"Potential Reconstruction for Two-Dimensional Discrete Schrödinger Equation"

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The wave motion on the lattices is described in discrete quantum mechanics by the finite-difference Schrödinger equation

$$-\frac{\psi_{i-1j}-2\psi_{ij}+\psi_{i+1j}}{\Delta_x^2}-\frac{\psi_{ij-1}-2\psi_{ij}+\psi_{ij+1}}{\Delta_y^2}+$$

$$+u_{ij}\psi_{ij}=\lambda\psi_{ij},\quad 1\le i\le M,\quad 1\le j\le N;$$

$$\psi_{0j}=\psi_{M+1j}=\psi_{i0}=\psi_{iN+1}=0.$$

When $u_{ij}=0$ (potential is zero),

$$\lambda_{mn}=\frac{4}{\Delta_x^2}\sin^2\frac{\pi m}{2(M+1)}+\frac{4}{\Delta_y^2}\sin^2\frac{\pi n}{2(N+1)},$$

$$1\le m\le M,\qquad 1\le n\le N.$$

The corresponding basic eigenvectors are

$$V_{mn}=2\sqrt{\Delta_x\Delta_y}\,\left\|\sin\frac{\pi im}{M+1}\sin\frac{\pi jn}{N+1}\right\|=$$

$$=\|v_{ij}\|,\quad 1\le i\le M,\quad 1\le j\le N.$$

We arrange eigenvalues

$$\lambda_1 \le \lambda_2 \le \dots \le \lambda_\ell, \quad \ell = M \times N$$

and consider vectors

$$E_\xi = \{ v_{11}(\xi), v_{12}(\xi) \dots, v_{1N}(\xi), \dots$$
$$v_{M1}(\xi), v_{M2}(\xi), \dots v_{MN}(\xi) \},$$

$$\lambda_\xi = \lambda_{mn}, \quad v_{ij}(\xi) = v_{ij}(m, n).$$

The eigenvalue problem

$$A \cdot E_\xi = \lambda_\xi E_\xi, \quad 1 \le \xi \le \ell,$$

where A is 5-diagonal matrix

$$A = \begin{bmatrix} \Phi & -I & 0 & 0 & \dots & 0 & 0 & 0 \\ -I & \Phi & -I & 0 & \dots & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \dots & \cdot & \cdot & \\ 0 & 0 & 0 & 0 & & -I & \Phi & -I \\ 0 & 0 & 0 & 0 & & 0 & -I & \Phi \end{bmatrix} M \cdot N$$

$M \cdot N$

$$\Phi = \begin{bmatrix} 4 & -1 & 0 & \dots & 0 & 0 \\ -1 & 4 & -1 & \dots & 0 & 0 \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ 0 & 0 & 0 & & -1 & 4 \end{bmatrix} N, \quad I = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \cdot & \cdot & \dots & \cdot \\ 0 & 0 & \dots & 1 \end{bmatrix} N$$

N $\qquad$ N

We state the problem of reconstruction 5-diagonal matrix of given different λ_ξ and given $e_i(\xi)$, $\sum_\xi e_i^2(\xi) = 1$, first components for all basic eigen-

S. I. Serdyukova "Inverse problem for symmetric tridiagonal matrices. Calculation of the system of discrete orthogonal polynomials with arbitrary weight" Russ. J. Numer. Anal. Math. Modelling, vol. 8, No. 3, 245-263 (1993)

$$\tilde{A} = \begin{bmatrix} \mathcal{D}_1 & C_1 & 0 & 0 & \dots & 0 & 0 & 0 \\ C_1 & \mathcal{D}_2 & C_2 & 0 & \dots & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \dots & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \dots & C_{N-2} & \mathcal{D}_{N-1} & C_{N-1} \\ 0 & 0 & 0 & 0 & & 0 & C_{N-1} & C_N \end{bmatrix}$$

te1 ww1

$$\mathcal{D}_1 = \begin{bmatrix} 4+\theta_1 & -1+u_1 & \dots & 0 & 0 \\ -1+u_1 & 4+\theta_2 & \dots & 0 & 0 \\ \cdot & \cdot & \dots & \cdot & \cdot \\ \cdot & \cdot & \dots & \cdot & \cdot \\ 0 & 0 & \dots & -1+u_{N-1} & 4+\theta_N \end{bmatrix}$$

vv1

$$C_1 = \begin{bmatrix} -1+v_1 & 0 & \dots & 0 \\ 0 & -1+v_2 & \dots & 0 \\ \cdot & \cdot & \dots & \cdot \\ 0 & 0 & & -1+v_N \end{bmatrix}$$

