

A constructive formula  
for function of a matrix.  
(Alternative to the Lagrange  
and Silvestre formula).

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Part I. Brief reminding.

Let  $\hat{M}$  be arbitrary matrix  
and  $\hat{I}$  be unit matrix of  
dimensionality  $N \times N$ .

$$\hat{M} = \begin{pmatrix} m_{11} & \dots & m_{1N} \\ \vdots & & \vdots \\ m_{N1} & \dots & m_{NN} \end{pmatrix}$$

$$\hat{I} = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$$

For any  $\tilde{M}$  there exists an identity (theorem of Hamilton and Cayley):

$$(\tilde{M})^n - \sigma_1(\tilde{M})\tilde{M}^{n-1} + \dots + (-1)^k \sigma_k(\tilde{M})\tilde{M}^{n-k} + \dots + (-1)^n \sigma_n \tilde{I} \equiv 0$$

where  $\sigma_1, \dots, \sigma_k, \dots, \sigma_n$  are coefficients of the characteristic polynomial of  $\tilde{M}$ :

$$\lambda^n - \sigma_1 \lambda^{n-1} + \dots + (-1)^k \sigma_k \lambda^{n-k} + \dots + (-1)^n \sigma_n = 0$$

or

$$(\lambda - \lambda_1)(\lambda - \lambda_2) \dots (\lambda - \lambda_k) \dots (\lambda - \lambda_n) = 0$$

where  $\lambda_1, \dots, \lambda_k, \dots, \lambda_n$  are roots of characteristic polynomial of matrix  $\tilde{M}$  (eigenvalues of matrix  $\tilde{M}$ )

We have analytical formulas

$$\sigma_k = f(\lambda_1, \dots, \lambda_k, \dots, \lambda_n)$$

there are Vieta formulas:

$$\sigma_1 = \lambda_1 + \lambda_2 + \dots + \lambda_k + \dots + \lambda_n \quad (3)$$

$$\sigma_2 = \lambda_1 \cdot \lambda_2 + \lambda_1 \cdot \lambda_3 + \dots + \lambda_p \cdot \lambda_q + \dots + \lambda_{n-1} \cdot \lambda_n$$

.....

$$\sigma_n = \lambda_1 \cdot \lambda_2 \cdot \lambda_3 \cdot \dots \cdot \lambda_k \cdot \dots \cdot \lambda_n$$

But in general case  
we have no analytical  
formulas:

$$\lambda_k = f(\sigma_1, \dots, \sigma_k, \dots, \sigma_n) = ?$$

We have analytical formulas for power sums

$$\lambda_1 + \dots + \lambda_k + \dots + \lambda_N = S_1 = \text{Tr}(\hat{M})$$

$$\lambda_1^2 + \dots + \lambda_k^2 + \dots + \lambda_N^2 = S_2 = \text{Tr}(\hat{M}^2)$$

$$\lambda_1^p + \dots + \lambda_k^p + \dots + \lambda_N^p = S_p = \text{Tr}(\hat{M}^p)$$

where  $\text{Tr}(\hat{M}^k)$  is a sum of diagonal elements of matrix  $\hat{M}^k$ .

(4)

But we have no analytical formulas in general case)

