Introduction of Mutually Unbiased Bases

and

Solution to the Mean King's problem

for arbitrary finite levels

Recent issues and prospects on quantum theory

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★ Prelude

Quantum Theory

Infinite or finite

◎ Realization ··· Approximation, System of Interest

◎ Application ··· Quantum Information Theory

→ Quantum Computer, Quantum Cryptography, et. al

◎ Foundation ··· To understand Quantum Mystery (?)

◎ Nonobviousness ··· Connections with various combinatorial problems

→ character in each dimension

Overview

★1 Mutually Unbiased Bases (MUBs)

- Complementarity in d level quantum systems
- State determination with d+1 MUBs
- Optimal state determination
- Open Problem with MUBs [Orthogonal Latin Squares (OLSs) ?]

★2 Mean King's Problem

- Introduction of Mean King's problem
- Review of HHH method [OLS]
- Solution for arbitrary levels [Orthogonal Arrays]
- Conclusion and discussion

★1 Mutually Unbiased Bases (MUBs)

★ Mutually Unbiased Bases [Complementarity 1]

Dim $(\mathcal{H}) = \infty$

$$[x, p] = i\mathbb{I} \quad \rightarrow \quad \langle x|p\rangle = \exp(ipx) \quad \downarrow \quad |\langle x|p\rangle|^2 = const$$

Dim $(\mathcal{H}) = d$

Maximum degree of incompatibility !!

~~$[x, p] = i\mathbb{I}$~~

Mutually Unbiased Bases

★ Mutually Unbiased Bases [Complementarity 2]

Consider d level quantum system: $\mathcal{H}_d$

Two orthonormal bases $\{|\psi_i\rangle\}_{i=1}^d$ $\{|\phi_j\rangle\}_{j=1}^d$

are said to be Mutually Unbiased if

$$p_j^{(i)} = |\langle\psi_i|\phi_j\rangle|^2 = \frac{1}{d}$$

for all i, j = 1,…,d

Let O_ψ and O_ϕ be observables
with Mutually Unbiased eigenvectors $\{|\psi_i\rangle\}_{i=1}^d$ $\{|\phi_j\rangle\}_{j=1}^d$

$\longrightarrow$ O_ψ and O_ϕ are also called mutually unbiased

For any eigenstate of O_ψ , information of O_ϕ is most random !!

$\longrightarrow$ " Complementary" pair of observables !!

Schwinger (1960)

★ Mutually Unbiased Bases [Entropic Uncertainty Relation]

Entropic Uncertainty Relation

$$H(p) + H(q) \geq \ln\left(\min_{i,j} \frac{1}{|\langle \phi_i | \psi_j \rangle|^2}\right)$$

$$= \ln d \quad \text{(Mutually Unbiased Obs)}$$

◎ Mutually Unbiased Observables $\Leftrightarrow$ maximum lower bound

Deutsch (1983), Kraus (1987), Maassen, Uffink (1988)

★ Mutually Unbiased Bases [state determination 1]

Consider a set of orthonormal bases: $\mu = 1, \ldots, n$

$$\{|\mu, i\rangle\}_{i=1}^{d}$$

Set of orthonormal bases are said to be mutually unbiased bases (MUBs)
if any pair of them are mutually unbiased !!

$$|\langle \mu, i|\nu, j\rangle|^2 = \frac{1}{d} \ (\mu \neq \nu)$$

$$\left(\Leftrightarrow \ |\langle \mu, i|\nu, j\rangle|^2 = \delta_{\mu\nu}\delta_{ij} + (1 - \delta_{\mu\nu})\frac{1}{d} \right)$$

Case $n = d + 1$ is important and intriguing !

★ Mutually Unbiased Bases [state determination 2]

Measurements of d+1 numbers of MUBs
determine (unknown) quantum state !!

$$p_{\mu i} = \langle \mu, i | \rho | \mu, i \rangle$$

Corresponding Density Operator

$$\boldsymbol{p} \equiv (p_{\mu i}) \to \rho(\boldsymbol{p}) = \sum_{\mu=1}^{d+1} \sum_{i=1}^{d} p_{\mu i} |\mu, i\rangle\langle \mu, i| - \mathbb{I}$$