$[E_1, \dots, E_\ell]$

$l_i(\xi) = E_\xi(i)$

$l_i \perp l_j$, $\|l_i\| =$

$$tei = 1+\theta_i = \sum_{\xi=1}^{\ell} \lambda_\xi \, l_i^2(\xi), \quad i = 1, \dots, N$$

$$wwi = -1+u_i = \sum_{\xi=1}^{\ell} \lambda_\xi \, l_i(\xi) \, l_{i+1}(\xi), \quad i = 1, \dots, N-1$$

Primitive Algorithm. Elements of $\tilde{C}$ and all basic eigenvectors can be computed from given λ_i, $i = 1, \dots, l = M \cdot N$ and given e_j, $j = 1, \dots, N$.

$$te1 \cdot e_1(\xi) + uu1 \cdot e_2(\xi) + vv1 \cdot e_{N+1}(\xi) = \lambda_\xi e_1(\xi)$$

$$te1 = \sum_{\xi=1}^{l} \lambda_\xi e_1^2(\xi) \qquad uu1 = \sum_{\xi=1}^{l} \lambda_\xi e_1(\xi) e_2(\xi)$$

$$vv1 = -\sqrt{\sum_{\xi} \underbrace{[(te1 - \lambda_\xi) e_1(\xi) + uu1 \cdot e_2(\xi)]^2}_{X_1(\xi)}}$$

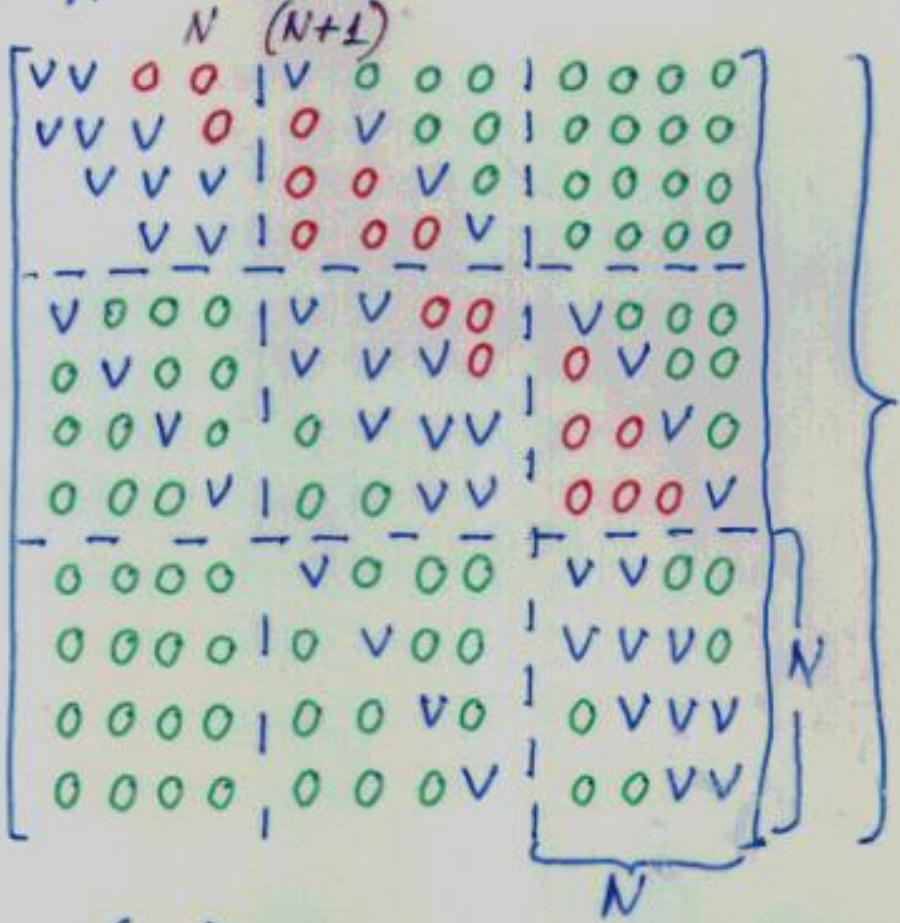
$$e_{N+1}(\xi) = -\frac{X_1(\xi)}{vv1}, \quad \xi = 1, \dots, l$$

$e_{N+1} \perp e_1 \to te1$, $e_{N+1} \perp e_2 \to uu1$

$$e_{N+1} \perp e_j \quad j = 3, \dots, N$$
$$\sum_{\xi=1}^{l} \lambda_\xi e_1(\xi) e_j(\xi) = 0$$

