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Quantum Brachistochrone

—Time-optimal quantum evolution-

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Plan of Talk

- Ⅰ. Introduction
- Ⅱ. Brachistochrone problem
- Ⅲ. Variational principle
- Ⅳ. Simple examples
- V. Summary
- VI. Recent development

Ⅰ. Introduction

- In ordinary quantum mechanics we solve the Schroedinger equation for a given Hamiltonian H and initial state $|\psi_i\rangle$ to predict the final state $|\psi_f\rangle$.

Here we consider the problem to find time optimal quantum evolution within a limited resource,

e.g. quantum computation

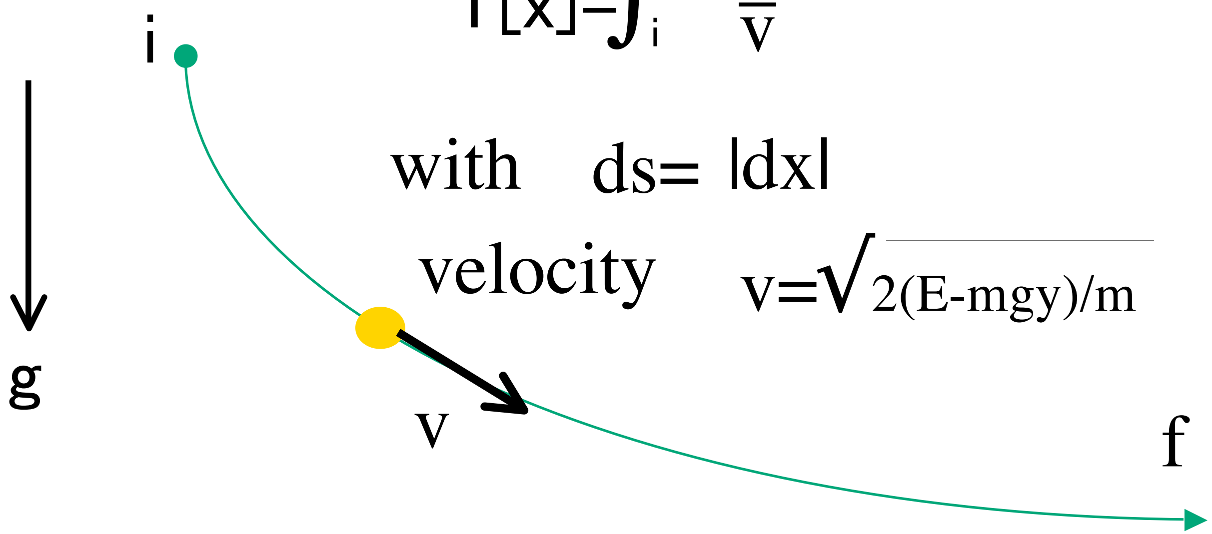
II Brachistochrone Problem

Minimize the time:

$$T[x]=\int_i^f \frac{ds}{v}$$

with $ds = |dx|$

velocity $v=\sqrt{2(E-mgy)/m}$

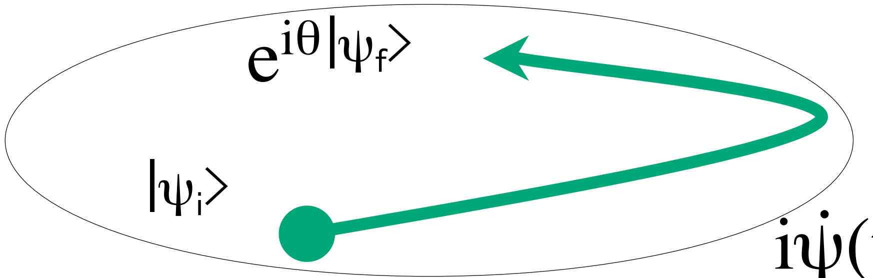


in classical mechanics

(Bernoullis) Answer: cycloid

Quantum Brachistochrone Problem

Find the action principle that gives the fastest quantum path from a given initial state $|\psi_i\rangle$ to a final state $|\psi_f\rangle$ up to phase by optimizing combination of available Hamiltonians $\{H\}$.



Hilbert space

$$i\dot{\psi}(t)=H(t)\psi(t)$$

$$\boxed{H(t)\in\text{available source}}$$

Metric in quantum space

The invariant metric for the phase transformation: $|\psi\rangle \dashrightarrow e^{i\theta(t)} |\psi\rangle$ is

Fubini-Study metric in CP^{n-1}:

$$ds = \| d|\psi\rangle - |\psi\rangle\langle\psi| d|\psi\rangle \|$$

Observe

$$\boxed{ds = \Delta E dt}$$

Schroedinger equation

Aharonov-Anandan ('90)

$$\Delta E^2 = \langle\psi| H^2 |\psi\rangle - \langle\psi| H |\psi\rangle^2$$

is the energy fluctuation.