[Example : d=2] 3 (=2+1) numbers of x, y and z Pauli Matrices are MUBs

・・・ Connection with Bloch Vector

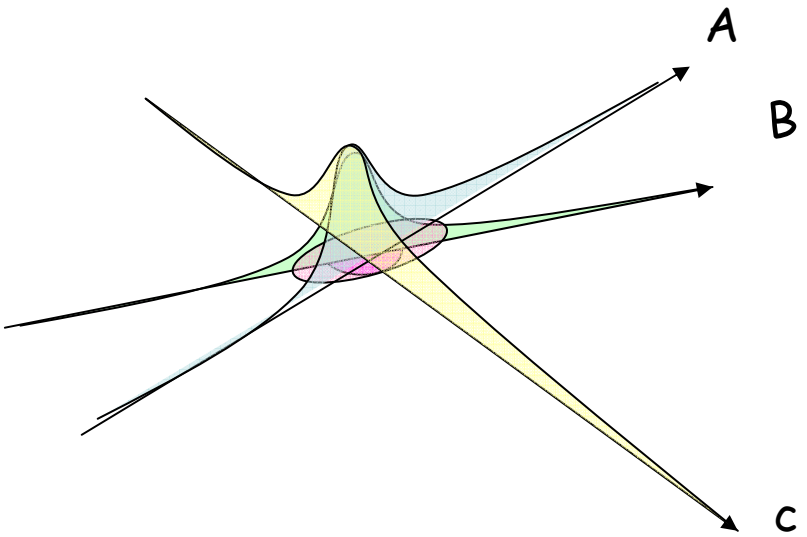
(Ivanovic 1981)

★ Mutually Unbiased Bases [optimal state determination]

d+1 numbers of MUBs provides an optimal state determination

in the sense that the effects of statistical errors are minimized !!

- In principle, ensembles are finite no matter how they are large
- In each measurement A,B, …, statistical errors appear
- Estimated state are in the intersection of their error ranges
- Find a set of observables with minimum intersection of error ranges



⟶ MUBs provides minimum int.

(Wootters & Fields 1989)

★ Mutually Unbiased Bases [open problem]

Existence of d+1 numbers of MUBs ?

★ at most d + 1 numbers !!
★ when d = p^m, d+1 numbers of MUBs do exist !!
★ Otherwise, unknown so far, even d = 6 !!

http://www.imaph.tu-bs.de/qi/problems/

Orthogonal Latin Squares (OLSs)

★ at most d - 1 numbers !!
★ when d = p^m, d-1 numbers of OLSs do exist !!
★ There are some d , e.g. , d = 6 and 10, less than d-1

Are there really some mathematical connection
between MUB problem and OLS problem ?

Zauner (1999),Klappenecker and Rotteler (2003), Wootters (2004)

★2 Mean King's Problem, with MUBs

★ Mean King's Problem [Introduction 1]

★ Alice has to guess King's output,
otherwise she will be executed !!

★ HHH found under the assumption of d+1 MUBs

Solution for Alice to survive with certainty

$\Leftrightarrow$ d-1 Maximum Numbers of OLSs

$\longrightarrow$ Alice might be executed in e.g., d=6 or 10 level systems !!!

Our Goal:
find solutions for Alice to survive with certainty
for arbitrary dimesion !!

Connection with Orthogonal Arrays !!

Hayashi, Horibe, & Hashimoto (2005)

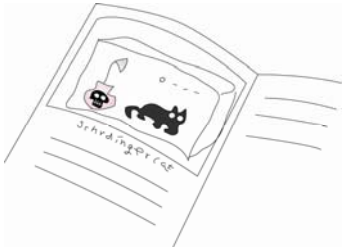
Mean King's Problem

Once upon a time,

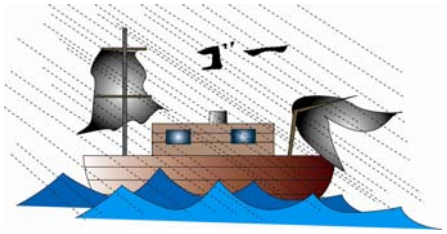
there lived a mean King who loved cats.

The King hated physicists

since he heard what had happened to Schroedinger's cat.



One day, a terrible storm came on, and



Alice, a physicist, got stranded on the island
that was ruled by the King.

The King called Alice to the royal laboratory
and gave her a challenge.

poor Alice …

★ Mean King's Problem [Introduction 2]

★1 King order Alice
to prepare d level quantum systems,
and hand it over to the King.

★2 The King will then secretly
measure it w.r.t. to one of d+1 MUBs
and return it to Alice.

I measure μ

Output i

★3 Alice is then allowed to perform
one more measurement on the system.

Output I

d

★4 Afterwards, the King reveals his measurement basis
and then Alice must immediately guess the correct
output of the King's measurement,

or she will be executed ...

I measured μ

King's output
was i !

L. Vaidman et al. (1987), Y. Aharonov & Englert (2001), etc..

★ Mean King's Problem [HHH method 1]

★ Alice has to guess King's output i ,
with her Output I and post information μ

$s(I,\mu)$: Estimation Function

★ No ways to survive with Prob. 1,
without entangling to some object

→ Use entanglement at the state preparation !!!!

HHH method $\mathbb{C}^d \otimes \mathbb{C}^d$ use the same d level system

$$|\Phi\rangle \equiv \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |i\rangle_A \otimes |i\rangle_K \in \mathbb{C}^d \otimes \mathbb{C}^d$$

maximal entangled state !!!

Allow Alice only projective measurement

→ Existence-Equivalence between
Alice's way to survive with certainty and d-1 OLSs

Hayashi, Horibe, & Hashimoto (2005)

★ Mean King's Problem [Our method 1]

@ She can perform not only projective measurement,
but also POVMs !!

