Steps towards full two-loop calculations for 2 fermion to 2 fermion processes

Precision calculations of massive particle production processes in the SM

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① Steps towards full two-loop SM calculations

Aim: so far little feeling for size of corrections from bosonic sector.

Very complex: electroweak SM: 57 vertices 11 types of lines $\Rightarrow$ "multiple factorial" growth of complexity

- **QED and QCD on electroweak processes: limited number of diagrams**
- **relatively small number of diagrams involving top or physical Higgs**
- **full gauge boson sector (incl. Higgs- and Fadeev-Popov ghosts) large number of diagrams**

Steps of technical complications: self–energies $\rightarrow$ form–factors $\rightarrow$ boxes

Complete calculations of observables available so far only for μ–decay

Awramik&Czakon, Onishchenko&Veretin

Full two–loop renormalization program: need full set of counter terms. e.g., on-shell renormalization scheme α, M_Z and M_W as basic parameters (QED–like scheme) $\rightarrow$ calculate gauge boson mass counter-terms (equiv. $\overline{\mathrm{MS}}$ vs. pole mass relation)

Theoretical issue: About the proper definition of masses of unstable particles

(Stuart, '91, Sirlin, '91, ..., Kniehl, Sirlin, '98)

The pole mass of the weak gauge bosons

(at two–loops)

The mass and width of a massive gauge boson V are defined via the position s_P of the pole of the full propagator (=zero of its inverse)

$$\boxed{s_P - m_V^2 - \Pi_V(s_P, m_V^2, \cdots) = 0,}$$

$\Pi_V(p^2, \cdots)$ **transversal part of the one-particle irreducible self-energy (depends on all SM parameters)**

- **bare amplitude in terms of bare parameters ($m_V \to m_{V,0},\ \Pi_V \to \Pi_{V,0}$)**
- **renormalized amplitude in terms of renormalized parameters, e.g., $\overline{\mathrm{MS}}$ (no index)**

Properties of the pole:

- **gauge invariant**
- **infrared finite**
- **complex in general**

Defines *pole (on–shell) mass* M and *width* Γ via

$$\boxed{s_P \equiv M^2 - \mathrm{i}M\Gamma.}$$

Renormalization

The renormalized amplitudes

$$\Pi_{V,r}(p^2, m^2_{V,r}, \cdots) = \Pi^{(1)}_{V,r}(p^2, m^2_{V,r}, \cdots) + \Pi^{(2)}_{V,r}(p^2, m^2_{V,r}, \cdots) + \cdots$$

to two-loops read (indices: 0=bare, r=renormalized)

$$\Pi^{(1)}_{V,r}(p^2, m^2_{V,r}, \cdots) = \left[\Pi^{(1)}_{V,0}(p^2, m^2_{V,0}, \cdots) + (\delta m^2_V)^{(1)} - (p^2 - m^2_{V,r})\,\delta Z^{(1)}_V\right]\Bigg|_{\substack{m^2_{j,0} = m^2_{j,r} \\ e_0 = e_r}}$$

$$\Pi^{(2)}_{V,r}(p^2, m^2_{V,r}, \cdots) = \left[\Pi^{(2)}_{V,0}(p^2, m^2_{V,0}, \cdots) + \left(\Pi^{(1)}_{V,0}(p^2, m^2_{V,0}, \cdots) + (\delta m^2_V)^{(1)}\right)\delta Z^{(1)}_V\right.$$

$$\left. + \sum_j (\delta m^2_j)^{(1)} \frac{\partial}{\partial m^2_{j,0}} \Pi^{(1)}_{V,0} + (\delta e)^{(1)} \frac{\partial}{\partial e_0} \Pi^{(1)}_{V,0} + (\delta m^2_V)^{(2)} - (p^2 - m^2_{V,r})\,\delta Z^{(2)}_V\right]\Bigg|_{\substack{m^2_{j,0} = m^2_{j,r} \\ e_0 = e_r}}$$

where in the $\overline{\mathrm{MS}}$ scheme order by order the mass-counter-term $(\delta m^2_V)^{(j)}$ subtracts the ϵ–poles at $p^2 = m^2_{V,r}$ and the wave-function renormalization counter-term $\delta Z^{(j)}_V$ subtracts the ϵ–poles remaining when $p^2 \neq m^2_{V,r}$

Strictly speaking the renormalization of the ghost sector (in particular of the gauge parameter) is not discussed here, because its not needed for what follows. In case of the Z the $\gamma - Z$-mixing is an additional complication (see below).

Pole mass and $\gamma - Z$-mixing

In the neutral gauge boson sector because of $\gamma - Z$-mixing we have to consider a 2×2 matrix propagator

$$D^{-1}(p^2) = \begin{pmatrix} p^2 - \Pi_{\gamma\gamma}(p^2) & \Pi_{\gamma Z}(p^2) \\ \Pi_{Z\gamma}(p^2) & p^2 - m_Z^2 - \Pi_{ZZ}(p^2) \end{pmatrix}$$

Position of Z-pole:

$$\boxed{s_P - m_Z^2 - \Pi_{ZZ}(s_P) - \frac{\Pi_{\gamma Z}^2(s_P)}{s_P - \Pi_{\gamma\gamma}(s_P)} = 0.}$$

- **Mixing term $\Pi_{\gamma Z}^2$ starts contributing at two-loops**
- **Photon term $\Pi_{\gamma\gamma}$ only contributes beyond two-loop**

Notation for self-energies Π_V ($V = W, Z$) with

$$\Pi_W(p^2, \cdots) = \Pi_{WW}(p^2, \cdots)$$

$$\Pi_Z(p^2, \cdots) = \Pi_{ZZ}(p^2, \cdots) + \frac{\Pi_{\gamma Z}^2(p^2, \cdots)}{p^2 - \Pi_{\gamma\gamma}(p^2, \cdots)} \, .$$

Formally, same formulae apply for W and Z.

Pole mass "master formula"

By iterative solution of the pole formula to two-loops we obtain our master formula:

$$
\begin{aligned}
s_P \quad = \quad & m^2 \\
& +\Pi^{(1)}(m^2, m^2, \cdots) \\
& +\Pi^{(2)}(m^2, m^2, \cdots) + \Pi^{(1)}(m^2, m^2, \cdots)\, \Pi^{(1)\prime}(m^2, m^2, \cdots) \\
& + \cdots
\end{aligned}
$$

which yields the pole mass M^2 and the width Γ at this order. $\Pi^{(L)}$ is the bare ($m = m_0$) or $\overline{\mathrm{MS}}$ -renormalized (m the $\overline{\mathrm{MS}}$ -mass) L-loop contribution to Π, and the prime denotes the derivative with respect to p^2. In this way we need to evaluate propagator type diagrams and their derivatives at $p^2 = m^2$.

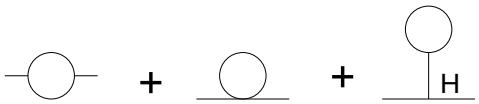
Note: the p^2–dependence has disappeared in this solution; it turned into a mass–dependence which cannot be disentangled from the original mass dependence of the off-shell amplitude.

Remark: the mixed $\Pi^{(1)}(m^2, m^2, \cdots)\, \Pi^{(1)\prime}(m^2, m^2, \cdots)$ term is crucial for getting a gauge–invariant result for the two–loop mass counter–term