Additional conditions on the initial data $e_1, \dots, e_N$!

row-vectors $l_{N+2}, \dots, l_{MN}$ are found of (5) 2-d, ..., $N(M-1)$-th equations of the spectra system. General number of additional conditions, guaranting orthogonality of computed row-vectors l_j, $j = N+1, \dots, L = M*N$, is equal to the number of zero-elements in lacuna between upper second and $(N+1)$-th diagonals, exept zero-elements in low N rows:



$$\frac{(N-2+1)(N-2)}{2}(M-1) + \frac{(N-1+1)(N-1)}{2}(M-1) =$$

$$= (N-1)^2(M-1) \leftarrow \text{lacuna!}$$

In one-dimensional case no such lacuna and no additional conditions

Besides additional conditions
N conditions of compatibility,
conditions providing satisfaction
of last N equations of the
spectral system are.

So not all $\ell_1, \dots, \ell_N$ (but only $\ell_1, \dots, \ell_k$, $k < N$) can be chosen arbitrary. k ?

The lacking $\ell_{k+1}, \dots, \ell_N$ must satisfy L conditions:

$$L = (N-1)^2(M-1) + N + (N-k) + \frac{(N-k)(k+N-1)}{2}$$

$$\|\ell_j\| = 1 \quad \ell_j \perp (\ell_1, \dots, \ell_{j-1})$$

$L \le (N-k)MN$ leads to $\rightarrow$

Lemma. Let $N \ge M \ge 2$ and X_1 is less of two positive roots of

$$P(X) = X^2 - (2MN-1)X + N(N-3) + 2(2N-1)(M-1),$$ then

$$k = [X_1].$$

$M=3, N=4 \rightarrow X_1 = 1,48\dots, \quad k=1$

$M=4, N=7 \rightarrow X_1 = 2, \quad k=2 \quad 1 \le k < N$

$M=N=2 \quad X_1 = 0.62, \quad k=0$ exeption!

All these were stated in concrete calculatings

$$\Delta_x = \Delta_y = 1, \quad M = 3, \quad N = 4, \quad k = 1,$$

original $\lambda_1 = 0.96775$... was perturbed $\tilde{\lambda}_1 = 0.96$. l_1 - first components of basic original eigenvectors. l_2, l_3, l_4 are found from solution of L conditions:

$$L = 9 + 18 + 4 = 31$$

The first 9 are orthonormality conditions: $(l_2, l_2) = 1$, $(l_3, l_3) = 1$, $(l_4, l_4) = 1$

$(l_1, l_2) = 0$

$(l_4, l_3) = 0$ $\quad (l_2, l_3) = 0$

(l_5, l_3) $\quad (l_1, l_4) = 0$ $\quad (l_2, l_4) = 0$ $\quad (l_3 l_4) = 0$

The last 4 equations of spectral system with $\lambda = \tilde{\lambda}_1$. The additional conditions: $\Sigma = \sum_{\xi=1}^{l}$, $l = M \cdot N = 12$

$\sum \lambda_1 l_1(\xi) l_3(\xi) = 0$ $\quad l_5 \perp l_3$

$\sum \lambda_1 l_1(\xi) l_4(\xi)$ $\quad l_5 \perp l_4$

$\sum \lambda_2 l_2(\xi) l_4(\xi) = 0$ $\quad l_6 \perp l_4$

$\sum \lambda_1^2 l_2(\xi) l_4(\xi) = 0$ $\quad l_8 \perp l_5$

$\sum \lambda_2^2 l_1(\xi) l_3(\xi) =$ ш.1 · ш.2 $\quad l_7 \perp l_5$

62 polynomial equations

(11)

```
Given spectrum.                Spetrum of found matrix.
                    precision 15; accuracy 9; iterations=10;
g:={0.96,                 sp:={0.959999999999521, time 2970700;
1.96775244887701,              0=1.96775244887522,
2.38196601125011,              0=2.38196601125278,
3.2038204263768,               0=3.20382042653208,
3.38196601125011,              0=3.38196601088368,
3.7961795736232,               0=3.79617957432827,
4.2038204263768,               0=4.20382042508211,
4.61803398874989,              0=4.61803399114552,
4.7961795736232,               0=4.79617957192296,
5.61803398874989,              0=5.6180339889683,
6.03224755112299,              0=6.03224755104145,
7.03224755112299}              0=7.03224755112361}$
```

green - eight digits!

