# Fluctuation Theorem and Quantum Measurements

S. Tasaki, T. Monnai (Waseda), T. Matsui (Kyushu)

1 Fluctuation Theorem in Classical Systems

2 Fluctuation Theorem in Quantum Systems

3 Fluctuation Theorem in Infinite Quantum Systems

4 Fluctuation Theorem and Measurement

5 Summary

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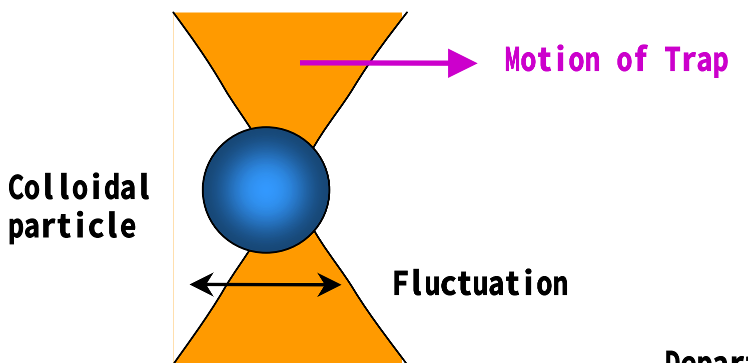
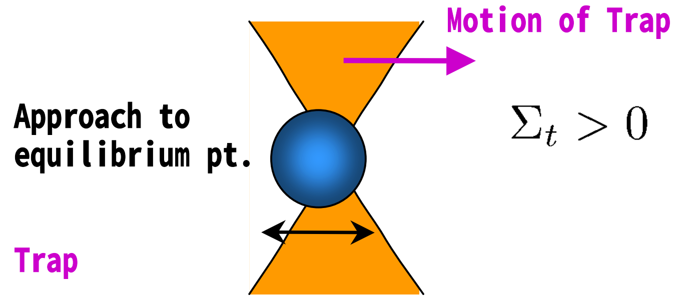
# 1 Fluctuation Theorem in Classical Systems

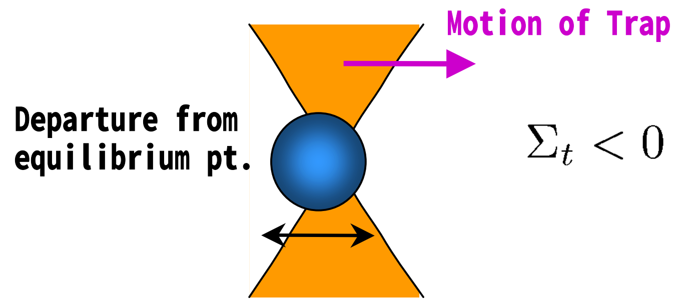
- **Experiment by** **Wang, Sevick, Mittag, Evans, Searles (02)**




$$\Sigma_t > 0$$

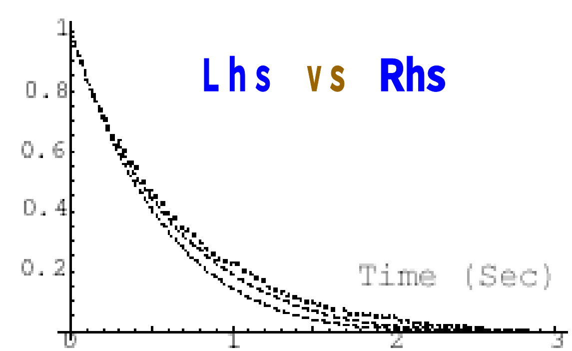
**Work done:** $\Sigma_t \equiv \frac{1}{k_B T} \int_0^t ds \; v_0 \cdot \{-k(x(s) - v_0 s)\}$




$$\Sigma_t < 0$$

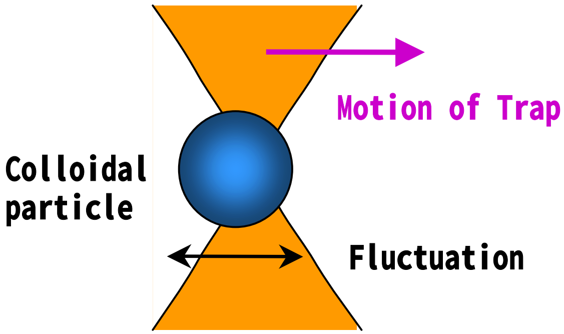
**FT**

$$\boxed{\frac{\Pr(\Sigma_t = a)}{\Pr(\Sigma_t = -a)} = e^a} \Leftrightarrow \frac{\Pr(\Sigma_t < 0)}{\Pr(\Sigma_t > 0)} = \langle e^{-\Sigma_t} \rangle_{\Sigma_t > 0}$$



