ミクロ・マクロ双対性
― 量子場をマクロデータから再構成する数学的方法 ―

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Abstract

The deep and important significance of the so-called "Quantum-Classical Correspondence" will be elaborated and explained from the mathematical-physical viewpoint of "Micro-Macro Duality". In the operator-algebraic formulation this will be shown to be implemented in the form of Takesaki duality of crossed products as a specific realization of the essence of Galois-Fourier duality.

Contents:

1 Quantum-Classical Correspondence and Micro-Macro Duality

1) Quantum-classical correspondence (q-c correspondence, for short) = *classical macro*-objects as *condensates of* ∞*-quanta* in *micro-quantum* system

"Micro-Macro Duality" [1]:

a) Emphasis on **bi-directionality** for controlling [Micro $\rightleftarrows$ Macro]

Bi-directionality in [deduction (top-down) vs. induction (bottom-up)]

Importance of bi-directionality as [accessibility to actual physical systems] + [**feedback**]

Cf. Standard approach = **[one-directional deductions]** of macro-phenomena from "ultimate" theory of microscopic world

$\Longrightarrow$ What basis for starting postulates in a theoretical deduction?
Starting *hypothesis* $\mathcal{TH}$: an *ad hoc* theoretical *postulate*
$\Longrightarrow$Predictions deduced from the theory

$$\updownarrow \longleftarrow \text{ measurement processes}$$

Experimental data $\mathcal{EX}$
$\Longrightarrow$ Comparison between theory & experiments: to check the starting $\mathcal{TH}$ to be "verified" as **one of the possible candidates** as sufficient condition*s* for

$$\begin{array}{lccc} & \mathcal{TH} & \searrow & \\ \mathcal{EX}: & \mathcal{TH}_1 & \longrightarrow & \mathcal{EX}+errors \\ & \vdots & \nearrow & \end{array}$$

b) **Duality**: mathematical formulation(s) of bi-directionality = **mathematical dualities** (or categorical adjunctions) between the algebras of Micro-scopic systems and Macro = its states or representations detectable macroscopically

e.g., Fourier-Pontryagin-Tannaka-Krein-Tatsuuma duality of [group $G \rightleftarrows \hat{G}$ or $Rep(G)$] [2]

Takesaki duality of crossed products [3, 4]: $(\mathcal{M} \rtimes G) \rtimes \hat{G} \simeq \mathcal{M} \otimes B(L^2(G))$

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{\rtimes G} & \mathcal{M} \rtimes G \\ | \wr & & | \wr \\ \mathcal{M}^G \rtimes \hat{G} & \underset{\rtimes \hat{G}}{\longleftarrow} & \mathcal{M}^G \end{array}$$

c) Mathematical possibility for bi-directionality suggested by sector theory in algebraic QFT:

$\mathfrak{F}$: field algebra $\curvearrowleft G$: group of symmetry
$\Longrightarrow$ observable algebra $\mathfrak{A} = \mathfrak{F}^G$ as fixed-point subalgebra of G-invariants

$\mathfrak{A}(= \mathfrak{F}^G)$ + **something** $\underset{\substack{\text{reconstruction} \\ \text{as } inverse \text{ problem}}}{\Longrightarrow}$ $[\mathfrak{F} \curvearrowleft G]$

What are "**something**" & [reconstruction in inverse direction]?:
"something"=**D**(oplicher-)**H**(aag-)**R**(oberts) **selection criterion** to select *physically relevant states* [5]

$\Longrightarrow$ DR category $\mathcal{T}(\subset End(\mathfrak{A})) \overset{[6]}{\simeq} Rep(G)$
$\longleftrightarrow \hat{G}$: sectors $\overset{\text{Tannaka-Krein duality}}{\Longrightarrow}$ G

Reconstruction [6] of $\mathfrak{F} \curvearrowleft G$ via *crossed product* $\mathfrak{F} = \mathfrak{F}^G \rtimes \hat{G} \curvearrowleft G = Gal(\mathfrak{F}/\mathfrak{A})$: Galois group

I.e., $\mathfrak{F} \curvearrowleft G$ (: *Micro*) *is recovered from "Macro data" $\mathcal{T}$ of sectors of* $\mathfrak{A}$ through Galois extension by *solving* DHR criterion as an "*equation*" involving $\mathfrak{A}$ as coefficient ring

Thus, we need, in two steps,

2) Classification of states-representations of algebra $\mathfrak{A}$ of physical variables into sectors + intrasectorial structures

Sec.1.1 [Q-C Correspondence I]: sectors = factor rep.'s = pure phases $\longleftrightarrow$ centre of a given representation $\pi(\mathfrak{A})'' = \mathcal{M}$ according to **quasi-equivalence** = macroscopic order parameters $\Longrightarrow$ **classifying space of sectors =** $Spec(\mathfrak{Z}(\mathcal{M}))$

[Q-C Correspondence II] : operationally meaningful determination of **intrasectorial** structures of a **factor**

$\mathcal{M}$ (describing a sector) by means of a **MASA** $\mathcal{A} = \mathcal{A}' \cap \mathcal{M} = \mathcal{U}''$

$\Downarrow$

Sec.2: Q-C Correspondence (II): How to Detect Inside of Sectors [1]

MASA $\mathcal{A}$ and measurement scheme by Kac-Takesaki operator $\rightarrow$ instrument:

$\rightarrow$ measurement **coupling** to make a composite system of $\mathcal{M}$ + measuring system **identified with** $\mathcal{A}$: determined by **K-T operator** of $\widehat{\mathcal{U}}$

$\rightarrow$ what is a K-T operator? $\rightarrow$ **instrument**

States (within a sector corresponding to $\mathcal{M}$) $\longleftrightarrow$ $Spec(\mathcal{A})$= **classifying space of intrasectorial states**

[: Analogy to roots of a semi-simple Lie algebra wrt a Cartan subalgebra]