$$\lambda_k = f(S_1, \dots, S_p, \dots) = ?$$

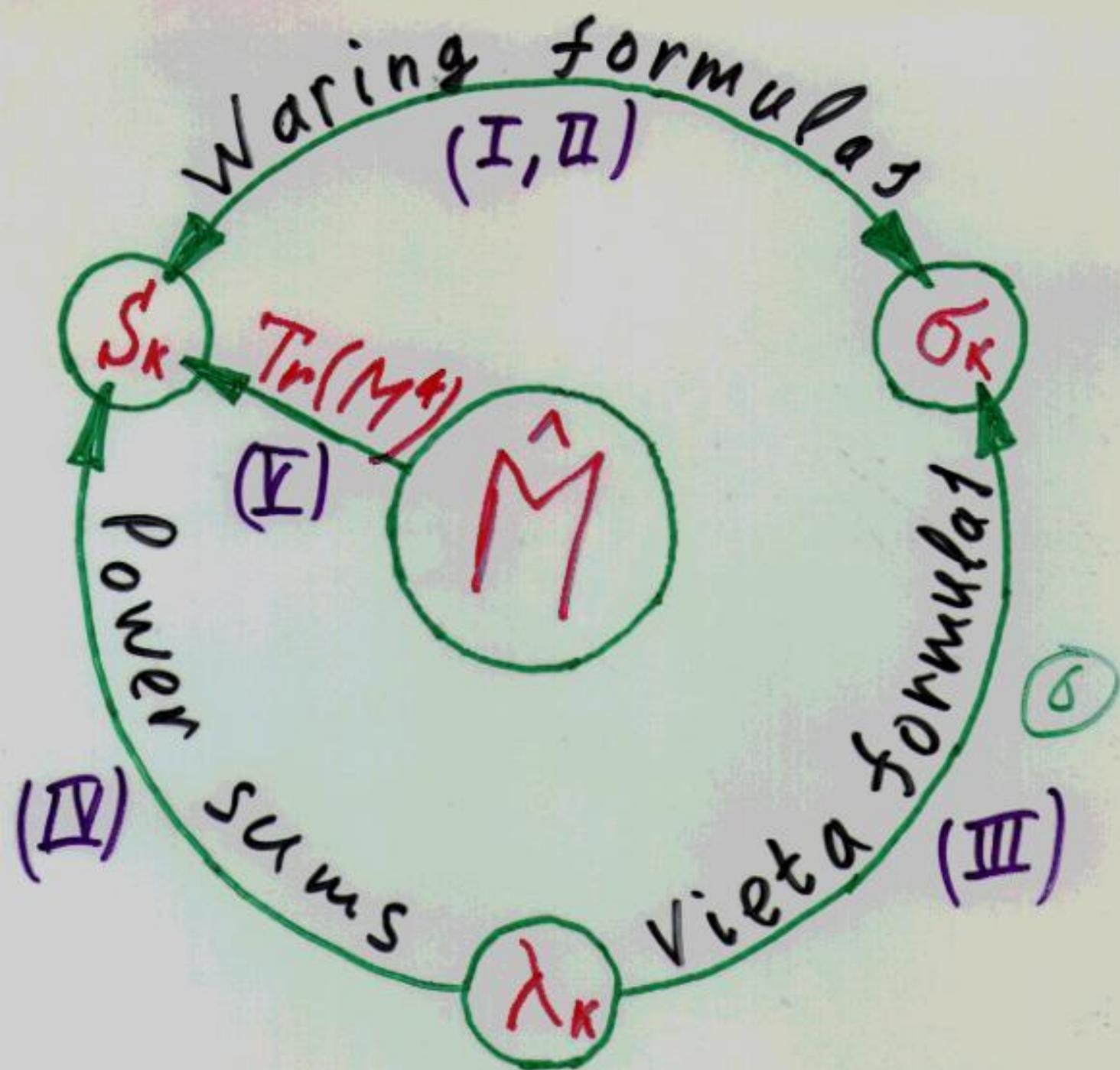
We have analytical formulas (Waring formulas) for relations between  $\sigma_k$  and  $S_k$

$$S_p = \sum_{k_1 + 2k_2 + \dots + nk_n = p} \frac{(k_1 + \dots + k_n - 1)! \alpha_1^{k_1} \dots \alpha_n^{k_n}}{(k_1)! \dots (k_n)!}$$

(5)

$$\alpha_p = \sum_{k_1 + 2k_2 + \dots + pk_p = p} \frac{(-1)^{k_1 + \dots + k_p + p} S_1^{k_1} \dots S_p^{k_p}}{1^{k_1} 2^{k_2} \dots p^{k_p} (k_1)! \dots (k_p)!}$$

rename:  $\alpha_p = (-1)^{p+1} \sigma_p$



The algorithms I, II, III, IV and V are finite sets of operations of addition and multiplication. But for  $\lambda_k$  this is incorrect!