# 1 Fluctuation Theorem in Classical Systems

**Langevin analysis of experiment by Wang, Sevick, Mittag, Evans, Searles (02)**




$$\zeta \frac{dx}{dt} + k(x - v_0 t) = f(t)$$

**Fluctuating force** $\langle f(t) f(t') \rangle = 2k_B T \zeta \delta(t - t')$

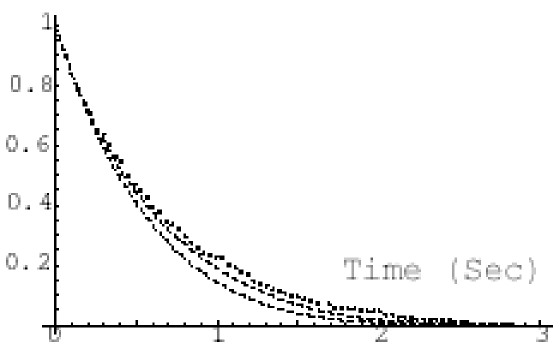
**Initial position distribution** $\propto e^{-k x_0^2/(2k_B T)}$

- Mazonka,Jarzynski (99): FT
- Tasaki,Terasaki,Monnai (02), van Zon, Cohen (03)

**Experiment by Wang et al.**

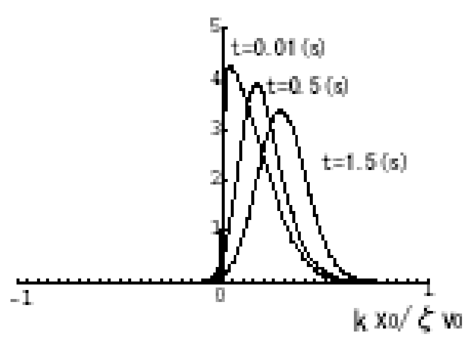
**FT-prediction**

$$\frac{\Pr(\Sigma_t<0)}{\Pr(\Sigma_t>0)} = \langle e^{-\Sigma_t} \rangle_{\Sigma_t>0}$$




$$\Sigma_t \equiv \frac{1}{k_B T} \int_0^t ds\, v_0 \cdot \{-k(x(s) - v_0 s)\}$$

**Initial state distr. for $\Sigma_t<0$**




**⇒ anti-thermodynamic behavior is mainly due to initial fluctuation**

# 1 Fluctuation Theorem in Classical Systems

**Example (detailed FT by Jarzynski):**

- **equilibrium to different equilibrium**

<Process>

1. A system is prepared to be in equilibrium.
2. Interaction V(t) is turned on at t=0 and is kept to be $V_f$ after t=T.

<Probability Distribution for Forward Process: $\Gamma_0 \rightarrow \Gamma_1$
Backward Process: $\Gamma'_0 \rightarrow \Gamma'_1$ >

$$P_F(\Gamma_0 \rightarrow \Gamma_1) \equiv \delta\left(\gamma(T,0:\Gamma_0) - \Gamma_1\right) \frac{e^{-\beta H_0(\Gamma_0)}}{Z_0}$$

$$P_B(\Gamma'_0 \rightarrow \Gamma'_1) \equiv \delta\left(\widetilde{\gamma}(T,0:\Gamma'_1) - \Gamma'_0\right) \frac{e^{-\beta H_1(\Gamma'_1)}}{Z_1}$$

$\Gamma'$: time reversal of $\Gamma$, $\gamma$(t,0): sol. of eq. of motion, $\tilde{\gamma}$(t,0): sol. for V(T-t)

<Detailed FT>

If $V(t)' = V(t)$ (time reversal symmetric) then

$$\boxed{\frac{P_F(\Gamma_0 \rightarrow \Gamma_1)}{P_B(\Gamma'_1 \rightarrow \Gamma'_0)} = e^{\Delta S(\Gamma_0)}}$$

$$\Delta S(\Gamma_0) = \beta\{F_1 - H_1(\gamma(T,0:\Gamma_0)) - F_0 + H_0(\Gamma_0)\}$$