Sec.3: Crossed Product $\mathcal{M} \rtimes_\alpha \mathcal{U}$ to Couple System & Apparatus [7]

Measurement coupling = **crossed product** $\mathcal{M} \rtimes_\alpha \mathcal{U}$ composed of object system + measuring system:
$\Longrightarrow$ determination $\mathcal{M} \rtimes_\alpha \mathcal{U} = \mathcal{A} \otimes B(L^2(\mathcal{U}))$ of algebra $\mathcal{A}$ of observables to be measured + reservoir system $B(L^2(\mathcal{U}))$ to amplify $\mathcal{A}$ with damping others (under the assumption of *semi-duality*)

$\rightarrow$ **Takesaki duality** $(\mathcal{M} \rtimes_\alpha \mathcal{U}) \rtimes_{\hat{\alpha}} \widehat{\mathcal{U}} = \mathcal{M} \otimes B(L^2(\mathcal{U})) = \mathcal{M}$: **Galois-Fourier duality** to generate measured values $Spec(\mathcal{A}) \subset \widehat{\mathcal{U}}$ + to recover the original system $\mathcal{M}$ from the measurement situation with $\mathcal{M} \rtimes_\alpha \mathcal{U}$ & **observed data structure**: $\mathcal{A} \subset (\mathcal{M} \rtimes_\alpha \mathcal{U}) \underset{\hat{\alpha}}{\curvearrowleft} \widehat{\mathcal{U}}$ (with the latter encoded in the corresponding instrument)

The same structure as above found also in sector theory:
field algebra $\mathfrak{F} \underset{\tau}{\curvearrowleft} G \Longrightarrow \mathfrak{F} \rtimes_\tau G \simeq \mathfrak{F}^G = \mathfrak{A}$: observable algebra $\overset{\text{DHR criterion}}{\Longrightarrow}$ sector data $= \hat{G} \underset{\hat{\tau}}{\curvearrowright} \mathfrak{A}$
$\Longrightarrow \mathfrak{A} \rtimes_{\hat{\tau}} \hat{G} = (\mathfrak{F} \rtimes_\tau G) \rtimes_{\hat{\tau}} \hat{G} = \mathfrak{F} \otimes B(L^2(G)) = \mathfrak{F}$

Sec.4: Reconstruction of Micro-Algebra $\mathcal{M}$ & its Type-Classification [7]

Starting from the problem of **state** determination (inside a sector): $\mathcal{M} \to \mathcal{M} \rtimes_\alpha \mathcal{U}$: measurement coupling

$\Longrightarrow$ operationally meaningful reconstruction of **algebra** $\mathcal{M}$ through crossed product $(\mathcal{M} \rtimes_\alpha \mathcal{U}) \rtimes_{\hat{\alpha}} \widehat{\mathcal{U}} = \mathcal{M}$

Classification scheme formulated in the **classifying space** $Spec(\mathcal{A}) \underset{\hat{\alpha}}{\curvearrowleft} \widehat{\mathcal{U}}$ reveals that the starting assumption of $\alpha = Ad^{\mathcal{U}}$($=$ external coupling only!) can reproduce only $\mathcal{M}$ of **type I**

$\Longrightarrow$ Operational detection of the intrinsic **dynamics** α with deviation from $Ad^{\mathcal{U}}$: necessary!
: conceptually very important observation to support the above *bi-directionality*

NB. Subtleties of *non-type I* algebras (e.g., typical cases of *type III* in relativistic QFT): a state within a sector cannot uniquely be specified by means of quantum observables for lack of *minimal projections*.

1.1 Q-C Correspondence (I) Sectors & centre = order parameters

[Q-C Correspondence I]= Major gap between Quantum Micro and Classical Macro, in terms of superselection sectors and *intersectorial* structures described by order parameters [8]:

At this level, clear-cut separation between quantum and classical is OK by order parameters to specify a sector

Sectors (or pure phases) $\overset{\text{def}}{\equiv}$ $\begin{array}{c}\textit{quasi-equivalence classes}\\ \textit{of factor states/rep.'s}\end{array}$: faithfully parametrized by Spec of centre of a relevant representation of algebra of micro-quantum observables to be interpreted as *macroscopic order parameters*

Quasi-equivalence [9]= unitary equivalence *up to multiplicity*

centre (of rep.'s) = *commutative* algebra of *macroscopic order parameters*

$\Longrightarrow$ (*superselection*) *sector structure* consisting of a family of *sectors* (or *pure phases*) described mathematically by factor states and representations

Totality of sectors $\Longrightarrow$ *mixed phase* involving both classical and quantum aspects.

Intersectorial structure = coexistence of and gap between quantum(=intrasectorial) and classical(=intersectoria aspects.

$\simeq Spec$(**centre**): **classifying space of sectors** to distinguish among different sectors

$\Longrightarrow$ At this level, Micro-Macro relation reduces to the relations between

Quantum (=*non-commutative*)	vs.	**Classical** (=*commutative*)

$\Longrightarrow$ A **unified scheme** for Micro-Macro relations can be formulated on the basis of selection criteria [8]:

$$\begin{array}{ccc} \text{(Micro)} & & \text{(Macro)} \\ \text{i)}\left[q: \begin{array}{c}\text{generic states} \\ \text{of } \textit{object system}\end{array}\right] & \underset{\uparrow}{\Longrightarrow} & \text{ii)}\left[c: \begin{array}{c}\textbf{reference sys.} \\ \text{with } \textbf{classifying} \\ \textbf{space} \text{ of } \textbf{sectors}\end{array}\right] \end{array}$$

iii) map to *compare* i) with ii)

$$\begin{array}{ccc} \Uparrow & & \Downarrow \\ \text{iv)}\left[\begin{array}{c}\textbf{selection} \\ \textbf{criterion}\text{:} \\ \text{ii)} \underset{c\text{-}q}{\Longrightarrow} \text{i)}\end{array}\right] & \overset{\textit{adjunction}}{\rightleftarrows} & \left[\begin{array}{c}\textit{classification \&} \\ \textit{interpretation} \text{ of i)} \\ \text{w.r.t. ii): i)} \underset{q\text{-}c}{\Longrightarrow} \text{ii)}\end{array}\right] \end{array}$$

as a natural generalization of

Example 1 *Manifold M with local charts $\{(U_\lambda, \varphi_\lambda : U_\lambda \to \mathbb{R}^n)\}$:*
i)= local neighbourhoods U_λ, ii)= $\mathbb{R}^n$,
iii)= local charts $\varphi_\lambda : U_\lambda \to \mathbb{R}^n$,
iv)= geometrical interpretation in terms of geometrical invariants such as homology, cohomology, homotopy, K-groups, characteristic classes, etc., etc.