Margolus and Levitan ('98)

time energy uncertainty $\Delta E \Delta t \geq h/4$

III Variational Principle

variable!

$$T[|\psi\rangle, H] = \int_i^f \frac{ds}{\Delta E}$$

is the total time duration from the initial state $|\psi_i\rangle$ to the final state $|\psi_f\rangle$ up to phase.

We minimize $T[|\psi\rangle, H]$ so that the Schroedinger equation holds and the Hamiltonian is in the available range. Intuitively the larger the energy fluctuation ΔE,the shorter the time duration T is.

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Action

The total action is therefore given by

ϕ, λ^a :Lagrange multipliers

$$S\,[|\psi\rangle, H, \langle\phi|, \lambda^a] = \int dt\, L,$$

$T\,[|\psi\rangle, H]$

$\bullet = d/dt$

$$L = \|\, |\dot{\psi}\rangle - |\psi\rangle\langle\psi|\dot{\psi}\rangle \| \,/\, \Delta E$$
$$+ i \langle\dot{\phi}|\psi\rangle + \langle\phi|H|\psi\rangle$$
$$+ \sum_a \lambda^a f_a(\tilde{H})$$

→ Schroedinger equation

→ constraints $f_a = 0$

Invariant under U(1) gauge transformation: $|\psi\rangle$, $\langle\phi|$, H

→ $e^{i\theta(t)}\,|\psi\rangle$, $\langle\phi|\,e^{-i\theta(t)}$, $H + \dot{\theta}(t)$ (U(1) gauge field")

$\tilde{H} = H - Tr(H)/n$ (traceless part of Hamiltonian)

(Remark 1)

We consider different quantum systems with different Hamiltonians H_i $(i=1,2, \ldots.)$.

Solving the Schroedinger equations $i\dot{\psi}_i(t)=H_i(t)\psi_i(t)$ with the initial and final states fixed up to phase and compare the time duration T_i to find the shortest path ψ_{opt} and corresponding optimal Hamiltonian H_{opt} .

(Remark 2)

Once H_{opt} is obtained we can set up a quantum system to which ordinary rules ,e.g. superposition principle, apply. Do experiments with H_{opt} ! If it is applied to the initial state $|\psi_i\rangle$, we can realize the optimal quantum computation.

(The gauge transformed Hamiltonian $H+\dot{\theta}(t)$ gives a different quantum system but in the same class as far as the optimality is concerned.)

Constraints

In practice the resource of Hamiltonian is limited. Available energy is finite and many body interaction more than two are not controllable.

Examples

constant

$f_1 = \mathrm{Tr}(H^2/2) - \omega^2 = 0,$ ← energy is limited

$f_a = \mathrm{Tr}(H g_a) = 0$ ← interaction is restricted

generator of SU($N=2^n$)

Define $\boxed{F := \sum_a \lambda^a \, \partial f_a(H)/\partial H}$

:a functional of H.

EOM and solution

$$\boxed{\begin{array}{ll} \delta H \rightarrow \ F = \{F,P\}, & \# \\ \delta|\psi\rangle \rightarrow \ (\dot{F}+i[H,F])|\psi\rangle = 0 & \#\# \end{array}}$$

$(P = |\psi\rangle\langle\psi|)$

and ## $\rightarrow$ $\boxed{\dot{F}+i\,[H,F]=0}$

Its formal solution is

$$\boxed{\begin{array}{l} F(t) = U(t)F(0)U^{+}(t)\,, \\ U(t) = T\exp[-i\int_0^t Hdt] \end{array}}$$

(F(0) is constrained by # at t=0)

$\rightarrow$ Hamiltonian H

A quick way to see how the key equation

$$\dot{F} + i\,[H,F] = 0.$$

comes out is consider an infinitesimal gauge transformation,

$\delta|\psi\rangle = i\ \ \alpha(t)\,|\psi\rangle$

$\delta\langle\phi| = -i\langle\phi|\,\alpha(t)$, $\qquad \alpha(t) \in su(N)$

$\delta H = \dot{\alpha}(t) + i[\alpha(t), H]$. (c.f. Fujikawa's talk)

The first two terms are in the action invariant,while the last constraint term directly gives the F-equation.