Entropy change for each initial condition

# 1 Fluctuation Theorem in Classical Systems

<Proof>

**Forward evolution and Backward evolution:**

$$\widetilde{\gamma}(t:\Gamma') = \gamma(T-t:\Gamma)'$$

$$e^{-\beta W(\Gamma_0)} P_F(\Gamma_0 \to \Gamma_1) = \delta\left(\gamma(T,0:\Gamma_0) - \Gamma_1\right) \frac{e^{-\beta H_1(\Gamma_1)}}{Z_0}$$
$$= \delta\left(\Gamma_0 - \gamma^{-1}(T,0:\Gamma_1)\right) \frac{e^{-\beta H_1(\Gamma_1)}}{Z_0} = \delta\left(\Gamma'_0 - \widetilde{\gamma}(T,0:\Gamma'_1)\right) \frac{e^{-\beta H_1(\Gamma'_1)}}{Z_0}$$
$$= \frac{Z_1}{Z_0} P_B(\Gamma'_1 \to \Gamma'_0) = e^{-\beta \Delta F} P_B(\Gamma'_1 \to \Gamma'_0)$$

**where** $W(\Gamma_0) = H_1(\gamma(T,0:\Gamma_0)) - H_0(\Gamma_0)$ **is work done externally**

<Corr>

**Jarzynski equality:** **work measurement ⇒ free energy difference**

$$\begin{aligned} \langle e^{-\beta W} \rangle_{\beta, H_0} &= \int d\Gamma_0 d\Gamma_1 e^{-\beta W(\Gamma_0)} P_F(\Gamma_0 \to \Gamma_1) \\ &= e^{-\beta \Delta F} \int d\Gamma'_0 d\Gamma'_1 P_B(\Gamma'_1 \to \Gamma'_0) = e^{-\beta \Delta F} \end{aligned}$$

# 1 Fluctuation Theorem in Classical Systems

valid even far from eq., includes (higher order) Onsager relations etc

## Fluctuation Theorem

Prob( σ ) : probability of observing an average entropy production rate/work externally done σ during time interval τ

$$\left( \lim_{\tau \to +\infty} \right) \frac{1}{\tau} \log \frac{\mathrm{Prob}(\sigma)}{\mathrm{Prob}(-\sigma)} = \sigma$$

**Classical Systems:**

τ →∞ : SSFT (Evans-Cohen-Morriss, Gallavotti-Cohen)
Stocahstic systems (Spohn-Lebowitz,Kurchan)
Chemical reaction (Gaspard)
Finite τ : TFT (Evans-Searles) 、 Detailed FT (Jarzynski) etc

**Jarzynski equality**

Classical systems (Jarzynski), Quantum systems (Yukawa)
Relation with Fluctuation Theorem (Crooks), Quantum version (Monnai)

**Quantum Systems:**

Finite system, finite τ (Kurchan) , Generalization (Maes-Netocny)
$C^*$-version (Matsui-ST)

# 2 Fluctuation Theorem in Quantum Systems

Prob( $\sigma$ ) : probability of observing an average entropy production rate/work externally done $\sigma$ during time interval $\tau$

$$\left(\lim_{\tau\to+\infty}\right)\frac{1}{\tau}\log\frac{\text{Prob}(\sigma)}{\text{Prob}(-\sigma)}=\sigma$$

To measure entropy change or work done externally,

[A] Measure energy, particle numbers etc. twice and evaluate the entropy change or work done as the difference of the two observed values.

(Kurchan's protocol)

[B] Measure flows of energy, particle numbers etc. and evaluate the entropy change or work done as accumulated values of the flows.

Classical Mechanically: [A] = [B]

Quantum Mechanically: [A] $\neq$ [B] <- noncommutativity of observables

# 2 Fluctuation Theorem in Quantum Systems

examples based on Kurchan's view:

**(a) equilibrium to different equilibrium (single system)**

<Process>

1. A systems is prepared to be in equilibrium. $\rho^{(i)} = e^{-\beta(H-\mu N-\Phi^{(i)})}$
2. Interaction V(t) is turned on at t=0.