Example 2 *Non-equilibrium local states characterized by localizing generalized equilibrium states with fluctuating thermal parameters [10, 11]:*
i) = set E_x of states ω with local energy bound $\omega((\mathbf{1}+H_{\mathcal{O}})^m)<\infty$,
ii) = classifying space B_K of thermodynamic phases with fluctuating parameters (β,μ) described by prob. meas.'s $\rho\in M_+(B_K)=:Th$.
iii) = comparison of unknown ω with known reference states $\omega_\rho=\mathcal{C}^*(\rho)=\int_{B_K}d\rho(\beta,\mu)\omega_{\beta,\mu}$ through examining criterion $\omega \underset{\mathcal{T}_x}{\equiv} \mathcal{C}^*(\rho)$ via "quantum fields at x" $\in\mathcal{T}_x$ (justified by energy bound in i)),
iv) = adjunction (as localized 0-th law of thermodynamics), $[\omega \underset{\mathcal{T}_x}{\equiv} \mathcal{C}^*(\rho)] \rightleftarrows [(\mathcal{C}^*)^{-1}(\omega) \underset{\mathcal{C}^*(\mathcal{T}_x)}{\equiv} \rho]$, or,

$$[E_x/\mathcal{T}_x]\,(\omega,\mathcal{C}^*(\rho)) \overset{q\rightleftarrows c}{\simeq} [Th/\mathcal{C}(\mathcal{T}_x)]\,((\mathcal{C}^*)^{-1}(\omega),\rho),$$

with q→c channel "$(\mathcal{C}^*)^{-1}$" adjoint to c→q channel $\mathcal{C}^*$ (from **c***lassical reference sys. to generic* **q***uantum states) achieves two goals, a) to identify a local thermal state ω and b) to give thermal interpretation $\rho \underset{\mathcal{C}^*(\mathcal{T}_x)}{\equiv} (\mathcal{C}^*)^{-1}(\omega)$ of ω in terms of known vocabulary $\rho\in Th$.*

2 Q-C Correspondence (II): How to Detect Inside of Sectors

Intrasectorial structures and measurement processes: To describe intrinsic quantum structures *within* a sector, not only theoretically but also operationally, necessarily involving quantum measurements.

2.1 Maximal abelian subalgebra

MASA $\mathcal{A} =$ maximal abelian subalgebra consisting of simultaneously measurable observables for specifying states inside a sector described by a **factor** $\mathcal{M}$ i.e., pointer positions on $Spec(\mathcal{A})$ of the measuring apparatus should determine a microscopic quantum state of $\mathcal{M}$ (**)
Reformulation necessary of the traditional MASA $\mathcal{A}' = \mathcal{A}$ to accommodate quantum systems with

∞-degrees of freedom (with non-type I representations) such as quantum fields: $\mathcal{A} = \mathcal{A}' \cap \mathcal{M}$

[**NB:** $\mathcal{A}' = \mathcal{A} \Longrightarrow \mathcal{M}$: type I $\because$) $\mathcal{A}' = \mathcal{A} \subset \mathcal{M} \Longrightarrow \mathcal{M}' \subset \mathcal{A}' = \mathcal{A} \subset \mathcal{M} \Longrightarrow \mathcal{M}' = \mathcal{M}' \cap \mathcal{M} = \mathfrak{Z}(\mathcal{M})$]
$\Longrightarrow$ measuring apparatus: **identified with** MASA $\mathcal{A} = \mathcal{A}' \cap \mathcal{M}$
$\Longrightarrow$ need to find a coupling between algebras $\mathcal{M}$ and $\mathcal{A}$ to realize (**)!!

To solve this problem, use $\exists\, \mathcal{U}$: abelian Lie group with Haar measure du which generates MASA $\mathcal{A}$:

$$\begin{aligned} \mathcal{U} &\subset \mathcal{U}(\mathcal{A}), \mathcal{A} = \mathcal{U}'' \\ &\Longrightarrow \mathcal{A} = \mathcal{M} \cap \mathcal{A}' = \mathcal{M} \cap \mathcal{U}' \\ &\Longrightarrow \text{W*-dynamical system } \mathcal{M} \underset{\alpha}{\curvearrowleft} \mathcal{U} \\ \text{with } \mathcal{A} &= \mathcal{M}^{\alpha(\mathcal{U})} \text{ and } \alpha = Ad. \end{aligned}$$

$\Longrightarrow$ relevance of **group duality** & **Galois extension (=Fourier-Galois duality)** expected naturally
MASA + measurement **coupling** to make a composite system of $\mathcal{M}$ + measuring system $\mathcal{A}$: determined by **Kac-Takesaki operator** [12, 4, 2] (**K-T operator**, for short) of dual group $\widehat{\mathcal{U}}$
$\rightarrow$ what is a K-T operator? $\rightarrow$ **instrument**