# Part II. The main problem.

The formula of Lagrange and Silvestre for function  $F$  of matrix  $\hat{M}$  is:

$$F(\hat{M}) = \sum_{k=1}^N \frac{(\hat{M} - \lambda_1 I) \dots (\hat{M} - \lambda_{k-1} I) (\hat{M} - \lambda_{k+1} I) \dots (\hat{M} - \lambda_n I)}{(\lambda_k - \lambda_1) \dots (\lambda_k - \lambda_{k-1}) (\lambda_k - \lambda_{k+1}) \dots (\lambda_k - \lambda_n)} F(\lambda_k)$$

Example 1.  $\hat{M}$  is 2D matrix:

$$F(\hat{M}) = \frac{\hat{M} - \lambda_1 I}{\lambda_2 - \lambda_1} F(\lambda_2) + \frac{\hat{M} - \lambda_2 I}{\lambda_1 - \lambda_2} F(\lambda_1)$$

Example 2.  $\hat{M}$  is 3D matrix:

$$F(\hat{M}) = \frac{(\hat{M} - \lambda_1 I)(\hat{M} - \lambda_2 I)}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)} F(\lambda_3) + \frac{(\hat{M} - \lambda_2 I)(\hat{M} - \lambda_3 I)}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)} F(\lambda_1) + \frac{(\hat{M} - \lambda_1 I)(\hat{M} - \lambda_3 I)}{(\lambda_2 - \lambda_1)(\lambda_2 - \lambda_3)} F(\lambda_2)$$

Arguments ( $\lambda_k$ ) of Lagrange and Silvestre formulas are badly-defined. But we can rewrite this formula

Example 1:  $F(\tilde{M}) = \frac{\lambda_2 F(\lambda_1) - \lambda_1 F(\lambda_2)}{\lambda_2 - \lambda_1} \tilde{I} + \frac{F(\lambda_2) - F(\lambda_1)}{\lambda_2 - \lambda_1} \tilde{M}$

and in general case:

$$F(\tilde{M}) = \Omega_0(\lambda_1, \dots, \lambda_n) \tilde{I} + \dots + \Omega_{n-1}(\lambda_1, \dots, \lambda_n) \tilde{M}^{n-1}$$

where  $\Omega_k(\lambda_1, \dots, \lambda_n)$  are symmetric function of set  $\lambda_k$ .

If  $F(\lambda) = A_0 + A_1 \lambda + A_2 \lambda^2 + \dots + A_k \lambda^k + \dots$  then  $\Omega_k(\lambda_1, \dots, \lambda_n)$  are symmetric polynomials of  $\lambda_k$ .

For symmetrical polynomial  
an equality: 9

$$\Omega_k(\lambda_1, \dots, \lambda_n) = \Phi_k(\sigma_1, \dots, \sigma_n)$$

is correct always!

According to basic theorem  
on symmetrical polynomials.

$\Phi_k(\sigma_1, \dots, \sigma_n)$  is a polynomial of  
the set  $\sigma_k$ . Now we have  
Lagrange and Silvestre formula  
in the form:

$$F(\hat{M}) = \Phi_0(\sigma_1, \dots, \sigma_n)I + \dots + \Phi_{n-1}(\sigma_1, \dots, \sigma_n)\hat{M}^{n-1}$$

Finally we need in explicit

form for  $\Phi_k(\sigma_1, \dots, \sigma_n) = ?$  in

general case  $F(\lambda) = \sum_k A_k \lambda^k$

for arbitrary coefficients  $A_k$

Part III. The main result.

$$F(\hat{M}) = \phi_0(\dots, \alpha_k, \dots) \hat{I} + \dots + \phi_m(\dots, \alpha_k, \dots) \hat{M}^m + \dots + \phi_{p-1}(\dots, \alpha_k, \dots) \hat{M}^{p-1} + \dots + \phi_p(\dots, \alpha_k, \dots) \hat{M}^p$$

If  $F(\lambda) = \sum_{k=0}^{\infty} A_k \lambda^k$  then

$$\Phi_k = \Psi_k - \sum_{p=1}^{k-1} \alpha_p \Psi_{k-p}$$

where

$$\Psi_p = \sum_{m=1}^{\infty} Z_{p,m} A_m$$

$$Z_{p,m} = \frac{(k_1 + \dots + k_N)!}{(k_1)! \dots (k_N)!} \alpha_1^{k_1} \alpha_2^{k_2} \dots \alpha_N^{k_N}$$

$$k_1 + 2k_2 + \dots + Nk_N + p = m$$

$$m \geq p$$

main functions!