This holds also for general case,e.g. mixed states

IV Simple Examples

$f = \mathrm{Tr}(H^2/2) - \omega^2 = 0$, ("isotropic constraint")

$F = H$, $(\langle H \rangle = 0)$

$$\boxed{\begin{array}{l} H = HP + PH \\ H' = 0 \end{array}} \quad \begin{array}{l} \# \\ \#\# \end{array}$$

$\#\# \longrightarrow$ H:time independent

$\# \longrightarrow$

$$H = HP + PH = i\,(|\dot{\psi}(0)\rangle\langle\psi(0)| - |\psi(0)\rangle\langle\dot{\psi}(0)|)$$

$$|\psi(t)\rangle = \cos\omega t\,|\psi(0)\rangle + (\sin\omega t)/\omega\,|\dot{\psi}(0)\rangle$$

A great circle in CP^{N-1}; geodesic (Grover's)

Miyake and Wadati (01)

IV Simple Examples

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$F = H$, $\quad (\langle H \rangle = 0)$

$H = HP + PH$ #

$H' = 0$ ## $\longrightarrow$ H:time independent

\# $\longrightarrow$

$$H = HP + PH = i\left(|\dot{\psi}(0)\rangle\langle\psi(0)| - |\psi(0)\rangle\langle\dot{\psi}(0)|\right)$$

$$|\psi(t)\rangle = \cos\omega t\,|\psi(0)\rangle + (\sin\omega t)/\omega\,|\dot{\psi}(0)\rangle$$

$|\dot{\psi}(0)\rangle$ is chosen in span$\{|\psi(0)\rangle, |\psi(T)\rangle\}$

The solution can be written as

$$\boxed{|\psi(t)\rangle = \cos\omega t\,|\psi(0)\rangle + \sin\omega t\,|\psi(T)\rangle_{\perp}}$$

Grover's (see, however,Discussion)

where $|\psi(T)\rangle_{\perp}$ is the Schmidt orthonormalized vector.

Two constraints, 1-qubit model

$f_1 = \mathrm{Tr}(H^2/2) - \omega^2 = 0,$	energy is limited
$f_2 = \mathrm{Tr}(H\sigma_z) = 0$	H is span$\{\sigma_x, \sigma_y\}$

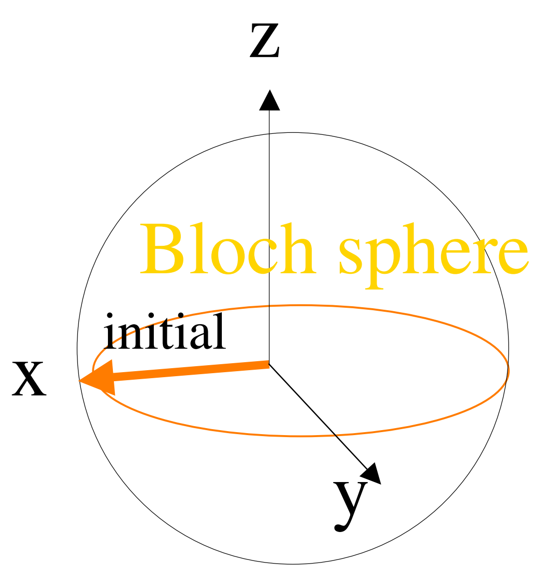
$F = \lambda^1 H + \lambda^2 \sigma_z$

Suppose the initial state is

$|\psi(0)\rangle\langle\psi(0)| = P(0) = (1+\sigma_x)/2$

$F(0) = \{F(0), P(0)\}$

$\longrightarrow$ $H(0) = -\omega\sigma_y$



Evolution operator

$$F(t)U(t) = i\,\lambda^1(t)\,\dot{U}(t) + \lambda^2(t)\,\sigma_z\,U(t)$$

$$= U(t)F(0) = U(t)[\lambda^1(0)\,H(0) + \lambda^2(0)\,\sigma_z\,] \qquad \#\#\#$$

It can be shown that the λ's are constants

we integrate ###

$$\boxed{\begin{aligned} &U(t)=\exp[-i\Omega\,\sigma_z\,t]\,\exp[-i\Omega'\,\sigma\,t] \\ &\sigma=(\omega\sigma_y+\Omega\,\sigma_z)/\Omega' \end{aligned}}$$

with $\Omega = \lambda^2/\lambda^1$ and $\Omega' = \sqrt{\omega^2 + \Omega^2}$

Hamiltonian H(t)

The time dependent Hamiltonian H(t) now becomes

$$H(t) = U(t)H(0)U^{+}(t) = -\mathbf{B}(t)\boldsymbol{\sigma},$$

$$\mathbf{B}(t) = \omega\,(\sin 2\Omega t, \cos 2\Omega t, 0)$$

i.e. Rotating magnetic field at the angular velocity 2Ω.