2.2 K-T operator & instrument [1]

i) *Hopf-(Kac-)von Neumann algebra* $M(\subset B(\mathfrak{H}))$ with Haar weight & coproduct $\Gamma : M \to M \otimes M$

$$\begin{array}{ccc} M & \overset{\Gamma}{\longrightarrow} & M \otimes M \\ \Gamma \downarrow & & \downarrow id \otimes \Gamma \\ M \otimes M & \underset{\Gamma \otimes id}{\longrightarrow} & M \otimes M \otimes M \end{array}$$

ii) **K-Toperator** $V \in \mathcal{U}((M \otimes M_*)^-)(\subset \mathcal{U}(\mathfrak{H} \otimes \mathfrak{H}))$: defined as the implementer of Γ,

$$\begin{aligned} \Gamma(x) &= V^*(\mathbf{1} \otimes x)V \quad \text{for } x \in M \\ V_{12}V_{13}V_{23} &= V_{23}V_{12} \text{ on } \mathfrak{H} \otimes \mathfrak{H} \otimes \mathfrak{H} \end{aligned}$$

called **pentagonal relation** to express coassociativity of Γ. (V_{ij} acts on i-th and j-th factors in $\mathfrak{H} \otimes \mathfrak{H} \otimes \mathfrak{H}$)

$$\begin{array}{rcl} \text{intertwining relation} & : & V(\lambda \otimes \iota) = (\lambda \otimes \lambda)V \\ \text{Fourier transform} & : & \lambda : M_* \ni \omega \longmapsto \lambda(\omega) \\ & : & = (\omega \otimes id)(V) \in \hat{M} \\ \text{s.t.} & & \lambda(\omega_1 * \omega_2) = \lambda(\omega_1)\lambda(\omega_2) \\ \text{convolution product in } M_* & : & \omega_1 * \omega_2 := \omega_1 \otimes \omega_2 \circ \Gamma \end{array}$$

Group duality can be formulated as Kac duality [2] $V \longleftrightarrow \hat{V} := \sigma V^* \sigma$ (with $\sigma(\xi \otimes \eta) = \eta \otimes \xi$).
For $M = L^\infty(G, dg)$ with locally compact group G with Haar measure dg, K-T operator V is given on $L^2(G \times G)$ by

$$(V\xi)(s,t) := \xi(s, s^{-1}t) \quad \text{for } \xi \in L^2(G\times G), s, t \in G,$$

or symbolically, $V|s,t\rangle = |s, st\rangle$ in Dirac-type notation.

iii) Comparison between $Spec(\mathcal{A})$) of MASA $\mathcal{A}$ and dual group $\widehat{\mathcal{U}}$:

$Spec(\mathcal{A})$: characters on abelian *algebra* $\mathcal{A} = \mathcal{U}''$
$\widehat{\mathcal{U}}$: group characters γ on abelian *group* $\mathcal{U} \subset \mathcal{A}$ s.t. $\gamma(u_1 u_2) = \gamma(u_1)\gamma(u_2)$, $\gamma(e) = 1$ (u_1, $u_2 \in \mathcal{U}$).

$Spec(\mathcal{A}) \hookrightarrow \widehat{\mathcal{U}}$ through $Spec(\mathcal{A}) \ni \chi \longmapsto \chi \restriction_{\mathcal{U}} \in \widehat{\mathcal{U}}$, but the opposite direction not guaranteed!

($Spec(\mathcal{A}) \simeq \widehat{\mathcal{U}}$ very close to $\mathcal{M}$: type I)

$\Longrightarrow$ **Distinction** between $\mathcal{U}$ as abstract group and unitary group $\mathcal{U} \subset \mathcal{M}$ embedded in $\mathcal{M}$ by group homomorphism $E : \mathcal{U} \hookrightarrow \mathcal{U}(\mathcal{M})$ associated with $\mathcal{A} \hookrightarrow \mathcal{M}$:

$$E(u) = \int_{\gamma\in\widehat{\mathcal{U}}} \gamma(u) dE(\gamma) \quad (u \in \mathcal{U}),$$

where dE is an $\mathcal{M}$-valued spectral measure $E(\Delta) = E(\chi_\Delta)$ defined for Borel sets Δ in $\widehat{\mathcal{U}}$.

iv) **K-T operator** V on $L^2(\widehat{\mathcal{U}}) \otimes L^2(\widehat{\mathcal{U}})$ in our context:

$(V\xi)(\gamma, \chi) := \xi(\gamma, \gamma^{-1}\chi)$ for $\gamma, \chi \in \widehat{\mathcal{U}},\ \xi \in L^2(\widehat{\mathcal{U}})$.

$\Longrightarrow$ **Representation** $E_*(V) = \int_{\gamma\in\widehat{\mathcal{U}}} dE(\gamma) \otimes \lambda_\gamma$ of V on $L^2(\mathcal{M}) \otimes L^2(\widehat{\mathcal{U}})$ defined by

$$E_*(V)(\xi_\Delta \otimes |\chi\rangle) = \int_{\gamma\in\Delta} dE(\gamma)\xi_\Delta \otimes |\gamma\chi\rangle, \qquad (1)$$

$$\text{for } \gamma, \chi \in \widehat{\mathcal{U}}, \quad \xi_\Delta \in E(\Delta)L^2(\mathcal{M}),$$

satisfying modified pentagonal relation,

$$E_*(V)_{12}E_*(V)_{13}V_{23} = V_{23}E_*(V)_{12}.$$

($L^2(\mathcal{M})$: standard-form representation space of $\mathcal{M}$)

Neutral position of measuring pointer = identity character $\iota \in \widehat{\mathcal{U}}$, $\iota(u) \equiv 1$ ($\forall u \in \mathcal{U}$).

(For non-cpt $\mathcal{U}$, $\nexists$vector $|\iota\rangle \in L^2(\mathcal{U})$ corresponding to $\iota \in \widehat{\mathcal{U}}$, but invariant mean $m_{\mathcal{U}}$ behaves as $\langle\iota|\cdots|\iota\rangle$ owing to amenability of abelian $\mathcal{U}$.)