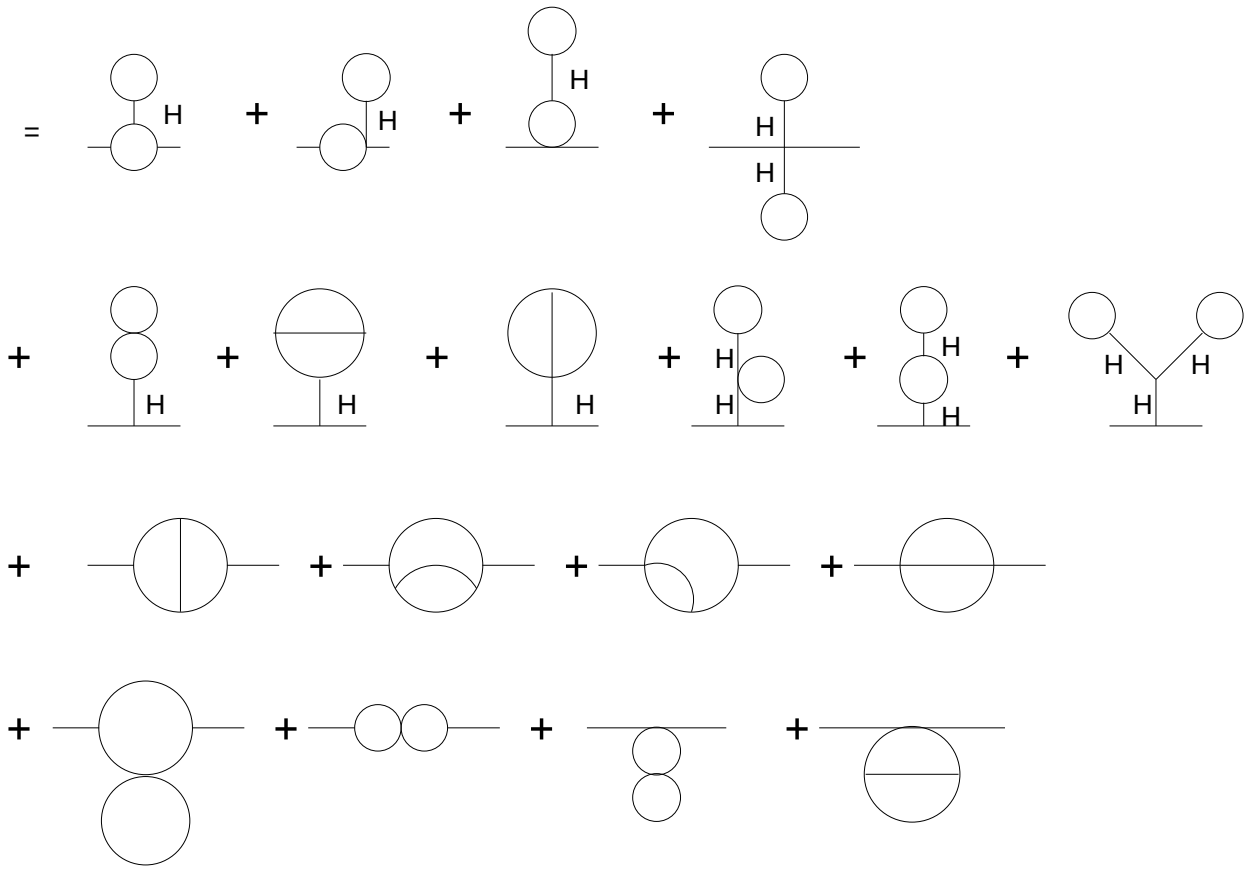
Diagrams and topologies

To be computed on-shell ($p^2 = m_V^2$): $\Pi(p^2) = \Pi_1(p^2) + \Pi_2(p^2) + \cdots$

$\Pi^{(1)}$ =



$\Pi^{(2)}$ =



Bosonic contribution				
Number of diagrams	**linear** R_ξ *gauge*		**nonlinear** R_ξ *gauge*	
one-loop :	~ 50			
two-loop :	$1PI$	$Total$	$1PI$	$Total$
Z	616	2348	410	1837
W	792	4084	537	2942
With one massive fermion family				
two-loop :	$1PI$	$Total$	$1PI$	$Total$
Z	802	4410	550	3631
W	990	7780	669	5604

K. Fujikawa '73

D.A. Dicus & C. Kao '94

Gauge

The gauge fixing Lagrangian is $L_{g.f.} = -\frac{1}{\xi_W} F^+ F^- - \frac{1}{2\xi} (\partial_\mu A^\mu)^2 - \frac{1}{2\xi_Z} (\partial_\mu Z^\mu - \xi_Z M_Z h)^2$

where for the linear R_ξ gauge is $F^\pm = \partial_\mu W^\pm_\mu \mp i\xi_W M_W \phi^\pm$

and the nonlinear R_ξ gauge is defined as $F^\pm = \partial_\mu W^\pm_\mu \mp i\xi_W M_W \phi^\pm \mp ieA_\mu W^\pm_\mu \pm ig\frac{\sin^2\theta_W}{\cos\theta_W} Z_\mu W^\pm_\mu$

The old vertices:	
$\{A_\mu, Z_\mu\} W^\pm_\mu \phi^\mp$	$absent$
$\{A_\sigma, Z_\sigma, A_\mu A_\nu, A_\mu Z_\nu, Z_\mu Z_\nu\} W^\pm_\mu W^\mp_\nu$	$modified$
$\{A_\mu, Z_\mu\} \overline{\eta}^\pm \eta^\pm$	$modified$
$W^\pm \overline{\eta}^\pm \{c_\gamma, \Delta_Z\}$	$modified$
The new vertices:	
$\{A_\mu A_\mu, A_\mu Z_\mu, Z_\mu Z_\mu\} \overline{\eta}^\pm \eta^\pm$	new
$\{A_\mu, Z_\mu\} W^\pm \overline{\eta}^\pm \{c_\gamma, \Delta_Z\}$	new

Evaluation of 2–loop self–energies

There exist a number of programs, which calculate the "bubble–diagrams" (analytical), obtained by low energy expansions, but also arbitrary self-energy diagrams (analytical and 1-dimensional integral-representations):

- **a class of massive 2–loop–integrals (Broadhurst 90, Fleischer, Kalmykov, Kotikov 99), which depend on one scale only have been implemented in *ONSHELL2* (Fleischer, Tarasov 92, Fleischer, Kalmykov 00),**
- **in general exact analytic results are not known and one has to resort to series expansions at low or high energies (Broadhurst, Fleischer, Tarasov, '93, Fleischer, Tarasov,'94, Fleischer, Kalmykov, Veretin, 98), which may be combined with methods of conformal mapping and Padé–resummation,**
- **a combined analytical-numerical program for 2–loop self-energy functions has been developed by the W"urzburg/Leiden–Collaboration (Bauberger, Berends, Böhm, Buza, '95, Bauberger, Böhm, '95),**
- **for the reduction of integrals to a basis of standard-integrals there exist packages which solve the systems of recurrence-relations (Tarasov '97). Utilizing relations between integrals in different dimensions D the problem of irreducible numerators could be solved (Tarasov '96b)**

- **For integrals showing up in a large mass expansion the package *TLAMM* (Avdeev et al., ’97) is available.**
- **Various expansions with respect to small parameters may by utilized (Smirnov, ’90, ’95, ’99 ,’01)**

Evaluation by expansion

Check of gauge invariance: R_ξ **gauge (independent gauge parameters** ξ_W**,** ξ_Z **and** ξ_γ**) Then there are several scales:** $m_W,\ \sqrt{\xi_W}m_W,\ m_Z,\ \sqrt{\xi_Z}m_Z,\ m_H$

We perform expansions in 3 steps:

1. Taylor (naive) expansion in $(\xi_V - 1)$**: i.e., propagators of the vector bosons and associated Higgs scalar ghosts look like**

$$D^V_{\mu\nu}(p) = \frac{i}{p^2 - m_V^2}\left(-g_{\mu\nu} + (1-\xi_V)\frac{p_\mu p_\nu}{p^2 - m_V^2} - (1-\xi_V)^2\frac{m_V^2 p_\mu p_\nu}{(p^2 - m_V^2)^2} + \dots\right)$$

$$\Delta_V(p) = \frac{i}{p^2 - m_V^2}\left(1 - (1-\xi_V)\frac{m_V^2}{p^2 - m_V^2} + (1-\xi_V)^2\left(\frac{m_V^2}{p^2 - m_V^2}\right)^2 + \dots\right)$$

where $V = W, Z$**.**

2. Expansion in the small parameter

$$\sin^2\Theta_W = 1 - \frac{m_W^2}{m_Z^2} \lesssim 0.24$$

by which $m_W^2 = m_Z^2\,(1 - \sin^2\Theta_W)$**; no–Higgs diagrams then are one–scale and can be calculated analytically with the *ONSHELL2* package.**

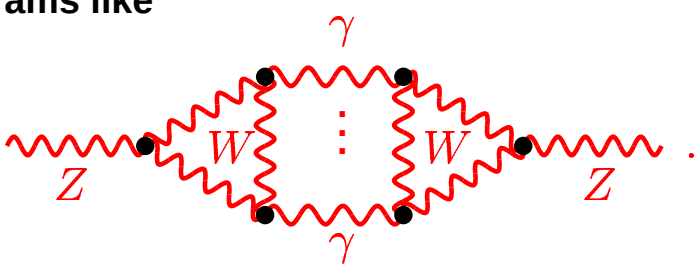
3. Diagrams with Higgs–lines are expanded for large m_H **in**

$$z \equiv m_Z^2/m_H^2 \lesssim 0.64$$

using the *TLAMM* package

Gauge invariance

As we know, resonant Z and W bosons decay mainly into fermion pairs. Indeed, if we switch off the fermions (as we do here) the gauge bosons are close to stable! For the purely bosonic contributions alone the imaginary part of $\Pi(p^2)$ on the mass-shell is zero at the two-loop level. This is due to the fact that in the bosonic sector we have the physical masses $m_\gamma = 0$, M_Z, M_W and M_H and by inspection of the possible two and three particle intermediate states one observes that all physical thresholds lie above the mass shells of the W and Z bosons, i.e., the self-energies of the massive gauge bosons develop an imaginary part only at $p^2 > M_V^2$ (to two loops in the SM). On kinematical grounds imaginary parts could show up from the Higgs or Faddeev-Popov ghosts, which have square masses $\xi_V M_V^2$, for small values of the gauge parameter. However, as we have verified, the two-loop on-shell self-energies are gauge independent. This implies that ghost contributions have to cancel and hence cannot contribute to the imaginary part. Thus $s_P = M_V^2$ in our case. In higher orders for the Z–propagator one gets an imaginary part as soon as $p^2 > 0$, from diagrams like



For the W–propagator an imaginary part is only possible for $p^2 > M_W^2$, because charge conservation requires at least one W in any physical intermediate state.

Drawback of our choice of expansion about $\xi_i = 1$: analytic structure (ghost thresholds) lost: do not get correct imaginary part from ghost contributions !