SPP

```
precision 25; accuracy 15;      Initial nonperturbed basic vectors
iterations=10; time 2871840 ms;    were used for matrix computation
sp:={0=0.9600000000000000000000001, sp := {la=0.9665730234841419237367531,
0=1.96775244887701010299377,      ...,la=7.028411685655399396123375}$
0=2.381966011250105151794829,     Solution of the system without 3 last
0=3.203820426376799799480456,     conditions of orthogonality e10,e11,e12
0=3.381966011250105151606389,     sp := {la=0.9859537736356722366983022,
0=3.79617957362320020095522,      ...,la=7.003158151728631308074687}$
0=4.203820426376799798761333,     Solution of "complete system" with
0=4.618033988749894849323346,     original basic vectors as initial data
0=4.79617957362320019982305,      sp := {0=1.007636440404440543026414,
0=5.618033988749894848279893,     ...,la=6.97504704940886626821059}$
0=6.032247551122989896980969,     Solution of "complete system" with
0=7.032247551122989897007021}$    solution of the system without 3 last
                    as initial data
                    sp := {0=0.9476678600730691307434605,
                    ...,la=7.032617631576653396822012}$

rt := {x1=0.291997448106443,x2=0.178928153992361,x3= - 0.186022645965438,
        .......        ......        .......
uu1=-0.998576168828071, u1= - 0.932780961152177, v1= - 0.925015887883282,
te2=4.02435602840292,   te3=4.04972793692043,   te4=4.0257821958051,
te5=3.96951493712544,   te6=3.98278970903635,   te7=3.99240483917565,
te8=4.0409547760039,    te9=3.98986439657223,   te10=3.97342847117483,
te11=3.96352710349044,  te12=3.98016499793694,  vd1=1.00223348440354,
vd2=1.03585243795544,   vd3=1.05540573999103,   vd4=0.951314233223938,
vd5=1.06073338832408,   vd6=0.973102630070526,  vd7=0.934027445097146,
vd8=0.982353002212367,  uu5= - 0.969487082474759, uu6= - 1.0107453464146,
uu7= - 1.03966627048992, uu9= - 1.03760836148713, uu10= - 1.06095785166668,
uu11= - 1.0224995740099, su2=19.135666402999,    su3=19.2399123533987, su4=17.9675754512291}$
```

+ GC time 1038800

$su_i = \sum (\lambda_j l_i(j))^2 \quad i = 2, 3, 4$

E_1 $\tilde{E}_1$

The first basic eigenvectors:

	original	found
1	0.18587401723009224506603 01	0.18587401723009224506603 01
2	0.30075047750377283682799 88	0.29199717588872973193778 07
3	0.30075047750377283682799 88	0.28143512863916208556441 83
4	0.18587401723009224506603 01	0.16323067339237042562683 25
5	0.26286555605956680301283 45	0.27281647409306813607949 12
6	0.42532540417601996609077 02	0.43119772714303463142609 7
7	0.42532540417601996609077 02	0.42277427916882614614995 19
8	0.26286555605956680301283 45	0.25238307259995952243696 14
9	0.18587401723009224506603 01	0.20430624239503478880923 56
10	0.30075047750377283682799 88	0.31768903977815170464164 18
11	0.30075047750377283682799 88	0.30702686523903831537115 53
12	0.18587401723009224506603 01	0.18603773025013549121112 63