$\Longrightarrow$ By Eq. (1) with $\chi = \iota$: neutral position $\iota \in \widehat{\mathcal{U}}$, **correlation** required for measurements ("perfect correlation" due to Ozawa [13]),

$$E_*(V)(\xi_\gamma \otimes |\iota\rangle) = \xi_\gamma \otimes |\gamma\rangle \qquad (\forall\gamma \in \widehat{\mathcal{U}}),$$

is established between states ξ_γ of Micro-system $\mathcal{M}$ and $|\gamma\rangle$ of measuring system $\mathcal{A}$ (if $\widehat{\mathcal{U}}$ is *discrete*). For generic state $\xi = \sum_{\gamma\in\widehat{\mathcal{U}}} c_\gamma \xi_\gamma$ of $\mathcal{M}$, *uncorrelated* initial state $\xi \otimes |\iota\rangle$ is transformed by $E_*(V)$ into *correlated* one:

$$E_*(V)(\xi \otimes |\iota\rangle) = \sum_{\gamma\in\widehat{\mathcal{U}}} c_\gamma \xi_\gamma \otimes |\gamma\rangle.$$

By this correlation, *one-to-one* correspondence is materialized between ξ_γ of $\mathcal{M}$ and measured data γ on the pointer.

v) Mathematical definition of **instrument** $\mathfrak{I} =$ (completely) positive operation-valued measure:

$$\mathfrak{I}(\Delta|\omega_\xi)(B) := (\omega_\xi \otimes m_{\mathcal{U}})(E_*(V)^*(B \otimes \chi_\Delta)E_*(V)) \\ = ((\langle\ \xi| \otimes \langle\iota|)E_*(V)^*(B \otimes \chi_\Delta)E_*(V)(|\xi\rangle \otimes |\iota\rangle$$

which unifies all ingredients relevant to a measurement. For an initial state $\omega_\xi = \langle\ \xi|\ (-)\xi\rangle$ of $\mathcal{M}$, $p(\Delta|\omega_\xi) = \mathfrak{I}(\Delta|\omega_\xi)(\mathbf{1})$: probability for measured values of observables in $\mathcal{A}$ to be found in a Borel set Δ and
$\mathfrak{I}(\Delta|\omega_\xi)/p(\Delta|\omega_\xi)$: final state realized after detection of measured values in Δ [14].

3 Crossed Product $\mathcal{M} \rtimes_\alpha \mathcal{U}$ = System + Apparatus [7]

i) Relevance of **Fourier-Galois duality**: By *Fourier transform* $\mathcal{F}$ (as unitary transf.: $L^2(\mathcal{U}, du) \to L^2(\widehat{\mathcal{U}}, d\gamma)$),

$$(\mathcal{F}\xi)(\gamma) := \int_{\mathcal{U}} \overline{\gamma(u)}\xi(u)du, \qquad \gamma \in \widehat{\mathcal{U}},$$

K-T operator V on $\widehat{\mathcal{U}}$ is transformed into K-T operator $W \in \lambda(\mathcal{U})'' \otimes L^\infty(\mathcal{U})$ on $\mathcal{U}$:

$$W =: (\mathcal{F} \otimes \mathcal{F})^{-1} V (\mathcal{F} \otimes \mathcal{F}),$$

$$(W\xi)(u, v) := \xi(vu, v) \quad \text{for } \xi \in L^2(\mathcal{U} \times \mathcal{U}), u, v \in \mathcal{U}$$

satisfying pentagonal and intertwining relations:

$$\begin{aligned} W_{12}W_{13}W_{23} &= W_{23}W_{12}, \\ W(\lambda_u \otimes \lambda_u) &= (I \otimes \lambda_u)W \quad (u \in \mathcal{U}) \end{aligned}$$

for regular representation $\lambda = \lambda^{\mathcal{U}}$ of $\mathcal{U}$.

ii) Representation EW of W through $E : \mathcal{A} \hookrightarrow \mathcal{M}$: $EW := (E \otimes id)(W) \in \mathcal{M} \otimes L^{\infty}(\mathcal{U})$ satisfies

$$\begin{aligned} (EW)_{12}(EW)_{13}W_{23} &= W_{23}(EW)_{12}, \\ EW(u \otimes \lambda_u) &= (I \otimes \lambda_u)EW. \end{aligned}$$

With adjoint action $\alpha = Ad$ of $\mathcal{U}$ on $\mathcal{M}$, we have *-automorphism $r(\alpha)$ in $\mathcal{M} \otimes L^{\infty}(\mathcal{U})$:

$$\begin{aligned} (r(\alpha)(X))(u) \quad &: = \alpha_u^{-1}(X(u)) \\ \text{for } X &\in \mathcal{M} \otimes L^{\infty}(\mathcal{U}), u \in \mathcal{U}, \end{aligned} \tag{2}$$

implemented by EW^*,

$$r(\alpha)(X) = (EW)X(EW)^*.$$

iii) (W*-)crossed product $\mathcal{M} \rtimes_{\alpha} \mathcal{U}$:
W*-crossed product $\mathcal{M} \rtimes_{\alpha} \mathcal{U}$ is defined by the von

Neumann algebra generated by $r(\alpha)(\mathcal{M}\otimes 1)$ and by $\mathbb{C}\otimes\lambda(\mathcal{U})''$:

$$\mathcal{M}\rtimes_{\alpha}\mathcal{U} := r(\alpha)(\mathcal{M}\otimes 1)\vee(\mathbb{C}\otimes\lambda(\mathcal{U})'').$$

$\Longrightarrow$ **Crossed product** $\mathcal{M}\rtimes_{\alpha}\mathcal{U}$ composed of object system + measuring system is due to Fourier transform $W=(\mathcal{F}\otimes\mathcal{F})^{-1}V(\mathcal{F}\otimes\mathcal{F})$ of K-T operator V responsible for measurement coupling + instrument