Examples:

1.) threshold $p^2 = 4\xi_Z M_Z^2$ of

$Z \to \phi^0\phi^0$ production, ϕ^0 the neutral Higgs ghost

which is below the Z mass-shell $p^2 = M_Z^2$ when $\xi_Z < \frac{1}{4}$

2.) threshold $p^2 = \xi_W M_W^2$ of

$W^\pm \to \phi^\pm\gamma$ production, $\phi^\pm$ the charged Higgs ghosts

which is below the W mass-shell $p^2 = M_W^2$ when $\xi_W < 1$.

Thus for small values of ξ we do not get correct imaginary part diagram by diagram. However, ghost contributions must cancel on-shell. Thus by gauge invariance, which we check we know that we get the correct result. At one–loop one may check this analytically.

(Fleischer, Jegerlehner, '81)

Cancellation is highly non-trivial: a consequence of the Slavnov-Taylor identities, which tell us how Higgs ghost, Faddeev-Popov ghosts and scalar components contained in the gauge boson fields decouple from physical amplitudes like the physical width.

For gauge parameters $\xi > 1$ the imaginary part of the W and Z self-energies in the bosonic sector up to two-loops is zero, by applying the Cutkowsky rules and inspecting all possible two and three particle intermediate states allowed by the SM Lagrangian. While for $\xi > 1$ the imaginary part is zero for each individual diagram, for small enough values of the gauge parameters a nontrivial cancellation must take place. An independent direct check of this is possible by considering the problem in the limit $\xi \to 0$, for example.

Results

- UV renormalization is exact (analytic);
  UV singularities (poles $1/\varepsilon^2$ and $1/\varepsilon$) are not affected by SSB;
  $\Rightarrow$ check against RG results within unbroken theory (Jones, '82, Machacek, Vaughn, '83) ✓
- confirms IR finiteness of on–shell mass for both Z and W
- gauge invariance of position of pole s_P;
  requires taking into account tadpoles ✓
- large m_H expansion breaks down at large m_H
  because of strong coupling problem (gets non-perturbative)
- relation between $\overline{\mathrm{MS}}$ and pole mass exhibits unphysical terms proportional to m_H^4, which violate Veltman's screening theorem: in observables at L–loops:

$$X(m_H) \to O((G_\mu m_H^2)^{L-1} \ln(m_H/M_W)^2) \text{ as } m_H \to \infty$$

  The fake terms drop in $\sin^2\Theta_W = 1 - \frac{M_W^2}{M_Z^2}$ as they should. ✓
- behavior for intermediate Higgs masses: looks O.K. down to
  about 130 GeV
- complete 2–loop calculation of fermionic corrections incl. QCD complete
- one of the main ingredients of full 2-loop corrections to μ–decay:
  Awramik&Czakon, Onishchenko&Veretin

Form of results

$$\frac{M_V^2}{m_V^2} = 1 + \left(\frac{e^2}{16\pi^2 \sin^2\theta_W}\right) X_V^{(1)} + \left(\frac{e^2}{16\pi^2 \sin^2\theta_W}\right)^2 X_V^{(2)},$$

$$X_V^{(2)} = \frac{m_H^4}{m_V^4} \sum_{k=0}^{5} \sin^{2k}\theta_W \, A_k^V .$$

$$A_i^V = \sum_{j=0}^{5} A_{i,j}^V \left(\frac{m_V^2}{m_H^2}\right)^j$$

All parameters in $\overline{MS}$ scheme.

Six coefficients calculated analytically. Expansion in powers and log's (i.e., is an asymptotic expansion not a naive Taylor expansion). Expansion coefficients $A_{i,j}$ given by a small set of transcendental constants like:

$$\begin{aligned}
S_0 &= \frac{\pi}{\sqrt{3}} \sim 1.813799365...,\\
S_1 &= \frac{\pi}{\sqrt{3}} \ln 3 \sim 1.992662272...,\\
S_2 &= \frac{4}{9}\frac{\mathrm{Cl}_2\left(\frac{\pi}{3}\right)}{\sqrt{3}} \sim 0.260434137632162...,\\
S_3 &= \pi \mathrm{Cl}_2\left(\frac{\pi}{3}\right) \sim 3.188533097...
\end{aligned}$$

$\overline{\mathrm{MS}}$ mass in terms of on–shell mass

Inverse of "master formula": express all $\overline{\mathrm{MS}}$ parameters in terms of on-shell ones:

$$m_V^2 = M_V^2 - \hat{\Pi}_V^{(1)} - \left\{ \Pi_V^{(2)} + \Pi_V^{(1)} \Pi_V^{(1)\prime} \right\}_{MS}$$
$$- \sum_j (\Delta m_j^2)^{(1)} \frac{\partial}{\partial m_j^2} \hat{\Pi}_V^{(1)} - (\Delta e)^{(1)} \frac{\partial}{\partial e} \hat{\Pi}_V^{(1)} \Big|_{m_j^2 = M_j^2,\, e = e_{\mathrm{OS}}} ,$$

sum runs over all species of particles $j = Z,\, W,\, H$

$$(\Delta m_j^2)^{(1)} = -\mathrm{Re}\hat{\Pi}_j^{(1)} \Big|_{m_j^2 = M_j^2,\, e = e_{\mathrm{OS}}} \equiv -M_V^2 \frac{e_{\mathrm{OS}}^2}{16\pi^2 \sin^2\theta_W} X_V^{(1)} \Big|_{m_j^2 = M_j^2}$$

stands for the self-energy of the jth particle at $p^2 = m_j^2$ in the $\overline{\mathrm{MS}}$ scheme and parameters replaced by the on-shell ones. Includes a change from the $\overline{\mathrm{MS}}$ to the on-shell scheme also for the electric charge

$$e(\mu^2) = e_{\mathrm{OS}} \left[1 + \frac{e_{\mathrm{OS}}^2}{16\pi^2} \left(\frac{7}{2} \ln\left(\frac{M_W^2}{\mu^2} \right) - \frac{1}{3} \right) \right]$$

with $e_{\mathrm{OS}}^2/4\pi = \alpha \sim 1/137 \rightarrow \hat{\Pi}^{(1)}$ depends on e by an overall factor e^2 only,

$$(\Delta e)^{(1)} \frac{\partial}{\partial e} \hat{\Pi}_V^{(1)} = \frac{e^2}{16\pi^2} \left[7 \ln\left(\frac{M_W^2}{\mu^2} \right) - \frac{2}{3} \right] \hat{\Pi}_V^{(1)} .$$

On-shell scheme mass counter-terms

Identifying $m_V^2 = m_{V,0}^2 = M_V^2 + \delta M_V^2$ **in inverse MF** $\Rightarrow$ **on-shell gauge-boson mass counter-terms** δM_V^2 **:**

$$\begin{aligned}
\delta M_V^2 &= -\mathrm{Re}\Bigg[\check{\Pi}_{V,0}^{(1)} + \check{\Pi}_{V,0}^{(2)} + \check{\Pi}_{V,0}^{(1)}\check{\Pi}_{V,0}^{(1)\prime} \\
&\quad + \sum_j (\delta M_j^2)^{(1)} \frac{\partial}{\partial m_{j,0}^2}\check{\Pi}_{V,0}^{(1)} + (\delta e)^{(1)} \frac{\partial}{\partial e_0}\check{\Pi}_{V,0}^{(1)}\Bigg]\Bigg|_{\substack{m_{j,0}^2 = M_j^2 \\ e_0 = e_{\mathrm{OS}}}} \\
&= \left(Z_{\overline{\mathbf{MS}}} \cdot Z_{\mathrm{OS}} - 1\right) M_V^2
\end{aligned}$$

in terms of the original bare on-shell amplitudes

$$\check{\Pi}_{V,0}^{(i)} = \Pi_{V,0}^{(i)}(p^2, m_{V,0}^2, \cdots)\Big|_{p^2=M_V^2, m_{j,0}^2=M_j^2,\, e_0=e_{\mathrm{OS}}}$$

and the bare on-shell counter-terms δM_j^2 **and** δe**.**

The second equality gives δM_V^2 **in terms of the singular factor** $Z_{\overline{\mathbf{MS}}} = m_{V,0}^2/m_V^2(\mu)$ **and the finite factor** $Z_{\mathrm{OS}} = m_V^2/M_V^2$**. These will be needed in two-loop calculations of observables in the on-shell scheme.**

Explicit expressions in (Jegerlehner, Kalmykov, Veretin, '01)

② The quark pole mass

The tensor decomposition of the one–particle irreducible self–energy of a massive fermion $\tilde{\Sigma}(p,m,\ldots)$ has the form

$$\tilde{\Sigma}(p,m,\ldots) = i\hat{p}\left[\tilde{A}(p^2,m,\ldots)-\gamma_5\tilde{C}(p^2,m,\ldots)\right] + m\left[\tilde{B}(p^2,m,\ldots)-\gamma_5\tilde{D}(p^2,m,\ldots)\right] ,$$