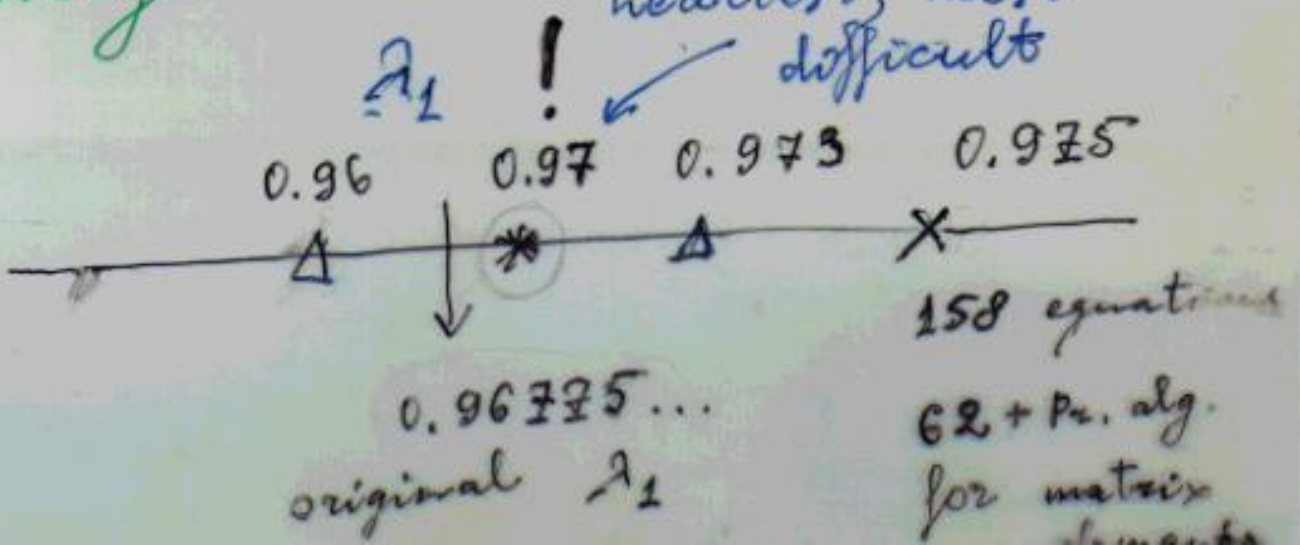
Herbert Melenk
Numeric package
Reduce 3.6

Instability.

$\lambda_1 = 0.967752\ldots$

$\tilde{\lambda}_1 = 0.96$; 0.973 perfect result!

But for $\tilde{\lambda}_1 = 0.97$; 0.975 the Newton method iterations diverge.



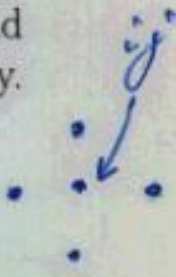
REMARK. *

Following to B.N.Zakhariev we say that the constructed perturbed symmetric five-diagonal matrix C corresponds to the discrete Schrödinger equation with the nonlocal potential

$$-\frac{\psi_{i-1,j}-2\psi_{i,j}+\psi_{i+1,j}}{h_x^2}-\frac{\psi_{i,j-1}-2\psi_{i,j}+\psi_{i,j+1}}{h_y^2}+U_{ij}\psi_{ij}+$$

$$+\xi_{ij}\psi_{i+1,j}+\xi_{i-1,j}\psi_{i-1,j}+\eta_{ij}\psi_{i,j+1}+\eta_{i,j-1}\psi_{i,j-1}=\lambda\psi_{i,j},$$

where $U_{ij}=\theta_{(i-1)N+j}$, $\xi_{ij}=v_{(i-1)N+j}$, $\eta_{ij}=u_{(i-1)N+j}$. Diagonal elements of C matrix are $4+\theta_\nu$ and the elements of second and $(N+1)-$th diagonals are $-1+u_\nu$ and $-1+v_\nu$ respectively.



* S. I. Serdyukova, B. N. Zakhariev
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*** N. S. Bakhvalov, K. K. Likharev, G. S. Kazacha, S. I. Serdyukova
"Statics and Dynamics of Single-Electron Solitons in Two-Dimensional Arrays of Ultrasmall Tunnel Junctions"
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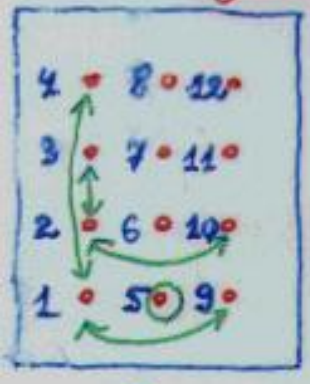
Inverse problem with symmetry conditions: symmetric five-diagonal matrix whose basic eigenvectors vectors satisfy the symmetry conditions can be reconstructed of given spectrum only. For $m = 3$, $n = 4$ the problem reduces to solving system of 36 equations (instead of 62!) But new difficulties arise with the statements of the problem. Consider the concrete example of $m = 3$, $n = 4$.

Symmetry conditions.