<'Entropy' measurement>

1. Measure energy, particle #, etc at t=0. $\epsilon_\lambda, \quad n_\lambda$
2. Measure energy, particle #, etc at $t_0$. $\bar{\epsilon}_\mu, \quad \bar{n}_\mu$

<Probability Distribution for Forward Process>

Distribution of $\eta_{\lambda\mu} \equiv \beta\left(\bar{\epsilon}_\mu - \mu\bar{n}_\mu - \Phi^{(f)} - \epsilon_\lambda + \mu n_\lambda + \Phi^{(i)}\right)$

$$P_F(\eta) \equiv \mathrm{Prob.}(\eta_{\lambda\mu} = \eta)$$

where $\begin{cases} \Phi^{(f)} = -\beta^{-1}\log \mathrm{Tr} e^{-\beta(H+V(t_0)-\mu N)} \\ \Phi^{(i)} = -\beta^{-1}\log \mathrm{Tr} e^{-\beta(H-\mu N)} \end{cases}$

<Backward Process>

Evolution generated by the interaction $V(t_0-t)$

Initially in $\rho^{(f)} = e^{-\beta(H+V(t_0)-\mu N-\Phi^{(f)})}$

$P_B(\eta)$: corresponding distribution of $\eta$ for backward process

If $\iota V(t)\iota = V(t)$ $\quad (\iota : \text{time reversal operation})$ then $\dfrac{P_F(\eta)}{P_B(-\eta)} = e^{\eta}$

## 2 Fluctuation Theorem in Quantum Systems

examples based on Kurchan's view:

**(b) nonequilibrium transient states**

<Process>

1. Two independent systems are prepared to be in different equilibria.
2. Interaction V is turned on at t=0.

<'Entropy' measurement>

1. Measure energy, particle #, etc at t=0. $\epsilon_{j\lambda}, \quad n_{j\lambda}$
2. Measure energy, particle #, etc at t. $\bar{\epsilon}_{j\mu}, \quad \bar{n}_{j\mu}$

<Probability Distribution>

Distribution of $\eta_{\lambda\mu} \equiv \sum_j \beta_j \left(\bar{\epsilon}_{j\mu} - \mu_j \bar{n}_{j\mu} - \epsilon_{j\lambda} + \mu_j n_{j\lambda}\right)$

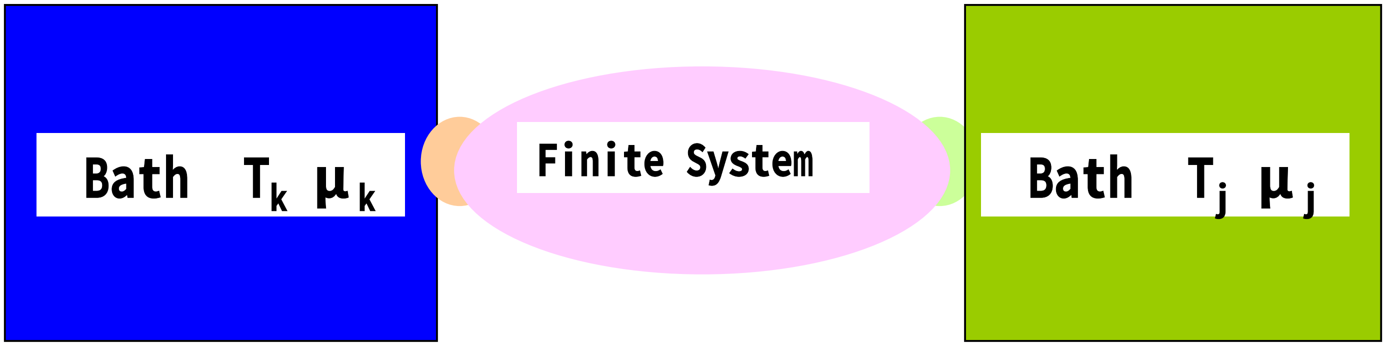
$$P_F(\eta) \equiv \mathrm{Prob.}(\eta_{\lambda\mu} = \eta)$$

If $\iota V \iota = V \quad (\iota : \text{time reversal operation})$ then $\dfrac{P_F(\eta)}{P_F(-\eta)} = e^{\eta}$

⇔ ∞-system version (Matsui, ST)

# 3 Fluctuation Theorem in Infinite Quantum Systems

FT In Infinitely extended quantum systems




SETTING: infinitely extended systems coupled by a bdd interaction

QUESTIONS: what would be the quantum analog of TFT?