3.1 Takesaki duality and physical meaning of crossed product

i) **Takesaki duality as Galois-Fourier duality:**

$$\begin{aligned}(\mathcal{M}\rtimes_{\alpha}\mathcal{U})\rtimes_{\hat{\alpha}}\widehat{\mathcal{U}} &= \mathcal{M}\otimes B(L^2(\mathcal{U})) \\ &= \mathcal{M}\ (\because\ \text{if properly}\ \infty)\end{aligned}$$

$\mathcal{M}\rtimes_{\alpha}\mathcal{U}$: to set up the measurement coupling to yield measured values $Spec(\mathcal{A})\subset\widehat{\mathcal{U}}$

$(\mathcal{M}\rtimes_{\alpha}\mathcal{U})\rtimes_{\hat{\alpha}}\widehat{\mathcal{U}}$ to recover the original system $\mathcal{M}$ from the coupling system $\mathcal{M}\rtimes_{\alpha}\mathcal{U}$ + **observed data**

structure: $\mathcal{A} \subset (\mathcal{M} \rtimes_\alpha \mathcal{U}) \underset{\hat{\alpha}}{\curvearrowleft} \widehat{\mathcal{U}}$ (encoded in the instrument)

Same structure found also in sector theory:
field algebra $\mathfrak{F} \underset{\tau}{\curvearrowleft} G \Longrightarrow \mathfrak{F} \rtimes_\tau G \simeq \mathfrak{F}^G = \mathfrak{A}$: observable algebra $\overset{\text{DHR criterion}}{\Longrightarrow}$ sector data $= \hat{G} \underset{\hat{\tau}}{\curvearrowright} \mathfrak{A}$
$\Longrightarrow \mathfrak{A} \rtimes_{\hat{\tau}} \hat{G} = (\mathfrak{F} \rtimes_\tau G) \rtimes_{\hat{\tau}} \hat{G} = \mathfrak{F} \otimes B(L^2(G)) = \mathfrak{F}$

$\Downarrow$

ii) Information on $\mathcal{M} \rtimes_\alpha \mathcal{U}$ is crucial not only for generating data to be measured but for recovering $\mathcal{M}$ from the observational data!

To proceed further, we tacitly assume a mathematical condition called *semi-duality* [4] of action α on $\mathcal{M}$: if $\exists$ unitary $v \in \mathcal{M} \otimes \lambda(G)''$ s.t. $\overline{\alpha}(v) = (v \otimes 1)(1 \otimes V')$, with a K-T operator V' defined by $(V'\xi)(g_1, g_2) = \xi(g_1 g_2, g_2)$ and $\overline{\alpha} := (\iota \otimes \sigma) \circ (\alpha \otimes \iota)$, a G-action α on $\mathcal{M}$ is called **semi-dual** $\Longrightarrow \mathcal{M} \rtimes_\alpha \mathcal{U} = \left[\mathcal{M} \otimes B(L^2(\mathcal{U}))\right]^{\alpha \otimes \lambda} = \mathcal{M}^\alpha \otimes B(L^2(\mathcal{U}))$

$\Longrightarrow$ determination $\mathcal{M} \rtimes_\alpha \mathcal{U} = \mathcal{M}^\alpha \otimes B(L^2(\mathcal{U}))$ of algebra $\mathcal{M}^\alpha = \mathcal{A}$ of observables to be measured + reservoir system $B(L^2(\mathcal{U}))$ to **amplify** $\mathcal{A}$ with damping other effects:
$\Longleftarrow$ strong Morita equivalence $\mathfrak{A}_1 \approx \mathfrak{A}_2$ of algebras $\mathfrak{A}_1$, $\mathfrak{A}_2 \Longleftrightarrow Rep_{\mathfrak{A}_1} \simeq Rep_{\mathfrak{A}_1}$
$\Longleftrightarrow$ stability $\mathfrak{A}_1 \otimes \mathcal{K} \simeq \mathfrak{A}_2 \otimes \mathcal{K}$.

Physically convenient for ensuring stability of the system against noise perturbations from its neglected surroundings.

Let $\mathcal{M}$ and $\mathcal{A} = \mathcal{A}' \cap \mathcal{M}$ be, respectively, a properly infinite v.N. algebra and its MASA generated by a loc. cpt. abelian unitary group $\mathcal{U} \subset \mathcal{A} = \mathcal{U}'' = \mathcal{M}^{\alpha(\mathcal{U})}$. If semi-duality holds with dual co-action $\hat{\alpha}$ of $\widehat{\mathcal{U}}$ on $\mathcal{A} \otimes B(L^\infty(\mathcal{U}))$, Takesaki duality [3] for $\mathcal{M}$,

$$(\mathcal{M} \rtimes_\alpha \mathcal{U}) \rtimes_{\hat{\alpha}} \widehat{\mathcal{U}} \simeq \mathcal{M} \otimes B(L^\infty(\mathcal{U})) \simeq \mathcal{M},$$

and for $\mathcal{A}$,

$$(\mathcal{A} \otimes B(L^\infty(\mathcal{U}))) \rtimes_{\hat{\alpha}} \widehat{\mathcal{U}}) \rtimes_\mu \mathcal{U} \simeq \mathcal{A} \otimes B(L^\infty(\widehat{\mathcal{U}})),$$

can be decomposed into the following isomorphisms:
(i) $\mathcal{M} \rtimes_\alpha \mathcal{U} \simeq \mathcal{A} \otimes B(L^\infty(\mathcal{U}))$[: amplification process],
(ii) $(\mathcal{A} \otimes B(L^\infty(\mathcal{U})) \rtimes_{\hat{\alpha}} \widehat{\mathcal{U}} \simeq \mathcal{M}$[: reconstruction].