$\tilde{A},\tilde{B},\tilde{C},\tilde{D}$ **Lorentz scalar functions depending on all parameters of the SM. At** $O(\alpha\alpha_s)$ $\tilde{C}=\tilde{D}=0$. **The position of the pole** $\tilde{M}$ **is given the zero of the inverse of the connected full propagator. By iterative solution we have up to 2-loops:**

$$\boxed{\frac{\tilde{M}}{m} = 1+\Sigma_1+\Sigma_2+\Sigma_1\Sigma_1'}$$

Σ_L **is the bare (**$m=m_0$**) or** $\overline{\mathrm{MS}}$ **- renormalized (**m **the** $\overline{\mathrm{MS}}$ **-mass)** L**-loop contribution to fermion self-energy, the prime denotes the derivative with respect to** p **and**

$$\tilde{\Sigma}(p,m,\ldots) = \tilde{\Sigma}\,\Big|_{\hat{p}=im_0} + (i\hat{p}+m_0)\left[\tilde{\Sigma}'\right]\Big|_{\hat{p}=im_0} + \cdots$$

and define dimensionless "on–shell" amplitudes Σ, Σ' **by**

$$\tilde{\Sigma}\Big|_{\hat{p}=im_0} = \left[-m_0\tilde{A}+m_0\tilde{B}\right]\Big|_{p^2=-m_0^2} \equiv -m_0\Sigma(m_0,\ldots)$$

and

$$\left[\tilde{\Sigma}'\right]\Big|_{\hat{p}=im_0} = \left[\left(\frac{\partial\tilde{\Sigma}}{\partial(i\hat{p})}\right)\right]\Bigg|_{\hat{p}=im_0} = \left[\tilde{A} + 2p^2\dot{\tilde{A}} + 2m_0^2\dot{\tilde{B}}\right]\Big|_{p^2=-m_0^2} \equiv \Sigma'(m_0,\ldots)$$

where $\dot{X}(p^2,\ldots)$ denotes the derivative of $X(p^2,\ldots)$ with respect to p^2.

In this way we need to evaluate propagator type diagrams and their derivatives at $p^2 = m^2$

What is the interpretation of the complex mass

$$\tilde{M} \equiv M' - \frac{i}{2}\Gamma'$$

M. C. Smith and S. S. Willenbrock, Phys. Rev. Lett. 1997

We define the pole mass M and the on–shell width Γ as in the bosonic case by (look at $|T|^2$)

$$\tilde{M}^2 = M^2 - iM\Gamma = M'^2 - \Gamma'^2/4 - iM'\Gamma'$$

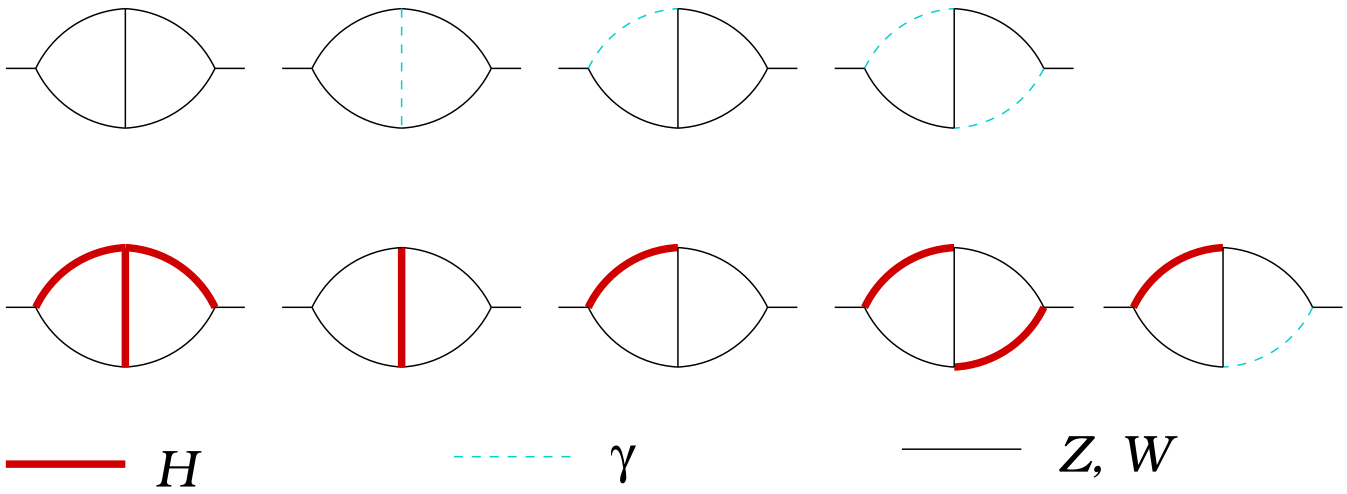
such that

$$M = \sqrt{M'^2 - \Gamma'^2/4} \;\; ; \quad \Gamma = \frac{M'}{M}\Gamma'$$

Since $M = M' + O(\alpha^2)$ and $\Gamma = \Gamma' + O(\alpha^2)$ for the $O(\alpha\alpha_s)$ terms considered in this paper we can identify $M = M'$ and $\Gamma = \Gamma'$ in the following.

③ Diagrams and topologies

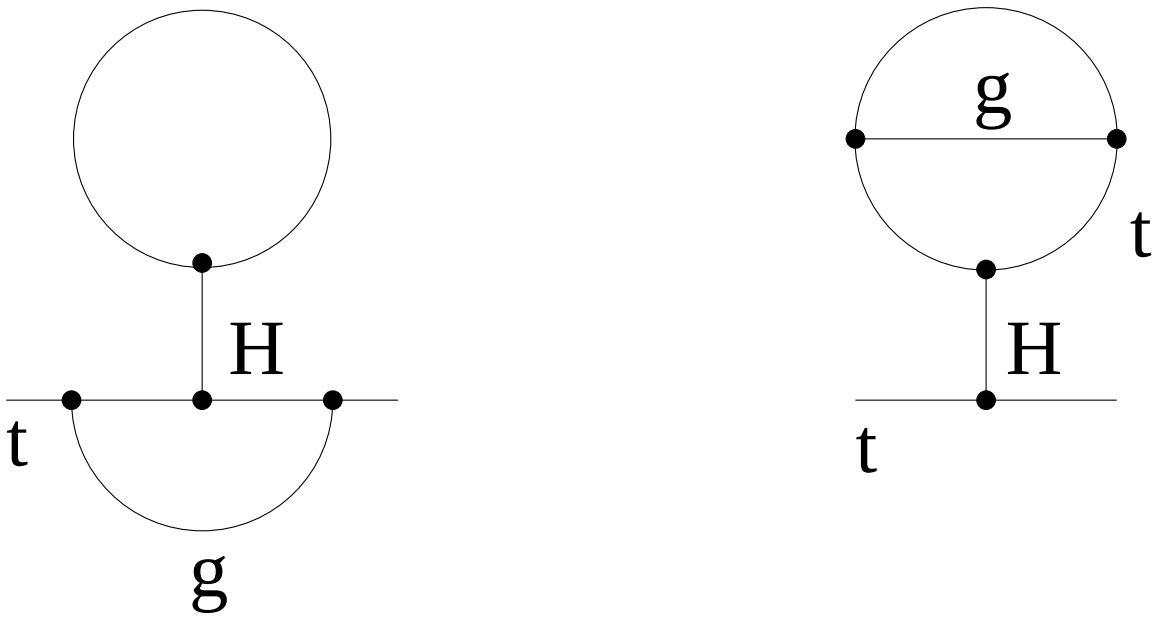
To be computed on-shell ($p^2 = m_t^2$):



The two-loop one-particle irreducible diagrams contributing to the pole mass of a quark.

ϕ_0 is the neutral pseudo-Goldstone boson and ϕ is charge pseudo-Goldstone boson.

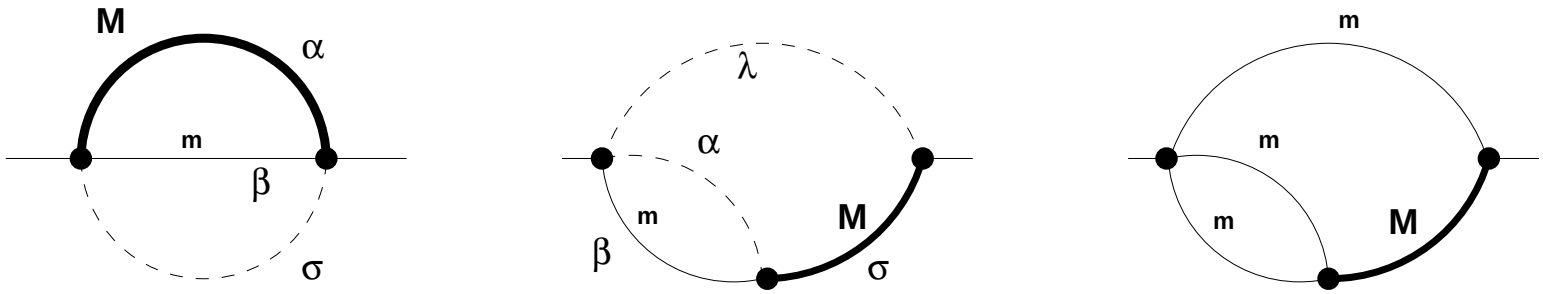
Tadpoles



The two-loop tadpole diagrams to be included for gauge and renormalization group invariance.