Basic eigenvectors of original nonperturbed problems satisfy

$l_2(j) = (-1)^{j-1} l_2(j)$

$l_1(j) = (-1)^{j-1} l_4(j) \quad j = 1:12$

$l_5(j) = (-1)^{j-1} l_8(j) \quad j = 1:12$

$l_5(j) = l_8(j) = 0 \quad j = 5:8$

$l_9(j) = (-1)^{j-1} l_{12}(j) \quad j = 1:12$

$l_{10}(j) = (-1)^{j-1} l_{11}(j)$

$l_1 = \begin{cases} l_9 & j = 1:4,\ 9:12 \\ -l_9 & j = 5:8 \end{cases}$ $\qquad l_1 = 1*(l_9)$

$l_2 = 1*(l_{10}), \quad l_3 = 1*(l_{11}), \quad l_4 = 1*(l_{12})$

To have all these symmetries it's sufficiently that part of them

chosen cond.

$$\begin{cases} l_1 = (-1)^{j-1} l_4, \quad l_2 = (-1)^{j-1} l_3, \quad j = 1:12; \\ l_5 = 0, \quad j = 5:8; \\ l_1 = \begin{cases} l_9, & j = 1:4, 9:12, \\ -l_9, & j = 5:8, \end{cases} \quad l_2 = \begin{cases} l_{10}, & j = 1:4, 9:12 \\ -l_{10}, & j = 5:8. \end{cases} \end{cases}$$

were satisfied. In addition we have symmetry of $\widetilde{M}$ and all its blocks with respect to the second large diagonal.

$sw_i = \sum (\lambda_2 l_i(\xi))^2$

The first $2 \to te1 = te4$, $te2 = te3$, $ww1 = ww3$, $sw1 = sw4$, $sw2 = sw3$, but

$sw1 = te1^2 + ww1^2 + vv1^2$ $\quad sw2 = ww1^2 + te2^2 + ww2^2 + vv2^2$

$sw4 = te4^2 + ww3^2 + vv4^2$ $\quad sw3 = ww2^2 + te3^2 + ww3^2 + vv3^2$

these lead to $vv1 = vv4$, $vv2 = vv3$. By using first and fourth spectral eq-ns

$$(te1 - \lambda) l_1 + ww1 \cdot l_2 + vv1 \cdot l_5 = 0$$

$$ww3 \cdot l_3 + (te4 - \lambda) l_4 + vv4 \cdot l_8 = 0$$

we find $l_5 = (-1)^j l_8$, $j = 1 = 12 \to te5 = te8$, $sw5 = sw8$. In the same way we get $l_6 = (-1)^{j-1} l_7$, $te6 = te7$, $sw6 = sw7 \to$

$vv5 = vv8$, $vv6 = vv7$, $vv1 = vv5$, $vv2 = vv6$.

Then we get $l_9 = (-1)^{j-1} l_{12}$, $l_{10} = (-1)^{j-1} l_{11}$ which lead to $te10 = te11 = te2$, $te9 = te12 = te1$ $ww9 = ww1$.

$$l_5(j) = l_8(j) = 0 \;,\; j = 5:8,$$

provide satisfying last 4 equations of the spectral system. The 1th and 9th equations are

$$(te1-\lambda)\, l_1 + uu1 \cdot l_2 + vv1 \cdot l_5 = 0,$$
$$vv5 \cdot l_5 + (te9-\lambda)\, l_9 + uu9 \cdot l_{10} = 0.$$

Using $te1 = te9$, $uu1 = uu9$, $vv1 = vv5$ and $l_1 = l_9$, $l_2 = l_{10}$, $j = 1:4, 9:12$, $l_1 = -l_9$, $l_2 = -l_{10}$, $l_5 = 0$, $j = 5:8$ we get that the 9th equations is satisfied of the first does. So this time compatibility conditions take uniform view $l_5 = 0$, $j = 5:8$.

We proved that if chosen symmetry conditions are satisfied the disturbed matrix has such form

$$\tilde{C} = \begin{bmatrix} A & W & 0 \\ W & B & W \\ 0 & W & A \end{bmatrix}, \quad A = \begin{bmatrix} te_1 & uu1 & 0 & 0 \\ uu1 & te_2 & uu2 & 0 \\ 0 & uu2 & te_2 & uu1 \\ 0 & 0 & uu1 & te_1 \end{bmatrix}$$

$$B = \begin{bmatrix} te5 & uu5 & 0 & 0 \\ uu5 & te6 & uu6 & 0 \\ 0 & uu6 & te6 & uu5 \\ 0 & 0 & uu5 & te5 \end{bmatrix} \quad W = \begin{bmatrix} vv1 & 0 & 0 & 0 \\ 0 & vv2 & 0 & 0 \\ 0 & 0 & vv2 & 0 \\ 0 & 0 & 0 & vv1 \end{bmatrix}$$

$\tilde{C}$ of order 12 has 10 different elements

In general, when spectrum is perturbed arbitrarily, nine-diagonal symmetric matrix with lacunas can be reconstructed of given spectrum and prescribed symmetry of basic eigenvectors. This matrix is symmetric not only with respect to the leading diagonal, but with respect to the second diagonal as well.