Spin motion

$(P(0)=(1+\sigma_x)/2)$

$$<\sigma> = Tr(P(t)\,\sigma) = Tr(U(t)P(0)U^{+}(t)\,\sigma)$$

$$= \begin{pmatrix} \cos 2\Omega t \cos 2\Omega' t + (\Omega/\Omega') \sin 2\Omega t \sin 2\Omega' t \\ -\sin 2\Omega t \cos 2\Omega' t + (\Omega/\Omega') \cos 2\Omega t \sin 2\Omega' t \\ (\omega/\Omega') \sin 2\Omega' t \end{pmatrix}$$

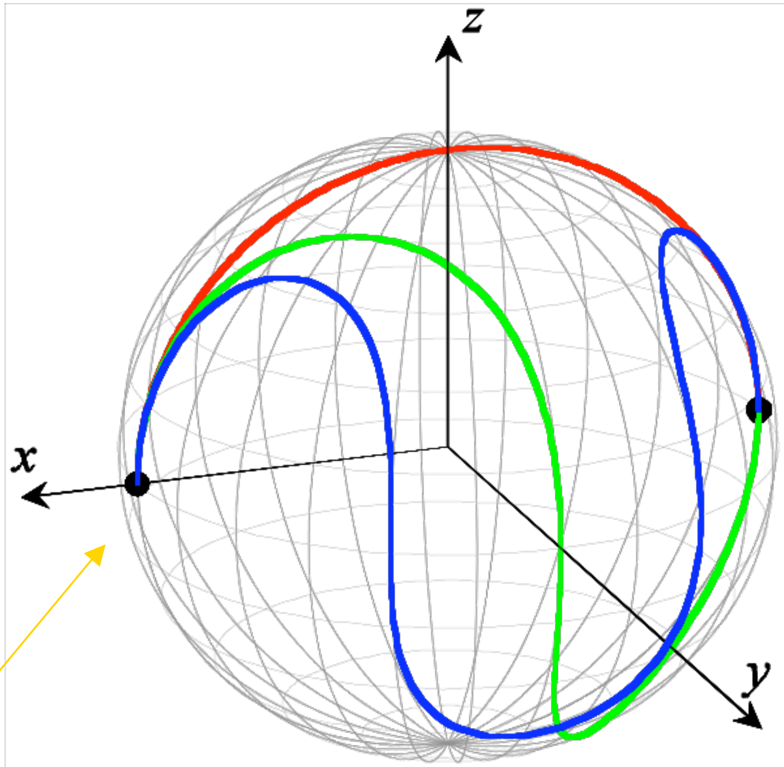
Suppose the final state is antipodal to the initial state.

Then the duration time is

$$2\Omega T = k\pi,\ 2\Omega' T = l\pi,\ k+l = \text{odd},\ l > k$$

Locally optimal paths

$\omega T=(\pi/2)\sqrt{l^2-k^2}$



$\pi/2$,(geodesic)

$(\pi/2)\sqrt{3}$,

$(\pi/2)\sqrt{5}$

The initial state is in the positive x direction, while the final state is its antipodal point.

Energy fluctuation

In the isotropic model, $\mathrm{Tr}(H^2/2) - \omega^2 = 0$

the energy fluctuation: $\Delta E = \omega =$ const.,

The action reduces to $S \propto \int_i^f ds$ so that the solution is geodesic.

In the second model with another constraint: $\mathrm{Tr}(Hg_a) = 0$,

the energy fluctuation is time dependent,

$$\Delta E(t) = \omega[1 - (\Omega \sin 2\Omega' t)/\Omega')^2]$$

so that solution is not geodesic.

Related works

☆ Alvarez and Gomes(‘99), Miyake and Wadati(01)

Grover’s algorithm follows geodesic in CP^{n-1}

a step for a single oracle calling remains to be specified

☆ Khaneja et al,(01),

1-qubit can be quick,while 2-qubit takes time,

→ Cartan decomposition of unitary operation

may become complicated for many qubits

We have reproduced their result for 3-qubits.

☆ Tanimura et al,

found optimal adiabatic solution in holonomic quantum computation

the formula for U(t) is similar but our case is not restricted to the adiabatic one

☆ There are growing number of works on time optimal quantum computation,

See, e.g. Nielsen, quant-ph/0502070

or Quantum Control Theory

V Summary

In analogy to the classical brachistochrone problem we have formulated a variational principle to find the optimal Hamiltonian H_{opt} and the optimal quantum evolution, for a given set of initial and final states and a set of available Hamiltonians.