④ Reduction to a set of master-integrals

In order to check gauge invariance we perform all calculations in the R_ξ gauge with three independent gauge parameters ξ_W, ξ_Z, ξ_γ. $\rightarrow$ Using Tarasov's recurrence relations we reduce all diagrams to a minimal set of master-integrals



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New master diagrams appearing in this two-loop calculation. Bold, thin and dashed lines correspond to off-shell massive, on-shell massive and to massless propagator, respectively.

Analytic result

$$\{\Sigma_2+\Sigma_1\Sigma_1'\}_{\overline{\rm MS}} = \lim_{\epsilon\to 0}\Bigg(\Sigma_{2,0}+\Sigma_{1,0}\Sigma_{1,0}'+\frac{\alpha_s}{4\pi}\frac{e^2}{16\pi^2\sin^2\theta_W}[\frac{1}{\epsilon}Z^{(2,1)}_{\alpha\alpha_s}+\frac{1}{\epsilon^2}Z^{(2,2)}_{\alpha\alpha_s}]$$

$$+\frac{\alpha_s}{4\pi}\frac{e^2}{16\pi^2\sin^2\theta_W}\frac{1}{\epsilon}\{Z^{(1,1)}_{\alpha}[1+2m_t^2\frac{\partial}{\partial m^2_{t,0}}]X^{(1)}_{\alpha_s,0}+Z^{(1,1)}_{\alpha_s}[1+2m_t^2\frac{\partial}{\partial m^2_{t,0}}]X^{(1)}_{\alpha,0}\}\Bigg)$$

$$=\frac{\alpha_s}{4\pi}\frac{e^2}{16\pi^2\sin^2\theta_W}(C^{(2,2)}_{\alpha\alpha_s}\ln^2\frac{m_t^2}{\mu^2}+C^{(2,1)}_{\alpha\alpha_s}\ln\frac{m_t^2}{\mu^2}$$

$$+\frac{1}{8}\ln\left(1-\frac{1}{\omega_t}\right)(1-\omega_t)\frac{(18\omega_t^2+21\omega_t+17)}{\omega_t}+\ln\omega_t[\frac{22\omega_t+17}{8\omega_t}]$$

$$+(1-\omega_t)\frac{(1+2\omega_t)(2+\omega_t)}{2\omega_t}\ln\omega_t\ln\left(1-\frac{1}{\omega_t}\right)+\frac{1+\omega_t-\omega_t^2}{2\omega_t}\ln^2\omega_t$$

$$-(1-\omega_t)^2\frac{4\omega_t+5}{8\omega_t}\ln^2\left(1-\frac{1}{\omega_t}\right)+\frac{(1+\omega_t)}{4\omega_t}(4\omega_t^2+7\omega_t-9)\mathtt{Li}_2\left(\frac{1}{\omega_t}\right)$$

$$-\frac{1}{\omega_t}(1-\omega_t)^2(1+2\omega_t)\{\frac{3}{2}\mathtt{S}_{1,2}\left(\frac{1}{\omega_t}\right)-\frac{3}{2}\mathtt{Li}_3\left(\frac{1}{\omega_t}\right)-\ln\omega_t\mathtt{Li}_2\left(\frac{1}{\omega_t}\right)$$

$$+\frac{1}{2}\ln\left(1-\frac{1}{\omega_t}\right)\mathtt{Li}_2\left(\frac{1}{\omega_t}\right)+\frac{1}{4}\ln\left(1-\frac{1}{\omega_t}\right)[\ln^2\omega_t+6\zeta_2]\}$$

$$+\frac{(1+y_Z)^2(1+y_Z^2)(17+41y_Z+17y_Z^2)}{18\omega_t y_Z^3}\{3(2\mathtt{Li}_3(y_Z)+\mathtt{Li}_3(-y_Z))-3\zeta_2\ln(1+y_Z)$$

$$-2\ln y_Z(2\mathtt{Li}_2(y_Z)+\mathtt{Li}_2(-y_Z))-\ln^2 y_Z(\ln(1-y_Z)+\frac{1}{2}\ln(1+y_Z))\}$$

$$+\frac{(1-y_Z)(1+y_Z)^3(17+41y_Z+17y_Z^2)}{18\omega_t y_Z^3}\times$$

$$\{2\ln y_Z(\ln(1-y_Z)+\frac{1}{2}\ln(1+y_Z))+2\mathtt{Li}_2(y_Z)+\mathtt{Li}_2(-y_Z)\}$$

$$-\frac{4}{9}\frac{(1+y_Z)}{y_Z^2}\left(5-4\frac{\omega_t y_Z}{(1+y_Z)^2}\right)\times$$

$$\{\frac{(1+y_Z^2)(1+4y_Z+y_Z^2)}{(1+y_Z)}[3(2\mathtt{Li}_3(y_Z)+\mathtt{Li}_3(-y_Z))-3\zeta_2\ln(1+y_Z)$$

$$-2\ln y_Z(2\mathtt{Li}_2(y_Z)+\mathtt{Li}_2(-y_Z))-\ln^2 y_Z(\ln(1-y_Z)+\frac{1}{2}\ln(1+y_Z))]$$

$$+(1-y_Z)(1+4y_Z+y_Z^2)[2\ln y_Z(\ln(1-y_Z)+\frac{1}{2}\ln(1+y_Z))+2\mathtt{Li}_2(y_Z)]$$

$$+\frac{9}{4}(1-y_Z)^2(1+y_Z)\ln(1+y_Z)+\frac{\left(1+2y_Z-24y_Z^2+2y_Z^3+y_Z^4\right)}{(1-y_Z)}\mathtt{Li}_2(-y_Z)$$

$$+\frac{3}{4}\frac{y_Z(1+3y_Z)(4-y_Z+y_Z^2)}{(1-y_Z)}\ln y_Z-\frac{1}{4}\frac{y_Z(2+9y_Z+3y_Z^2+16y_Z^3+6y_Z^4)}{(1+y_Z)(1-y_Z)}\ln^2 y_Z\}$$

$$-\frac{(1+y_Z)^3(9+32y_Z+9y_Z^2)}{4\omega_t(1-y_Z)y_Z^2}\mathtt{Li}_2(-y_Z)+\frac{447}{16}+\frac{125}{9}\frac{(1+y_Z^2)}{y_Z}$$

$$+\frac{32}{3}[1-\omega_t\frac{y_Z}{(1+y_Z)^2}]\{\zeta_3-4\zeta_2\ln 2\}+\frac{(1+y_Z)^4(17y_Z^2-19y_Z+17)}{8\omega_t y_Z^3}\ln(1+y_Z)$$

$$-\frac{1}{\omega_t}\ln^2 y_Z\{-\frac{685}{36}+\frac{17}{36y_Z^2}+\frac{67}{24y_Z}-\frac{335}{24}y_Z-\frac{497}{72}y_Z^2-\frac{17}{12}y_Z^3+\frac{25}{1-y_Z}\}$$

$$+\frac{(1+y_Z)}{24\omega_t y_Z^2(1-y_Z)}[51y_Z^5+113y_Z^4+134y_Z^3+237y_Z^2+197y_Z+68]\ln y_Z$$

$$-\frac{1}{3}\zeta_2\{\frac{20-39y_Z-y_Z^2-80y_Z^3-20y_Z^4}{y_Z(1-y_Z)}-\omega_t\frac{7-49y_Z+17y_Z^2-55y_Z^3-16y_Z^4)}{(1-y_Z)(1+y_Z)^2}\}$$

$$-\frac{1}{\omega_t}\{\frac{4157}{72}+\frac{425(1+y_Z^4)}{72y_Z^2}+\frac{2561(1+y_Z^2)}{96y_Z}\}$$

$$+\frac{1}{\omega_t}\zeta_2\{\frac{187}{3}+\frac{17}{6y_Z^2}+\frac{133}{12y_Z}+\frac{535}{12}y_Z+\frac{211}{12}y_Z^2+\frac{17}{6}y_Z^3-\frac{50}{1-y_Z}\}$$

$$-3\omega_t\ln\omega_t\frac{(1+y_H+y_H^2)}{(1+y_H)^2}+\frac{3}{2\omega_t}\frac{y_H}{(1+y_H)^2}\frac{(1+y_Z)^4}{y_Z^2}\ln\frac{(1+y_Z)^2}{y_Z}$$

$$-\frac{1}{\omega_t}\frac{y_H}{(1+y_H)^2}\{\frac{11(1+y_H^2)(1+y_H)^2}{8y_H^2}+8N_c+\frac{1}{2}\frac{(1+y_Z)^4}{y_Z^2}\}$$

$$+\frac{1}{\omega_t}\zeta_2\{\frac{3}{2y_H}+\frac{9}{2}y_H+\frac{3}{4}y_H^2\}-\frac{\omega_t}{36}\frac{(625+1286y_H+625y_H^2)}{(1+y_H)^2}$$

$$+\frac{1}{\omega_t}\frac{(1-y_H)^2}{y_H^2}\ln y_H[\ln(1-y_H)+\frac{1}{2}\ln(1+y_H)][(1-y_H^2)-\frac{1}{2}(1+y_H^2)\ln y_H]$$