$$N = M * N, \qquad M = 2 * k + 1,$$

$$\begin{bmatrix}
A_1 & W_1 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 \\
W_1 & A_2 & W_2 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & 0 & 0 \\
\vdots & \vdots & W_{k-1} & A_k & W_k & 0 & \vdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & W_k & A_{k+1} & W_k & 0 & \vdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & W_k & A_k & W_{k-1} & 0 & \vdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & 0 & W_2 & A_2 & W_1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & 0 & W_1 & A_1
\end{bmatrix}$$

$$N = M * N, \qquad M = 2 * k,$$

$$\begin{bmatrix}
A_1 & W_1 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 \\
W_1 & A_2 & W_2 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & 0 & 0 \\
\vdots & \vdots & W_{k-2} & A_{k-1} & W_{k-1} & 0 & \vdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & W_k & A_k & W_k & 0 & \vdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & W_k & A_k & W_{k-1} & 0 & \vdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & 0 & W_2 & A_2 & W_1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & 0 & W_1 & A_1
\end{bmatrix}$$

$$A_i = \begin{bmatrix}
a_1^i & b_1^i & 0 & \dots & 0 \\
b_1^i & a_2^i & b_2^i & \dots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \vdots & 0 & b_1^i & a_1^i
\end{bmatrix}, \qquad W_i = \begin{bmatrix}
v_1^i & w_1^i & 0 & \dots & 0 \\
w_1^i & v_2^i & w_2^i & \dots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \vdots & 0 & w_1^i & v_1^i
\end{bmatrix}.$$

The blocks A_i, W_i are three-diagonal matrices symmetric with respect to the both large diagonals. Very simple calculation shows, that the number of spectra points N are equal the number of different elements of the matrix.

$M=3,\ N=4$ 12×12 matrix $\tilde{\mathfrak{M}}=$

$$=\begin{bmatrix} A & W & O \\ W & B & W \\ O & W & A \end{bmatrix},\quad W=\begin{bmatrix} -vv1 & ww1 & 0 & 0 \\ ww1 & -vv2 & ww2 & 0 \\ 0 & ww2 & -vv2 & ww_1 \\ 0 & 0 & ww1 & -vv1 \end{bmatrix}$$

$$A=\begin{bmatrix} te_1 & uu1 & 0 & 0 \\ uu1 & te2 & uu2 & 0 \\ 0 & uu2 & te2 & uu1 \\ 0 & 0 & uu1 & te1 \end{bmatrix}\quad B=\begin{bmatrix} te5 & uu5 & 0 & 0 \\ uu5 & te6 & uu6 & 0 \\ 0 & uu6 & te6 & uu5 \\ 0 & 0 & uu5 & te5 \end{bmatrix}$$

$\tilde{\mathfrak{M}}$ has 12 different elements and can be reconstructed of given spectrum, being original spectrum disturbed arbitrarily.

The symmetry of matrix elements and the symmetry of basic eigenvectors of $\tilde{\mathfrak{M}}$ follow of

$$\begin{cases} l_1(j)=(-1)^{j-1}l_2(j),\quad l_2(j)=(-1)^{j-1}l_3(j),\ j=1:12; \\ l_5(j)=0,\quad j=5:8; \\ l_1(j)=\begin{cases} l_9(j) & j=1:4,\ 9:12, \\ -l_9(j) & j=5:8, \end{cases} \\ l_2(j)=\begin{cases} l_{10}(j), & j=1:4,\ 9:12, \\ -l_{10}(j), & j=5:8. \end{cases} \end{cases}$$

chosen symmetry conditions providing all symmetries

The rest 8 equations are obtained of 5-th and 6-th equations of the spectral systems, when $j = 1:4$, after substituting into them $l_9(j) = l_1(j)$ and $l_{10}(j) = l_2(j)$. (21)

1. One λ_j is disturbed

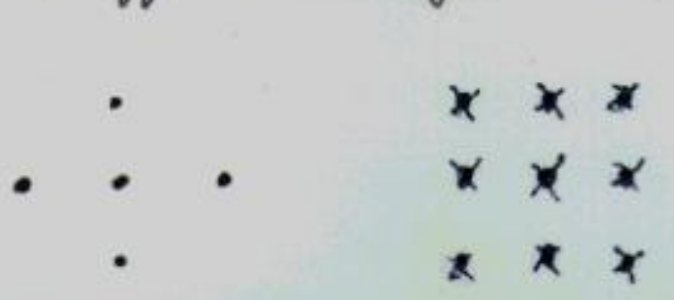
$\tilde{\lambda}_2 = 0.96$, $\tilde{\lambda}_1 = 0.97$, $\tilde{\lambda}_1 = 0.975$