Proving by use induction method.

$$\left. \begin{aligned} \phi_p &= \psi_p - \sum_{m=1}^{N-p-1} \alpha_m \psi_{p+m} \\ \psi_p &= \sum_{q=0}^{\infty} Z_{p,q} A_q \end{aligned} \right\} \rightarrow \phi_p = \sum_{q=0}^{\infty} Y_{p,q} A_q$$

(11)

where  $Y_{p,q} = Z_{p,q} - \sum_{m=1}^{N-p-1} \alpha_m Z_{p+m,q}$

Theorem of H-C:  
 $\hat{M}^N = \alpha_1 \hat{M}^{N-1} + \alpha_2 \hat{M}^{N-2} + \dots + \alpha_N \hat{I}$

For function  $F(\hat{M}) = \hat{M}^q$  ( $q > N$ )  
 suppose that:

$$F(\hat{M}) = \hat{M}^q = Y_{0,q} \hat{I} + Y_{1,q} \hat{M} + \dots + Y_{N-1,q} \hat{M}^{N-1}$$

therefore:  $\hat{M}^{q+1} = Y_{0,q+1} \hat{I} + \dots + Y_{N-1,q+1} \hat{M}^N$

with helping H-C  
 and properties of  $Z_{p,q}$ :

$$Z_{0,q} = 1 + \alpha_1 Z_{1,q} + \alpha_2 Z_{2,q} + \dots + \alpha_N Z_{N,q}$$

Example 1:  $\hat{M}$  is 2D matrix.

$$\hat{M}^2 = \alpha_1 \hat{M} + \alpha_2 \hat{I} \quad (12)$$

$$F(\hat{M}) = \sum_{k=0}^{\infty} A_k \hat{M}^k = \phi_0 \hat{I} + \phi_1 \hat{M}$$

$$\phi_2 = \psi_1; \quad \phi_0 = \psi_0 - \alpha_1 \psi_1, \quad \psi_1 = A_0 + \alpha_2 \psi_2$$

$$\psi_0 = A_0 + \alpha_1 \psi_1 + \alpha_2 \psi_2$$

$$\psi_0 = \sum_{q=0}^{\infty} A_q Z_{0,q}; \quad \psi_1 = \sum_{q=0}^{\infty} Z_{1,q} A_q; \quad \psi_2 = \sum_{q=0}^{\infty} Z_{2,q} A_q$$

$$Z_{0,q} = \sum_{\substack{k_1 + 2k_2 = q \\ q \geq 0}} \frac{(k_1 + k_2)!}{(k_1)! (k_2)!} \alpha_1^{k_1} \alpha_2^{k_2}$$

$$Z_{1,q} = \sum_{\substack{k_1 + 2k_2 + 1 = q \\ q \geq 1}} \frac{(k_1 + k_2)!}{(k_1)! (k_2)!} \alpha_1^{k_1} \alpha_2^{k_2}$$

$$Z_{2,q} = \sum_{\substack{k_1 + 2k_2 + 2 = q \\ q \geq 2}} \frac{(k_1 + k_2)!}{(k_1)! (k_2)!} \alpha_1^{k_1} \alpha_2^{k_2}$$

In general case of 2D matrix  $M$ :

$$F(\hat{M}) = (A_0 + \alpha_2 \sum_{k_1+2k_2+2=9} \frac{(k_1+k_2)!}{(k_1)! (k_2)!} \alpha_1^{k_1} \alpha_2^{k_2} A_2) I +$$