# 3 Fluctuation Theorem in Infinite Quantum Systems

**Theorem：Relative Entropy** (Ojima et al.(88,89),Jaksic Pillet (01,02))

The relative entropy $S(\omega|\omega_t)$ between the initial and present states is given by

$$S(\omega|\omega_t) = \sum_{j=1}^{N} \beta_j \int_0^t \omega_s(J_j^q)ds - \omega_t(D_S) + \omega(D_S)$$

where $\omega_s \equiv \omega \circ \tau_s$ and $J_j^q$ corresponds to the heat flow to the $j$th reservoir:

$$J_j^q \equiv -\tilde{\delta}_j(V) + \sum_{\lambda=1}^{L} \mu_\lambda^{(j)} \tilde{g}_\lambda^{(j)}(V) \left( \sim \frac{d}{dt}\tau_t\left(H_j\right) - \sum_{\lambda=1}^{L} \mu_\lambda^{(j)} \frac{d}{dt}\tau_t\left(N_j^{(\lambda)}\right) \right) .$$

Moreover if $\tau_t$ is asymptotically abelian,

(i) $Ep(\omega_t) \equiv \frac{d}{dt}S(\omega|\omega_t) \to Ep(\omega_{\pm\infty}) = \sum_j \beta_j \omega_{\pm\infty}(J_j^q)$ for $t \to \pm\infty$,

(ii) $Ep(\omega_{\pm\infty})$ does not depend on $e^{-D_S}$,

(iii) $Ep(\omega_{+\infty}) \geq 0$ and $Ep(\omega_{-\infty}) \leq 0$.

**NB1:Araki's Relative Entropy：** $C^*$-extension of

$S(\rho_2 | \rho_1) = \mathrm{Tr}\ \{\rho_1 (\log\rho_1 - \log\rho_2)\}$

**NB2:NESS Entropy Production is independent of system division.**

# 3 Fluctuation Theorem in Infinite Quantum Systems

## GNS representation and Relative modular operator

### GNS representation

| conventional view | | | GNS view (Hilbert space) | |
|---|---|---|---|---|
| Density matrix | $\rho^{1/2}$ | $\Leftrightarrow$ | $\|\Omega\rangle$ | 'Hilbert' space vector |
| Observable | $A$ | $\Leftrightarrow$ | $A\|\Omega\rangle \leftrightarrow A\rho^{1/2}$ | Left multiplication |
| | | | $\tilde{A}\|\Omega\rangle \leftrightarrow \rho^{1/2}A$ | Right multiplication |

NB. Similarity with Thermofield dynamics of Umezawa et al. (Ojima)

| conventional view | | | GNS view (Hilbert space) | |
|---|---|---|---|---|
| Density matrix | $\rho^{1/2}$ | $\Leftrightarrow$ | $\|\Omega\rangle$ | 'Hilbert' space vector |
| Observable | $A$ | $\Leftrightarrow$ | $A\|\Omega\rangle$ | Non-tilde operator |
| | | | $\tilde{A}\|\Omega\rangle$ | Tilde operator |

**Relative modular (super)operator:** $\Delta_{\Omega_t\Omega}|\Omega'\rangle \Leftrightarrow \rho_t\rho'^{1/2}\rho^{-1}$

$$\log\Delta_{\Omega_t\Omega}|\Omega'\rangle \Leftrightarrow \log\rho_t\rho'^{1/2} - \rho'^{1/2}\log\rho$$

$$\langle\Omega_t|\log\Delta_{\Omega_t\Omega}|\Omega_t\rangle \Leftrightarrow \mathrm{Tr}\{\rho_t(\log\rho_t - \log\rho)\} = S(\rho|\rho_t)$$

# 3 Fluctuation Theorem in Infinite Quantum Systems

**Theorem：Transient Fluctuation Theorem** (T.Matsui, ST 03)

Let $\Omega$ be a vector representation of the initial state $\omega$ in the GNS representation, where $\omega_t$ corresponds to the vector $\Omega_t$, and $\Delta_{\Omega_t,\Omega}$ be the relative modular operator of the two states. Then, Araki's relative entropy is given by $S(\omega|\omega_t) = (\Omega_t, \ln \Delta_{\Omega_t,\Omega}\Omega_t)$.

Define the probability of finding the values of *the mean relative entropy production* within the interval $[a, b]$ as

$$\Pr([a,b];\omega_t) = (\Omega_t, \{P_t(b) - P_t(a-0)\}\,\Omega_t) \qquad (t > 0)$$

where $P_t(b)$ is the spectral family satisfying

$$\frac{1}{t}\ln \Delta_{\Omega_t,\Omega} = \int_{-\infty}^{+\infty} \lambda dP_t(\lambda)$$