[Notice: She can use another ancilla
with higher dimension d' to realize the POVMs]

Theorem 1 [G.K., H. Tanaka, M. Ozawa] For any d level system,

By using maximal entangled state, Alice can find a POVM
to guess King's output with probability 1 !

with a certain mathematical connection with Orthogonal Arrays

Alice can survive with certainty !!

★ Mean King's Problem [Proof]

★ prepare $\mathbb{C}^{d'} \otimes \mathbb{C}^{d}$ systems $\quad d' \geq d$

with the same maximal entanglement state

$$|\Phi\rangle \equiv \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |i\rangle_A \otimes |i\rangle_K \in \mathbb{C}^{d'} \otimes \mathbb{C}^{d}$$

$$= \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} \overline{|\mu, i\rangle}_A \otimes |\mu, i\rangle_K, \qquad (\overline{|\mu, i\rangle}_A \equiv \sum_{i=0}^{d-1} \overline{\langle i|\mu, i\rangle_K} |i\rangle_A)$$

$\xrightarrow{\mu}$ post measurement state

$$|\Phi_{\mu,i}\rangle \equiv \overline{|\mu, i\rangle}_A \otimes |\mu, i\rangle_K$$

$$\langle \Phi_{\mu,i} | \Phi_{\nu,j} \rangle = |\langle \mu, i | \nu, j \rangle_K|^2$$ MUB cond.

★ Mean King's Problem [Proof]

Find a projective measurement $\{|I\rangle\}_{I=0}^{dd'-1}$ on $\mathbb{C}^{d'} \otimes \mathbb{C}^{d}$

and an estimation function $s(I,\mu) \in \{0,\ldots,d-1\}$

s.t. (*) $\langle I|\Phi_{\mu,i}\rangle = 0$ whenever $s(I,\mu) \neq i$

Lemma 1

Given $s(I,\mu)$, there exists an orthonormal basis $\{|I\rangle\}$ satisfying (*)

iff there is a dd' × d(d+1) matrix $H(I;\mu,i)$ s.t.

(*1) $H(I;\mu,i) = 0$ whenever $s(I,\mu) \neq i.$

(*2) $H^{\dagger}H(\mu,i;\nu,j) = \delta_{\mu\nu}\delta_{ij} + (1-\delta_{\mu\nu})\frac{1}{d}$

★ Mean King's Problem [Orthogonal Array]

◎ Orthogonal Array $\mathrm{OA}_n(k,d)$

(degree k, order d and index n) $\Leftrightarrow$

an $nd^2 \times k$ array with entries from {0,1,…,d-1}
s.t. in each (ordered) pair of distinct columns,
every (ordered) pair occurs exactly n times.

★ every symbol occurs nd times in each column

For any (k,d), we can always find
a suitable n and OAn(k,d)

$\mathrm{OA}_1(4,3)$

0	0	0	0
0	1	1	1
0	2	2	2
1	0	1	2
1	1	2	0
1	2	0	1
2	0	2	1
2	1	0	2
2	2	1	0

Let $s(I,\mu)$ form an OAn(k,d)

$$\frac{1}{nd}\sum_{I=0}^{nd^2-1}\delta_{i,s(I,\mu)}\delta_{j,s(I,\nu)} = \delta_{\mu,\nu}\delta_{i,j} + (1-\delta_{\mu,\nu})\frac{1}{d}$$

★ Mean King's Problem [Proof]

◎ Let Alice prepare d' = nd Ancila

◎ Set $$H(I;\mu,i) \equiv \frac{1}{\sqrt{nd}}\delta_{i,s(I,\mu)}$$

where Alice use the estimation function $s(I,\mu)$
which forms an OAn(d+1,d)

Condition in Lemma 1 holds

(* 1) $H(I;\mu,i) = 0$ whenever $s(I,\mu) \neq i.$

(* 2) $H^{\dagger}H(\mu,i;\nu,j) = \delta_{\mu\nu}\delta_{ij} + (1-\delta_{\mu\nu})\frac{1}{d}$

From Lemma 1, Alice finds a suitable measurements $\{|I\rangle\}_{I=0}^{nd^2-1}$
and estimation function $s(I,\mu)$,
from which she can guess King's outputwith certainty !!

★ Mean King's Problem [Proof]

Let $P : \mathbb{C}^{d'} \otimes \mathbb{C}^d \longrightarrow \mathbb{C}^d \otimes \mathbb{C}^d$

and $M_I \equiv P|I\rangle\langle I|P$

Positive Operator Valued Measure in $\mathbb{C}^d \otimes \mathbb{C}^d$

(I) positivity $M_I \geq 0$

(II) normalization $\sum_{I=0}^{dd'-1} M_I = P^2 = P$

Q.E.D

★ Mean King's Problem [Conclusion]

We showed a solution of Mean King's problem always exists for any d

Using maximal entangled state and
a suitable POVM in $d \times d$ system

Mathematical connection between MUB
and Orthogonal Array

Unfortunately, this result tells nothing about
the existence of maximum numbers of MUBs

Possible Application: Quantum Cryptography

If it is safe, our protocol provides an efficient scheme for QC

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