Isomorphism ii) $\Longrightarrow$ *recovery of unknown* micro-algebra $\mathcal{M}$ of quantum observables via crossed product $(\mathcal{A} \otimes B(L^\infty(\mathcal{U})) \rtimes_{\hat{\alpha}} \widehat{\mathcal{U}}$ from *macroscopically visible* MASA $\mathcal{A}$ and $B(L^\infty(\mathcal{U}))=$ CCR consisting of $\mathcal{U}$ + *measured data* $\widehat{\mathcal{U}}$ coupled through $\widehat{\mathcal{U}}$-action $\hat{\alpha}$.
i)$\Longleftrightarrow$ii): mutually equivalent by *Takesaki duality* [3], and controlled by K-T operators V and $W = (\mathcal{F} \otimes \mathcal{F})^{-1} V (\mathcal{F} \otimes \mathcal{F})$ generating the measurement coupling $E_* V$. For instrument $\mathfrak{I}$, we have

$$\mathfrak{I}(\Delta|\omega_\xi)(B) := (\omega_\xi \otimes m_{\mathcal{U}})(r(\hat{\alpha})(B \otimes \chi_\Delta))$$

with $B \otimes \chi_\Delta \in \mathcal{M} \rtimes_\alpha \mathcal{U}$. Thus,
1st crssd prdct $\mathcal{M} \underset{\alpha}{\rtimes} \mathcal{U} =$ Micro-Macro composite system with $\widehat{\mathcal{U}}$-action $\underset{\hat{\alpha}}{\curvearrowright} (\mathcal{M} \underset{\alpha}{\rtimes} \mathcal{U}) \overset{\text{instrument } \mathfrak{I}}{\Longrightarrow}$ readable data $\in Spec(\mathcal{A}) \subset \widehat{\mathcal{U}}$ of $\mathcal{A}$
$\Longrightarrow$ 2nd crssd prdct $(\mathcal{M} \underset{\alpha}{\rtimes} \mathcal{U}) \underset{\hat{\alpha}}{\rtimes} \widehat{\mathcal{U}} \simeq \mathcal{M}$: Micro system recovered from Macro data $\widehat{\mathcal{U}}$ of $\mathcal{A} \Longrightarrow$ bi-directionality OK!

4 Reconstruction of Micro-Algebra $\mathcal{M}$ & its Type-Classification

Recovery of $\mathcal{M}$ from dyn. sys. $\mathcal{A}\otimes B(L^{\infty}(\mathcal{U})) \underset{\hat{\alpha}}{\curvearrowleft} \widehat{\mathcal{U}}$ through $(\mathcal{A} \otimes B(L^{\infty}(\mathcal{U}))) \rtimes_{\hat{\alpha}} \widehat{\mathcal{U}} \simeq \mathcal{M}$

$\Downarrow$

i) Data of dynamical system $\mathcal{A} \otimes B(L^{\infty}(\mathcal{U})) \underset{\hat{\alpha}}{\curvearrowleft} \widehat{\mathcal{U}}$

$\downarrow$

ii) modular data of $(\mathcal{A} \otimes B(L^{\infty}(\mathcal{U}))) \rtimes_{\hat{\alpha}} \widehat{\mathcal{U}}$

$\downarrow$

iii) structure of $\mathcal{M}$ ([7])

States (within a sector corresponding to $\mathcal{M}$) $\longleftrightarrow$ $Spec(\mathcal{A})$= **classifying space of intrasectorial states** in harmony with our general strategy in Sec.2 in the sense of **intrasectorial** analysis = [**sector analysis** of coupled system].

i) Dynamical system $\mathcal{N} \underset{\hat{\alpha}}{\curvearrowleft} G$ with $\mathcal{N} := \mathcal{A} \otimes B(L^{\infty}(\mathcal{U}))$ and $G = \widehat{\mathcal{U}}$ and its central part: $\mathcal{A} \underset{\beta}{\curvearrowleft} G$ with $\mathfrak{Z}(\mathcal{N}) = \mathcal{A}$.

Proposition 3 *For W*-crossed product $\mathcal{Q} = \mathcal{A} \rtimes_\beta G$ of an abelian dynamical system $\mathcal{A} \underset{\beta}{\curvearrowleft} G$:*
(i) action β: free $\Longleftrightarrow$ $\mathcal{A}$: maximally abelian in $\mathcal{Q}$: $\mathcal{A} = \mathcal{Q} \cap \mathcal{A}'$;
(ii) when β is free, $\mathcal{Q}$ is a factor $\Longleftrightarrow$ β is ergodic. In this case, $\mathfrak{Z}(\mathcal{Q}) = \mathcal{A}^\beta$.

Central ergodicity of $\hat{\alpha}$ is related with factoriality of crossed product $\mathcal{M} = \mathcal{N} \rtimes_{\hat{\alpha}} G$:

For $\hat{\alpha}$ free on the centre, the equalities hold

$$\mathfrak{Z}(\mathcal{M}) = \mathfrak{Z}(\mathcal{Q}) = r(\hat{\alpha})(\mathfrak{Z}(\mathcal{N})^{\hat{\alpha}} \otimes 1),$$

and hence, the following conditions are equivalent:
(i) action $\hat{\alpha}$ is ergodic on the centre;
(ii) $\mathcal{M} = \mathcal{N} \rtimes_{\hat{\alpha}} G$: factor;
(iii) $\mathcal{Q} = \mathcal{A} \rtimes_\beta G$: factor.

ii) Modular data of $\mathcal{M} = \mathcal{N} \rtimes_{\hat{\alpha}} G$:

$$\left(\widetilde{\Delta}^{it}\xi\right)(s) = \Delta^{it}_{\phi\circ\hat{\alpha}_s,\phi}\xi(s),$$
$$\left(\widetilde{J}\xi\right)(s) = U_\phi(s)J_\phi\xi(s^{-1}), \quad \xi \in L^2(G, \mathfrak{H}_\phi), s \in G.$$