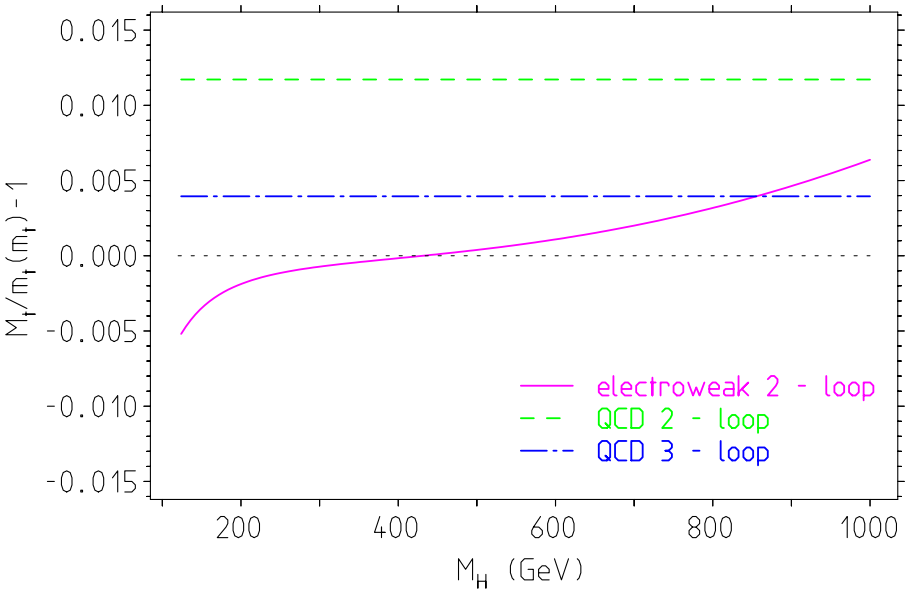
$$
\begin{aligned}
&-\frac{1}{8}\frac{1}{\omega_t}\frac{2+8y_H-10y_H^2-3y_H^3}{y_H}\ln^2 y_H+\frac{1}{8}\frac{1}{\omega_t}\frac{(1+y_H)(6-63y_H+5y_H^2)}{y_H}\ln y_H\\
&-\frac{1}{8}\frac{1}{\omega_t}\frac{(1+y_H)^2(5-62y_H+5y_H^2)}{y_H^2}\ln(1+y_H)-\frac{3}{2}\frac{1}{\omega_t}\zeta_2\ln(1+y_H)\frac{(1-y_H)^2(1+y_H^2)}{y_H^2}\\
&+\frac{1}{\omega_t}\frac{(1-y_H)(1+y_H)}{y_H^2}\{\frac{(5-28y_H+5y_H^2)}{4}\mathtt{Li}_2(-y_H)+(1-y_H)^2\mathtt{Li}_2(y_H)\}\\
&+\frac{1}{\omega_t}\frac{(1-y_H)^2(1+y_H^2)}{y_H^2}\{\frac{3}{2}[2\mathtt{Li}_3(y_H)+\mathtt{Li}_3(-y_H)]-\ln y_H[2\mathtt{Li}_2(y_H)+\mathtt{Li}_2(-y_H)]\})\ ,
\end{aligned}
$$

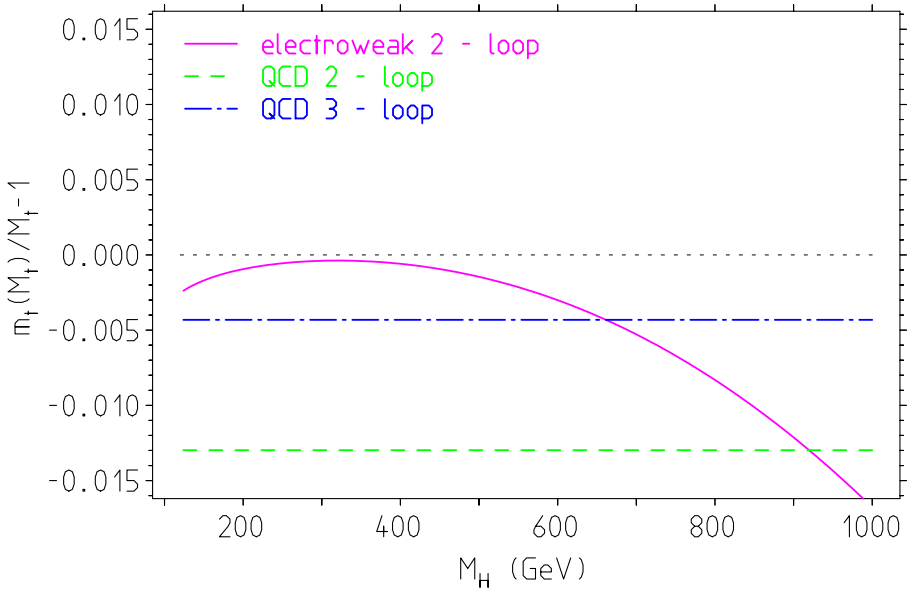
where

$$
\begin{aligned}
\omega_t &= \frac{m_W^2}{m_t^2}\ ,\\
y_A &= \frac{1-\sqrt{1-\frac{4m_t^2}{m_A^2}}}{1+\sqrt{1-\frac{4m_t^2}{m_A^2}}}\ ,A=H,Z\ \ .
\end{aligned}
$$

In limit of zero mass gauge bosons $O(\alpha\alpha_s)$ and $O(\alpha^2)$: Faisst, Kühn, Seidensticker, Veretin 03