Then, if the initial state $\omega$ is time reversal symmetric, for $0 \le a \le b$, the probability satisfies

$$a \le \frac{1}{t}\log\frac{\Pr([a,b];\omega_t)}{\Pr([-b,-a];\omega_t)} \le b$$

**NB: Natural extension of Evans-Searles transient fluctuation Theorem**

**NB: Relative entropy is not a usual observable, but can acquire meaning in connection with twice measurements.**

# 4 Fluctuation Theorem and Measurements

## Example with the effect of noncommutativity

⇔ Quantum version of Externally dragged Langevin equation

$$\zeta \frac{dx}{dt} + k(x - v_0 t) = f(t)$$

Fluctuating force $\langle f(t) f(t') \rangle = 2 k_B T \zeta \delta(t - t')$

Initial position distribution $\propto e^{-k x_0^2 / (2 k_B T)}$

- Mazonka,Jarzynski (99): FT
- Tasaki,Terasaki,Monnai (02), van Zon,E.G.D. Cohen (03)

Analysis of experiment by Wang, Sevick, Mittag, Evans, Searles (02)

# 4 Fluctuation Theorem and Measurements

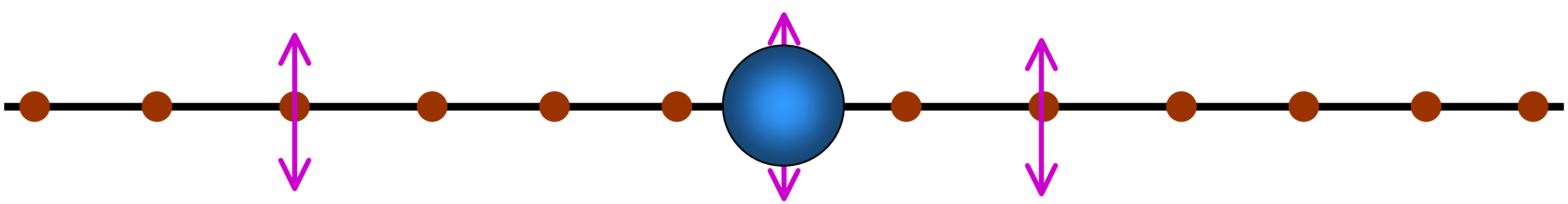
## Example with the effect of noncommutativity

⇔ Quantum version of Externally dragged Langevin equation (ST,Monnai)

**Model:**

$$
\begin{aligned}
H(t) &= H_0 + \frac{k}{2} f(t)\,(-2q + f(t)) \;, \qquad (f(t) \equiv 0 \qquad (t<0)) \\
H_0 &= \frac{p^2}{2m} + \frac{k}{2} q^2 + \frac{1}{2} : \int d\lambda ((p_\lambda - \kappa_\lambda q)^2 + \omega_\lambda^2 q_\lambda^2) :
\end{aligned}
$$

**Initial state** $\rho_i \propto e^{-\beta H_0}$



# 4 Fluctuation Theorem and Measurements

## Example with the effect of noncommutativity

⇔ Quantum version of Externally dragged Langevin equation

quantity of interest = work externally done/temperature

### [A] Twice measurement case

- Backward evolution is identical to the forward evolution.
- Fluctuation theorem holds:

$$\int e^{i(-\xi+i)\eta} P_F(\eta) d\eta = \int e^{i\xi\eta} P_F(\eta) d\eta$$

- Distribution is not simple. Particularly, it is not Gaussian.
- In classical limit, it becomes Gaussian.

### [B] Flow measurement case

- Distribution is Gaussian.
- Mean value and variance are Identical to those in case [A].
- It does not satisfy Fluctuation Theorem.
- In the classical limit, it agrees with that of case [A].

$$\log \frac{P(\Sigma_\tau = A)}{P(\Sigma_\tau = -A)} = \left(1 + \hbar^2 \epsilon_2 + \mathrm{O}(\hbar^4)\right) A$$

$$\epsilon_2 = -\frac{\beta^3 k^2}{48m} \int d\lambda \frac{\omega_\lambda v_\lambda^2}{|\eta_+(\omega_\lambda)|^2} \left| \int_0^{t_0} ds e^{i\omega_\lambda t} \dot{f}(s) \right|^2$$

# 5 Summary

- The distribution of entropy production or work externally done obeys a simple symmetry relations. ⇔ Fluctuation Theorem (FT)

- In Classical case, FT holds for a wide range of systems such as the stochastic dynamics, Hamiltonian dynamics, etc.

- In Quantum case, the measurement procedures do influence the final results.

  ⇒ Problem of selecting appropriate method is an open problem.

- Entropy production and fluctuation theorem for infinite systems

  Entropy production = time derivative of Araki's relative entropy
  Fluctuation theorem for this entropy production holds.
  Modular operator ⇔ Observable via twice measurements