2. 5 λ_j are disturbed

$$\lambda_j = \lambda_1 + 1.005(\lambda_j - \lambda_1) \quad j = 2:6$$

In all cases package NUMERIC, REDUCE 3.6 was used and original values of variables were chosen as initial data → results on next sheet. (21)

Supports of nonlocal potentials:



5-diagonal $\tilde{M}$ — 9-diagonal $\tilde{M}$ — 7-diagonal $\tilde{M}$

↓

the most proper polynomial system!

iterations diverge

Nine-diagonal matrix.


```
EXAMPLE I : first eigenvalue is disturbed.
{0.96,25,9,10,62}$
18 digits after the point coinside in given and found spectra after
10 iterations!
Time: 85790 ms  plus GC time: 370 ms comt;
spi := {                                sp := {
0.96,                                   la=0.959999999999999999999979,
1.96775244887701010299 3724,            la=1.96775244887701010299 37_18,
2.38196601125010515179 5413,            la=2.38196601125010515179 5_52,
3.20382042637679979940 2898,            la=3.20382042637679979 9_387688,
3.38196601125010515179 5413,            la=3.38196601125010515 1_831839,
3.79617957362320020059 7102,            la=3.79617957362320020 5_34146,
4.20382042637679979940 2898,            la=4.20382042637679979 4_97111,
4.61803398874989484820 4587,            la=4.61803398874989484 8_085841,
4.79617957362320020059 7102,            la=4.79617957362320020 0_662779,
5.61803398874989484820 4587,            la=5.61803398874989484 820_8109,
6.03224755112298989700 6276,            la=6.03224755112298989 700_2855,
7.03224755112298989700 6276}$           la=7.03224755112298989 006_724}$

EXAMPLE II : first eigenvalue is disturbed.
Result is not so good as above, 13 digits after the point
coinside after 10 iterations, but iterations converge!
Time: 70850 ms  plus GC time: 250 ms comt;
{0.97,25,9,10,62}$
spi := {                                sp := {
0.97,                                   la=0.97000000000000 00_18158187,
1.96775244887701010299 3724,            la=1.96775244887701 01_3870931,
2.38196601125010515179 5413,            la=2.38196601125010 51_70504529,
3.20382042637679979940 2898,            la=3.20382042637679 9_831600376,
3.38196601125010515179 5413,            la=3.38196601125010 5_684441331,
3.79617957362320020059 7102,            la=3.79617957362320 _1072437313,
4.20382042637679979940 2898,            la=4.20382042637680 0_105065042,
4.61803398874989484820 4587,            la=4.61803398874989 4_227994763,
4.79617957362320020059 7102,            la=4.79617957362320 0_144687512,
5.61803398874989484820 4587,            la=5.61803398874989 4_773400815,
6.03224755112298989700 6276,            la=6.03224755112298 9_343023721,
7.03224755112298989700 6276}$           la=7.03224755112 29_91015372569}$

EXAMPLE III : 5 eigenvalues are disturbed.
Initial spectrum with lacuna, 2-6 eigenvalues are disturbed - pressed
to the right <g(j)=g(1)+1.005*(g(j)-g(1))>$
Time: 85430 ms  plus GC time: 370 ms comt;

{0.96775244887701010299 37244,25,9,10,62}$

g(7)-g(6) is equal 0.4 approximately.

  After 10 iterations given spi and found sp spectra coinside with
14 digits after the point.
sp := {                                   spi := {
0.96775244887701 0_0993888727,            0.96775244887701010299 37244,
1.97275244887701 0_056623155,             1.9_72752448877010102993724,
2.38903707906197 0_24502234,              2.38_90370790619706270394 22,
3.21500076626429 87_27202151,             3.2_15000766264298747884944,
3.39403707906197 06_30825944,             3.3_94037079061970627039422,
3.81032170924693_0785390339,              3._81032170924693111510851 19,
4.20382042637679 9_53439042,              4.20382042637679979940 2898,
4.61803398874989_5443711349,              4.61803398874989484820 4587,
4.79617957362320 0_554856528,             4.79617957362320020059 7102,
5.61803398874989 4_919553097,             5.61803398874989484820 4587,
6.03224755112298 9_594998732,             6.03224755112298989700 6276,
7.03224755112298 98_1394093}$             7.03224755112298989700 6276}$
```