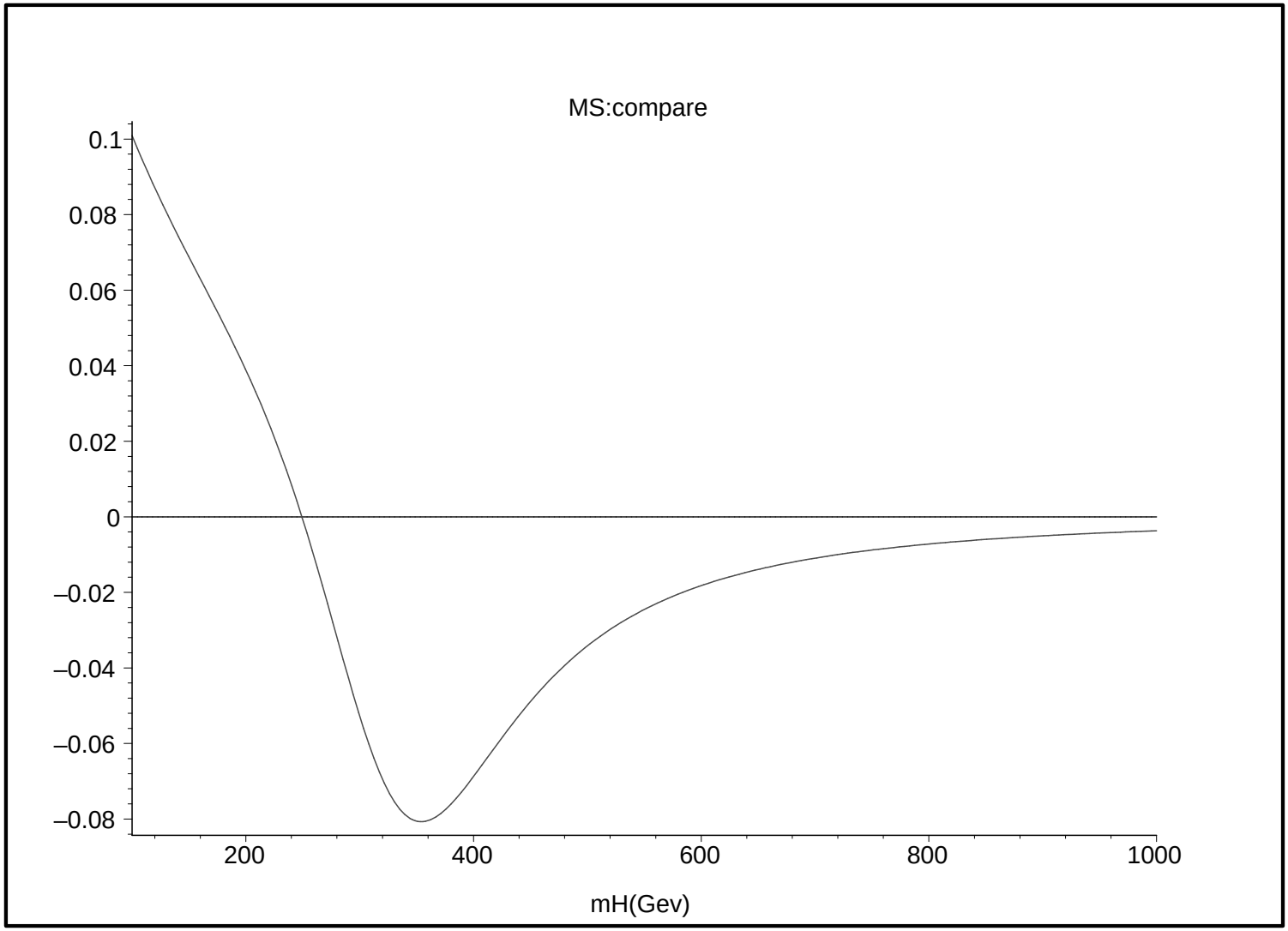
⑤ Numerical illustration



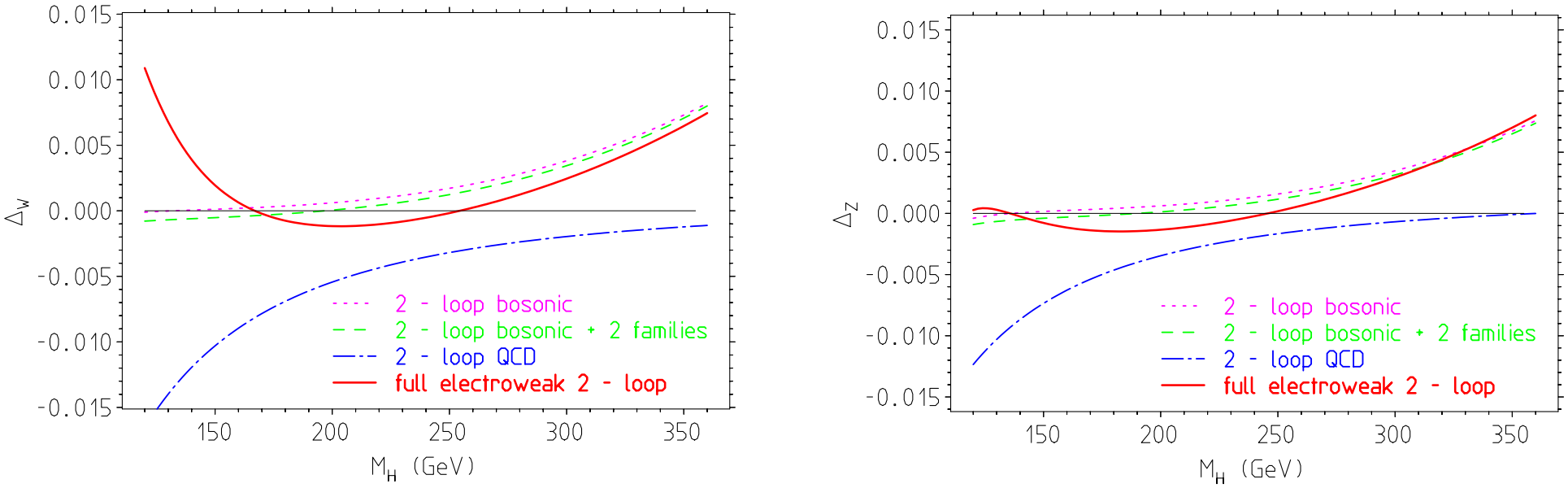


Electroweak $O(\alpha\alpha_s)$ correction to $M_t/m_t(m_t)-1$ [left] and $m_t(M_t)/M_t-1$ [right], in comparison with $O(\alpha_s^2)$ and $O(\alpha_s^3)$ QCD corrections as a function of the Higgs boson mass M_H.

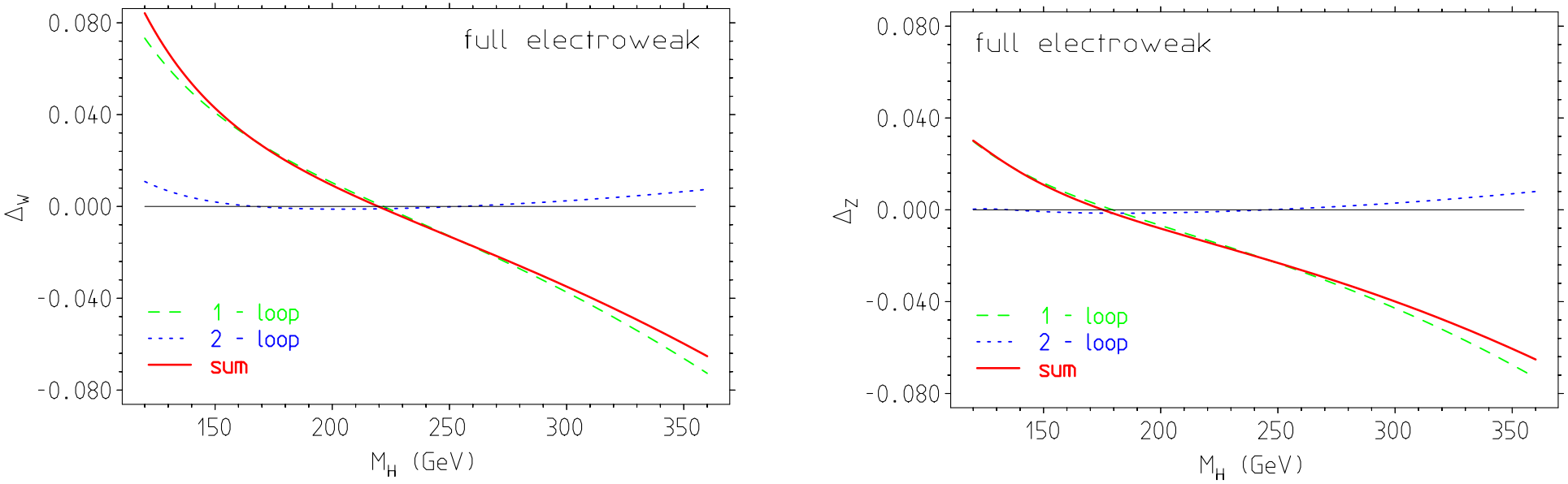
Gaugeless limit result (FKSV03) vs. full result



Electroweak $O(\alpha\alpha_s)$ correction gaugeless vs. full correction.



Various two-loop corrections to the relation $\Delta_V \equiv M_V^2/m_V^2(M_V) - 1$ as a function of the Higgs mass M_H for intermediate Higgs masses.



The complete one- and two-loop correction to the relation $\Delta_V \equiv M_V^2/m_V^2(M_V) - 1$ as a function of the Higgs mass M_H for intermediate Higgs masses.

Numerical comments

For the two-loop calculation we have to take into account the part proportional to ε of the one-loop propagator type integral

$$J = \int \frac{\mathrm{d}^d q}{\left(q^2 - m_1^2 + \mathrm{i}0\right)\left((k-q)^2 - m_2^2 + \mathrm{i}0\right)} ,$$

where $d = 4 - 2\varepsilon$. Part of J linear in ε is

$$J = \mathrm{i}\pi^{2-\varepsilon} \frac{\Gamma(1+\varepsilon)}{2(1-2\varepsilon)} \left\{ \frac{m_1^{-2\varepsilon} + m_2^{-2\varepsilon}}{\varepsilon} + \frac{m_1^2 - m_2^2}{\varepsilon\, k^2} \left(m_1^{-2\varepsilon} - m_2^{-2\varepsilon}\right) \right.$$

$$-2 \frac{\sqrt{-\lambda(m_1^2, m_2^2, k^2)}}{k^2} \Bigg[$$

$$\arccos\left(\frac{m_1^2 + m_2^2 - k^2}{2 m_1 m_2}\right) \left(1 - \varepsilon \ln\left(\frac{-\lambda(m_1^2, m_2^2, k^2)}{k^2}\right)\right)$$

$$\left. +2\varepsilon \Big(\mathrm{Cl}_2(\tau_1) - \mathrm{Cl}_2(\pi - \tau_1) + \mathrm{Cl}_2(\tau_2) - \mathrm{Cl}_2(\pi - \tau_2) \Big) \Bigg] + \mathcal{O}(\varepsilon^2) \right\},$$

where $\lambda(m_1^2, m_2^2, k^2) = m_1^4 + m_2^4 + k^4 - 2m_1^2 k^2 - 2m_2^2 k^2 - 2m_1^2 m_2^2$ and the angles τ_i are defined via

$$\cos\tau_1 = \frac{k^2 - m_1^2 + m_2^2}{2 m_2 \sqrt{k^2}} , \quad \cos\tau_2 = \frac{k^2 + m_1^2 - m_2^2}{2 m_1 \sqrt{k^2}} .$$

$\mathrm{Cl}_2(\theta)$ is the Clausen function $\mathrm{Cl}_2(\theta) = \frac{1}{2\mathrm{i}} \left[\mathrm{Li}_2\left(e^{\mathrm{i}\theta}\right) - \mathrm{Li}_2\left(e^{-\mathrm{i}\theta}\right)\right]$. This expansion is directly applicable in the region where $\lambda \le 0$, i.e. when $(m_1 - m_2)^2 \le k^2 \le (m_1 + m_2)^2$.