Application: find H_{opt} by classical computer ,then run quantum computer!

Thank you!

Discussions

Open questions:

(1) Generalization of optimal Hamiltonian to optimal POVM's.(in progress,see next slide)

(2) Relation of time complexity and gate complexity

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Recent Progress

☆ Generalization to G/H space

relevant in the case e.g. NMR computation

where 1-qubit operations can be done instantly

☆ Generalization to mixed state

adopt the Lindblad master equation for the density

operator and the Petz monotone metric

relevant to the optimal CP maps, measurement basis

computation

$\dot{F}+i\,[H,F]=0$ seems universal !

G/H space (c.f. Tanimura's talk)

ds/ /dt ΔE

Lagrangian given by

$$L=[Tr[\dot{U}^\dagger(\dot{U}-P(\dot{U}))]/ Tr[U^\dagger H(HU-P(HU))]]^{1/2}$$

$$+Tr[\Lambda^\dagger(-i\dot{U}+HU)] +\sum_a \lambda^a f_a(H)$$

P: projection to G/H

is invariant under

U--->hU

Λ---->h Λ

H---->$hHh^\dagger + i\dot{h}h^\dagger$, $h\in H$

Mixed state ρ

The Lagrangian $L(\rho, H, L_j, \sigma, \lambda^a)$ is

$$L = ds/\,/dt\,\Delta E + \mathrm{Tr}[\sigma(\dot{\rho} + i[H, \rho] - \sum_j (L_j \rho L_j^\dagger - \{\rho, L_j^\dagger L_j\}/2))]$$

$$+\sum_a \lambda^a f_a(H)$$

the Petz metric

$$(ds/dt)^2 = \mathrm{Tr}[\dot{\rho} C_\rho(\dot{\rho})]$$

Variation w.r.t σ gives the Lindblad master equation

$$\dot{\rho} + i[H, \rho] - \sum_j (L_j \rho L_j^\dagger - \{\rho, L_j^\dagger L_j\}/2) = 0,$$

where L_j is the j-th POVM.

$$\dot{F} + i\,[H,F] = 0.$$

The F equation seems universal in the sense that it holds for mixed state case as well as pure state case.

This simplifies the optimization problem, because we can solve e.g.,the equation for the Lindblad operator L_j to obtain an optimal CP maps after finding the optimal Hamiltonian.

Lie Algebra method

Recently we have found a systematic way to solve the F equation using the Lie algebraic method .

Let the Cartan subalgebra of su(2^n) be H^j ($j=1\sim 2^n - 1$) and the generators E_α corresponding to the positive root vectors α, which satisfy $[H^j, E_\alpha]= \alpha^j E_\alpha$.

$H^j = 1\otimes 1\otimes .. \otimes\sigma_z$, $1\otimes 1\otimes .. \sigma_z\otimes 1, \ldots\ldots\ldots \sigma_z\otimes\sigma_z\otimes .. \otimes\sigma_z$

$E_\alpha = \sigma_+\otimes \sigma_+\otimes \ldots\otimes \sigma_{+\,,}(1\pm\sigma_z)/2 \otimes \sigma_+\otimes_{\ldots\ldots}\otimes \sigma_{+\,,}$

Expand the Hamiltonian H and F in the generators H^j, E_α.

The constraints to H imply that they are mutually exclusive.

Let

$$\boxed{\begin{aligned} H &= \textstyle\sum_\alpha h_\alpha(t)\, E_\alpha + h.c. + \sum_i a^i H^i, \\ F &= c_0 H + \textstyle\sum f_\beta(t)\, E_\beta + h.c. + \sum_j b^j H^j \end{aligned}}$$

$$Tr(E_\alpha^* E_\beta) = \delta_{\alpha\beta} \quad , \; Tr(H^{i*} H^j) = \delta_{ij}$$

$$\dot{h}_\alpha + i\alpha\cdot b\, h_\alpha + \Sigma_\beta\, h_{\alpha-\beta}(t)\, f_\beta = 0$$

$$\dot{f}_\beta - i\beta\cdot a\, f_\beta + \Sigma_\beta\, h_\alpha(t)\, f_{\beta-\alpha} = 0$$

a^i and b^j are constant

This gives an operational way to obtain the optimal Hamiltonian H_{opt}, which is time dependent in general. Structure of the equations is similar to the one for the falling cat problem in control theory (c.f. Tanimura's talk).