(12)

$$+ \left( \sum_{k_1+2k_2+1=9} \frac{(k_1+k_2)!}{(k_1)! (k_2)!} \alpha_1^{k_1} \alpha_2^{k_2} A_2 \right) M$$

If  $\alpha_1 = \text{Tr}(M) = 0$

then:

$$F(\hat{M}) = (A_0 + \sum_{2k_2+2=9} \alpha_2^{k_2+1} A_2) I + \left( \sum_{2k_2+1=9} \alpha_2^{k_2} A_2 \right) M$$

If  $\alpha_1 = \text{Tr}(\hat{M}) = 0$  and  $F(\lambda) = \exp(\lambda)$

then  $A_1 = \frac{1}{1!}$  and:

$$\exp(\hat{M}) = \left( \sum_{k_2=0}^{\infty} \frac{(\sqrt{\alpha_2})^{2k_2}}{(2k_2)!} \right) I + \left( \frac{1}{\sqrt{\alpha_2}} \sum_{k_2=0}^{\infty} \frac{(\sqrt{\alpha_2})^{2k_2+1}}{(2k_2+1)!} \right) \hat{M}$$

3/

$$\exp(\hat{M}) = \cos(\sqrt{\alpha_2}) I + \frac{\sin(\sqrt{\alpha_2})}{\sqrt{\alpha_2}} \hat{M}$$

$$\text{Tr}(\hat{M}) = 0$$

In the reality:

$$\textcircled{5} \quad \mathcal{H} \rightarrow \hat{M}^2 = \alpha_1 \hat{M} + \alpha_2 \hat{I} \quad \textcircled{2}$$

$$M^3 = (\alpha_1^2 + \alpha_2) \hat{M} + \alpha_1 \alpha_2 \hat{I}$$

$$M^4 = (\alpha_1^3 + 2\alpha_1 \alpha_2) \hat{M} + \alpha_2 (\alpha_1^2 + \alpha_2) \hat{I}$$

$$M^5 = (\alpha_1^4 + 3\alpha_1^2 \alpha_2 + \alpha_2^2) \hat{M} + \alpha_2 (\alpha_1^3 + 2\alpha_1 \alpha_2) \hat{I}$$

$$M^6 = (\alpha_1^5 + 4\alpha_1^3 \alpha_2 + 3\alpha_1 \alpha_2^2) \hat{M} + \alpha_2 (\alpha_1^4 + 3\alpha_1^2 \alpha_2 + \alpha_2^2) \hat{I}$$

$$M^7 = (\alpha_1^6 + 5\alpha_1^4 \alpha_2 + 6\alpha_1^2 \alpha_2^2 + \alpha_2^3) \hat{M} + \alpha_2 (\alpha_1^5 + 5\alpha_1^3 \alpha_2 + 6\alpha_1 \alpha_2^2 + \alpha_2^3) \hat{I}$$

$$F(M) = \frac{\lambda_2 F(\lambda_1) - \lambda_1 F(\lambda_2)}{\lambda_2 - \lambda_1} \hat{I} + \frac{F(\lambda_2) - F(\lambda_1)}{\lambda_2 - \lambda_1} \hat{M}$$

$$F(\lambda) = \sqrt{\lambda}$$

$$\sqrt{\hat{M}} = \frac{\lambda_2 \sqrt{\lambda_1} - \lambda_1 \sqrt{\lambda_2}}{\lambda_2 - \lambda_1} \hat{I} + \frac{\sqrt{\lambda_2} - \sqrt{\lambda_1}}{\lambda_2 - \lambda_1} \hat{M}$$

$$\sqrt{\hat{M}} = \pm \left( \sqrt{\frac{\sigma_2}{\sigma_1 + 2\sqrt{\sigma_2}}} \hat{I} + \frac{1}{\sqrt{\sigma_1 + 2\sqrt{\sigma_2}}} \hat{M} \right)$$

$$\sigma_1 = \lambda_1 + \lambda_2 \quad \sigma_2 = \lambda_1 \lambda_2$$

$$\hat{M}^2 = \sigma_1 \hat{M} - \sigma_2 \hat{I} \quad (16)$$