$\Delta_{\phi\circ\hat{\alpha}_s,\phi}$: relative modular operator from n.f.s. weight ϕ of $\mathcal{N}$ to $\phi\circ\hat{\alpha}_s$

Connes cocycle derivative: $V_t = (D(\phi\circ\hat{\alpha}_s) : D\phi)_t = \Delta^{it}_{\phi\circ\hat{\alpha}_s,\phi}\circ\Delta^{-it}_{\phi}$

Dual weight $\widehat{\phi}$ of $\mathcal{N}\rtimes_{\hat{\alpha}} G$ is defined as such a n.f.s. weight given for $X\in\mathcal{N}_+$ by

$$\widehat{\phi}(X) = \begin{cases} \|\xi\|^2, & X = \pi_l(\xi)^*\pi_l(\xi), \quad \xi\in\mathfrak{B}, \\ +\infty, & \end{cases}$$

where $\mathfrak{B}$ is the set of left bounded vector (with left action π_l).

Modular automorphism group $\sigma^{\widehat{\phi}}$ of dual weight $\widehat{\phi}$ is given by $\sigma^{\widehat{\phi}}_t(X) = \widetilde{\Delta}^{it}X\widetilde{\Delta}^{-it}$ for $X\in\mathcal{N}\rtimes_{\hat{\alpha}} G$, whose action on $\mathcal{N}\rtimes_{\hat{\alpha}} G$ can be specified by

$$\sigma^{\widehat{\phi}}_t(r(\hat{\alpha})(X\otimes 1)) = r(\hat{\alpha})(\sigma^{\phi}_t(X)\otimes 1), \quad X\in\mathcal{N}, t\in\mathbb{R},$$

$$\sigma^{\widehat{\phi}}_t(\lambda(s)) = \lambda(s)r(\hat{\alpha})((D\phi\circ\hat{\alpha}_s : D\phi)_t\otimes 1), \ s\in G.$$

iii) Structure of $\mathcal{M} = \mathcal{N} \rtimes_{\hat{\alpha}} G$:

Theorem 4 *$\mathcal{M} = \mathcal{N} \rtimes_{\hat{\alpha}} G$ of* **centrally ergodic** *dynamical system $(\mathcal{N} \underset{\hat{\alpha}}{\curvearrowleft} G)$ has the same von Neumann factor type as $\mathcal{Q} = \mathfrak{Z}(\mathcal{N}) \rtimes_{\hat{\alpha}} G$. Namely, the following criteria hold:*
(i) $\mathcal{M}$ is of type I $\Longleftrightarrow$ dyn. sys. $(\mathfrak{Z}(\mathcal{N}) \underset{\hat{\alpha}}{\curvearrowleft} G)$ on the centre is isomorphic to the flow on $L^\infty(G)$: $(\mathfrak{Z}(\mathcal{N}) \underset{\hat{\alpha}}{\curvearrowleft} G) \cong (L^\infty(G) \underset{Ad\lambda_G}{\curvearrowleft} G)$;
(ii) $\mathcal{M}$ is of type II $\Longleftrightarrow$ $(\mathfrak{Z}(\mathcal{N}) \underset{\hat{\alpha}}{\curvearrowleft} G)$ is not isomorphic to $(L^\infty(G) \underset{Ad\lambda_G}{\curvearrowleft} G)$ and $\mathfrak{Z}(\mathcal{N})$ admits $\hat{\alpha}$-inv. measure with support $\mathfrak{Z}(\mathcal{N})$;
(iii) $\mathcal{M}$ is of type III $\Longleftrightarrow$ $\mathfrak{Z}(\mathcal{N})$ admits no $\hat{\alpha}$-inv. measure with support $\mathfrak{Z}(\mathcal{N})$.

"Feedback" necessary here!: in view of the above i), our starting assumption [$\mathcal{M} \underset{\alpha}{\curvearrowleft} \mathcal{U}$ with $\alpha = Ad$: adjoint action of $\mathcal{U}$] was too restrictive to recover $\mathcal{M}$ of *non type I* !!

However, the assumption $\alpha = Ad$ is simply due to an oversimplification (*common* in measurement theory) to neglect **intrinsic dynamics of microsystem** $\mathcal{M}$ keeping only coupling terms between $\mathcal{M}$ and apparatus $\mathcal{A}$! So, if intrinsic dynamics of $\mathcal{M}$ is retained, the above results allow us to recover a generic Micro-Algebra $\mathcal{M}$. I.e., we have such a flow chart as [states $\rightarrow$ algebra $\rightarrow$ dynamics $+$ classifying space].

For $\mathcal{M}$ of type III: modular structure of $\mathcal{M}$ is completely determined by that of $\mathcal{A}$

$\Longrightarrow$ modular spectrum $S(\mathcal{M})$ depends on $\mathcal{A}$ and $\widehat{\mathcal{U}}$: namely, modular spectrum is given by

$$S(\mathcal{M}) = \bigcap \{Spec(\Delta_{\phi\circ\hat{\alpha}_\gamma,\phi}) : \phi \in \mathcal{W}_\mathcal{A}\}$$

where $\mathcal{W}_\mathcal{A}$ is the set of all normal semi-finite faithful weights on $\mathcal{A}$ and $\Delta_{\phi\circ\hat{\alpha}_\gamma,\phi} = [D(\omega \circ \hat{\alpha}_\gamma) : D\omega]_t \, \Delta_\phi$.

By Connes theory:

(1) $\mathcal{M}$: type III$_\lambda$, ($0 < \lambda < 1$), $\Longleftrightarrow S(\mathcal{M}) = \{\lambda^n : n \in \mathbb{Z}\} \cup \{0\}$,

(2) $\mathcal{M}$: type III$_0$ $\Longleftrightarrow S(\mathcal{M}) = \{0, 1\}$,

(3) $\mathcal{M}$: type III$_1$ $\Longleftrightarrow S(\mathcal{M}) = \mathbb{R}_+$.

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