_q^2)$ spin-independent corrections** [Pineda&Vairo('01)]
⇒ momentum-dependent corrections [Barchielli etal.('88)]

$$V_{\rm SI}(r) = \frac{1}{m_1^2}\left(\frac{1}{2}\left\{\vec{p}_1^{\,2}, V_{p^2}^{(2,0)}(r)\right\} + \frac{V_{l^2}^{(2,0)}(r)}{r^2}\vec{l}_1^{\,2} + V_r^{(2,0)}(r)\right) + (1\to 2) + \frac{1}{m_1m_2}\left(-\frac{1}{2}\left\{\vec{p}_1\cdot\vec{p}_2, V_{p^2}^{(1,1)}(r)\right\} - \frac{V_{l^2}^{(1,1)}(r)}{2r^2}\left(\vec{l}_1\cdot\vec{l}_2 + \vec{l}_2\cdot\vec{l}_1\right) + V_r^{(1,1)}(r)\right)$$

where

$$V_{p^2}^{(2,0)} = V_d - \frac{2}{3}V_e , \quad V_{l^2}^{(2,0)} = V_e , \quad V_{p^2}^{(1,1)} = -V_b + \frac{2}{3}V_c , \quad V_{l^2}^{(1,1)} = -V_c$$

$$V_b(r) = -\frac{\delta_{ij}}{3}\int_0^\infty dt\, \tau^2 \frac{\langle \mathcal{E}_i(0,t_1)\ \mathcal{E}_j(r,t_2)\rangle_W^c}{\langle W(r,t)\rangle}$$

$$V_d(r) = \frac{\delta_{ij}}{6}\int_0^\infty dt\, \tau^2 \frac{\langle \mathcal{E}_i(0,t_1)\ \mathcal{E}_j(0,t_2)\rangle_W^c}{\langle W(r,t)\rangle} \quad ......$$

4. Numerical procedures

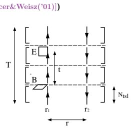
▶ **Simulation details**
- Wilson gauge action
- set lattice spacing by Sommer scale $r_0 = 0.5$ fm

$\beta = 6/g^2$	a [fm]	$V = L^3T$	$N_{\rm tsl}$	$N_{\rm iupd}$	$N_{\rm conf}$
5.85	0.122	24^4	3	50000	77
6.0	0.093	$20^3 40$	4	7000	33
6.2	0.068	$24^3 32$	5	10000	33
6.3	0.059	24^4	6	6000	39
5.85	0.122	24^4	3	50000	133
6.0	0.093	$24^3 32$	4	50000	100
6.2	0.068	$30^3 40$	5	50000	33

- NEC SX5, SX8, SX9 @ RCNP Osaka University
- Field strength operator with Huntley-Michael improvement
  $ga^2F_{\mu\nu} \equiv (U_{\mu\nu} - U_{\mu\nu}^\dagger)/(2i)$
- use Polyakov loop correlator as the quark-antiquark source
- use multi-level algorithm [Lüscher&Weisz ('01)]
- use transfer matrix theory

▶ **Multilevel algorithm** (original [Lüshcer&Weisz('01)])

(1) Compute the component of the Polyakov loops with the field strength insertion in each time slice
(2) Compute sublattice correlators
(3) Take sublattice average of correlators through internal update (iupd)
large memory required
(4) Construct correlation functions from sub-lattice correlators.
(5) Average over spatial sites, possible combinations of two fields insertion for given t
⇒ **1 conf**

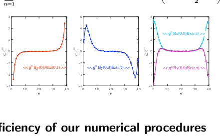


eg. for V_1' spin-orbit

▶ **Transfer matrix theory for field strength correlators**
[Koma&Koma('07)]

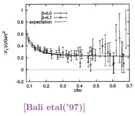
(example) spin-orbit correction $V_1'(r)$

$$\langle \mathcal{B}_y(0,0)\mathcal{E}_z(0,\tau)\rangle_{PP^*}/\langle P(0)P(r)^*\rangle = 2\sum_{n=1}^{\infty}\langle 0(r)|\mathcal{B}_y|n(r)\rangle\langle n(r)|\mathcal{E}_z|0(r)\rangle e^{-(\Delta E_{n0})\frac{T}{2}}\sinh\left((\Delta E_{n0})(\frac{T}{2}-\tau)\right)$$

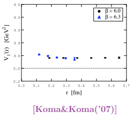


▶ **Efficiency of our numerical procedures**

(ex) spin-orbit correction $V_1'(r)$



[Bali etal('97)]



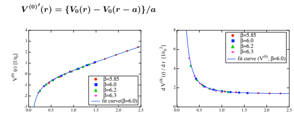
[Koma&Koma('07)]

5. Numerical results

▶ **Static potential and force**

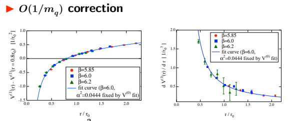
$$V^{(0)}(r) = -\frac{1}{T}\ln\langle P(0)P(r)^*\rangle + O(e^{-(\Delta E_{10}(r))T})$$

$$V^{(0)\prime}(r) = \{V_0(r) - V_0(r-a)\}/a$$



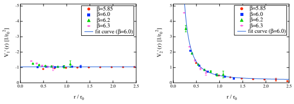
$$V^{(0)}(r) = -\frac{c}{r} + \sigma r + \mu \quad \Rightarrow \quad c = 0.297(1),\ \sigma a^2 = 0.0468(2)$$

▶ **$O(1/m_q)$ correction**

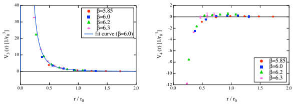


$$V^{(1)}(r) = -\frac{\alpha^2}{r^2} + c'\ln r \quad \Rightarrow \quad c' r_0^2 = 0.49(1) \quad (\alpha = (3/4)c)$$

▶ **$O(1/m_q^2)$ spin-dependent corrections**



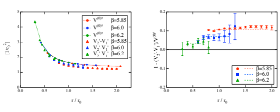
spin-orbit



spin-spin

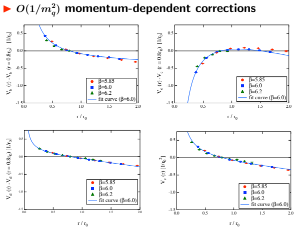
▶ **Gromes relation** [Gromes('84), Brambilla,Gromes&Vairo('01)]

$$V^{(0)\prime}(r) = V_2'(r) - V_1'(r)$$



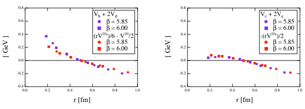
⇒ existence of nonperturbative contribution
⇒ satisfied in the continuum limit (Lorentz invariance restoration)

▶ **$O(1/m_q^2)$ momentum-dependent corrections**



▶ **BBMP relations** [Barchielli,Brambilla,Montaldi&Prosperi('88)]

$$V_b(r) + 2V_d(r) = -\frac{V^{(0)}(r)}{2} + \frac{r}{6}V^{(0)\prime}(r) \qquad V_c(r) + 2V_e(r) = -\frac{r}{2}V^{(0)\prime}(r)$$



⇒ seem to be satisfied

6. Summary

▶ We have investigated the heavy quark potential including relativistic corrections from lattice QCD (nonperturbative inputs in potential NRQCD)
⇒ static potential
⇒ $O(1/m_q)$ correction
⇒ $O(1/m_q^2)$ spin-dependent corrections
⇒ $O(1/m_q^2)$ momentum-dependent corrections

▶ We have confirmed long-range nonperturbative contributions to these corrections and observe a reasonable scaling behavior.

▶ References

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