For the region $\lambda > 0$ we need the proper analytic continuation.

$$J = \mathrm{i}\pi^{2-\varepsilon}\,\frac{\Gamma(1+\varepsilon)}{2(1-2\varepsilon)}\left(\frac{m_1^{-2\varepsilon}+m_2^{-2\varepsilon}}{\varepsilon}+\frac{m_1^2-m_2^2}{\varepsilon k^2}\left(m_1^{-2\varepsilon}-m_2^{-2\varepsilon}\right)\right.$$
$$-\mathrm{i}\frac{\left[-\lambda(m_1^2,m_2^2,k^2)\right]^{1/2-\varepsilon}}{(k^2)^{1-\varepsilon}}\left\{\frac{2\Gamma^2(1-\varepsilon)}{\varepsilon\Gamma(1-2\varepsilon)}\,(1-\rho_1-\rho_2)\right.$$
$$+\frac{1}{\varepsilon}\sum_{i=1}^{2}\left(\rho_i(-z_i)^{-\varepsilon}-(1-\rho_i)(-z_i)^{\varepsilon}\right)$$
$$\left.\left.+2\varepsilon\sum_{i=1}^{2}\left(\rho_i(-z_i)^{-\varepsilon}\mathtt{Li}_2(z_i)-(1-\rho_i)(-z_i)^{\varepsilon}\mathtt{Li}_2(1/z_i)+\mathcal{O}(\varepsilon)\right)\right\}\right),$$

where ρ_1 and ρ_2 are some numbers, which we will define later and

$$z_1 = \frac{\left[\sqrt{\lambda(m_1^2,m_2^2,k^2)}+m_1^2-m_2^2-k^2\right]^2}{4m_2^2k^2},$$
$$z_2 = \frac{\left[\sqrt{\lambda(m_1^2,m_2^2,k^2)}-m_1^2+m_2^2-k^2\right]^2}{4m_1^2k^2}.$$

Firstly, we note that the causal prescription amounts to the following rule for λ ($\lambda > 0$)

$$\ln(-\lambda(m_1^2,m_2^2,k^2)) = \ln(\lambda(m_1^2,m_2^2,k^2))-\mathrm{i}\pi,$$
$$\sqrt{-\lambda(m_1^2,m_2^2,k^2)} = -\mathrm{i}\sqrt{\lambda(m_1^2,m_2^2,k^2)}.$$

The function $\mathtt{Li}_2(z)$ is real for real z and $|z| \leq 1$. For real z and $|z| > 1$ we change argument $z \to 1/z$ using the relation

$$\mathtt{Li}_2(z) + \mathtt{Li}_2\left(\frac{1}{z}\right) = -\frac{1}{2}\ln^2(-z) - \zeta_2,$$

by which an imaginary part shows up. This change of variables can be done from the very beginning by an appropriate choice of the values of the coefficients ρ_j:

$$\begin{aligned} 0 < z_j < 1 \quad &\Rightarrow \quad \rho_j = 1; \quad \ln(-z_j) = \ln(z_j) + \mathrm{i}\pi, \\ z_j > 1 \quad &\Rightarrow \quad \rho_j = 0; \quad \ln(-z_j) = \ln(z_j) - \mathrm{i}\pi. \end{aligned}$$

Assuming $m_1 < m_2$ in the following we have

- for $k^2 < (m_1 - m_2)^2 \Rightarrow z_1 < 1, z_2 > 1$

$$J = \mathrm{i}\pi^{2-\varepsilon}\,\frac{\Gamma(1+\varepsilon)}{2(1-2\varepsilon)}\left(\frac{m_1^{-2\varepsilon}+m_2^{-2\varepsilon}}{\varepsilon} + \frac{m_1^2-m_2^2}{\varepsilon k^2}\left(m_1^{-2\varepsilon}-m_2^{-2\varepsilon}\right)\right.$$
$$+\frac{\sqrt{\lambda(m_1^2,m_2^2,k^2)}}{k^2}\left\{\ln(z_1z_2)+\varepsilon\left[2\mathtt{Li}_2\left(\frac{1}{z_2}\right)-2\mathtt{Li}_2(z_1)-\frac{1}{2}\ln^2 z_1+\frac{1}{2}\ln^2 z_2\right.\right.$$
$$\left.\left.\left.-\ln(z_1z_2)\ln\left(\frac{\lambda(m_1^2,m_2^2,k^2)}{k^2}\right)+\mathcal{O}(\varepsilon^2)\right]\right\}\right)$$

- for $k^2 > (m_1 + m_2)^2 \Rightarrow z_1 < 1, z_2 < 1$

$$J = \mathrm{i}\pi^{2-\varepsilon}\,\frac{\Gamma(1+\varepsilon)}{2(1-2\varepsilon)}\left(\frac{m_1^{-2\varepsilon}+m_2^{-2\varepsilon}}{\varepsilon} + \frac{m_1^2-m_2^2}{\varepsilon k^2}\left(m_1^{-2\varepsilon}-m_2^{-2\varepsilon}\right)\right.$$
$$+\frac{\sqrt{\lambda(m_1^2,m_2^2,k^2)}}{k^2}\left\{\ln(z_1z_2)+\varepsilon\left[-8\zeta_2-2\mathtt{Li}_2(z_1)-2\mathtt{Li}_2(z_2)\right.\right.$$
$$\left.-\frac{1}{2}\ln^2 z_1-\frac{1}{2}\ln^2 z_2-\ln(z_1z_2)\ln\left(\frac{\lambda(m_1^2,m_2^2,k^2)}{k^2}\right)\right]$$
$$\left.\left.+\mathrm{i}\pi\left[2-2\varepsilon\ln\left(\frac{\lambda(m_1^2,m_2^2,k^2)}{k^2}\right)\right]+\mathcal{O}(\varepsilon^2)\right\}\right).$$

In particular, the imaginary part of J in each order of ε coincides with that obtained from the exact result

$$\mathrm{Im}J = \mathrm{i}\pi\theta\left(k^2-(m_1+m_2)^2\right)\frac{\sqrt{\lambda(m_1^2,m_2^2,k^2)}}{k^2}\left(\frac{\lambda(m_1^2,m_2^2,k^2)}{k^2}\right)^{-\varepsilon}\frac{\Gamma(1-\varepsilon)}{\Gamma(2-2\varepsilon)}.$$

In the limit, when one of the masses vanishes, the result is

$$J|_{m_1=0,\,m_2\equiv m} = \mathrm{i}\pi^{2-\varepsilon}m^{-2\varepsilon}\frac{\Gamma(1+\varepsilon)}{(1-2\varepsilon)}\left\{\frac{1}{\varepsilon}\right.$$

$$\left.-\frac{1-u}{2u\varepsilon}\left[(1-u)^{-2\varepsilon}-1\right]-\frac{(1-u)^{1-2\varepsilon}}{u}\varepsilon\mathtt{Li}_2(u)+\mathcal{O}(\varepsilon^2)\right\},$$

with $u=k^2/m^2$.

The transition from the bare parameters to the renormalized ones requires differentiations of the one-loop propagators with respect to all parameters, couplings, masses and external momentum. The integrals obtained thereby can be reduces again to integrals of the original type J plus simpler bubble integrals.

Quite universal representation follows from the one-fold integral representation: in normalization $\frac{1}{\mathrm{i}\pi^{2-\varepsilon}\Gamma(1+\varepsilon)}$

$$B_0(m_1^2, m_2^2; p^2) = \frac{1}{\varepsilon}\int_0^1 dx \frac{(\mu^2)^\varepsilon}{\left[m_1^2 x + m_2^2(1-x) - p^2 x(1-x) - \mathrm{i}0\right]^\varepsilon}$$

On-shell condition is $p^2 = M^2$. So that finite part F and linear in ε parts L are

$$F_0(m_1^2, m_2^2; p^2) = -\int_0^1 dx \ln\left(\frac{m_1^2}{\mu^2}x + \frac{m_2^2}{\mu^2}(1-x) - \frac{p^2}{\mu^2}x(1-x) - \mathrm{i}0\right)$$

$$L_0(m_1^2, m_2^2; p^2) = \frac{1}{2}\int_0^1 dx \ln^2\left(\frac{m_1^2}{\mu^2}x + \frac{m_2^2}{\mu^2}(1-x) - \frac{p^2}{\mu^2}x(1-x) - \mathrm{i}0\right)$$

To get numerically stable results it is necessary to work on MAPLE with sufficiently high accuracy (our experience: we get an accuracy of 40 decimals) (wenn calculating with 100 decimals). The $''\mathrm{i}0''$ causal prescription is introduced in program as small number 10^{-80}.

Two-loop bubble type diagram.

Another type of integral appeared in the diagrams with Higgs and top-quark is the finite part of two-loop bubble master integrals $\Phi(z)$ with two-different non-zero mass scales. The standard representation for this integral is

$$\Phi(z) = \begin{cases} 4\sqrt{\frac{z}{1-z}}\,\mathrm{Cl}_2\left(2\arcsin\sqrt{z}\right) \quad , & z < 1 \\ \frac{1}{\lambda}\left[-4\mathrm{Li}_2\left(\frac{1-\lambda}{2}\right) + 2\ln^2\left(\frac{1-\lambda}{2}\right) - \ln^2(4z) + 2\zeta_2\right] , & z > 1 \end{cases}$$

where

$$\lambda = \sqrt{1 - \frac{1}{z}}.$$

This representation is not stable numerically at $z \sim 1$.

The universal representation, stable for any value of z is

$$\Phi(z) = 2\frac{1-\eta}{1+\eta}\left[\mathrm{Li}_2(\eta) - \mathrm{Li}_2\left(\frac{1}{\eta}\right)\right] , \quad \eta = \frac{1-\sqrt{\frac{z}{z-1}}}{1+\sqrt{\frac{z}{z-1}}}.$$

Connection with RG functions of unbroken phase

In the SM it is interesting to compare the RG equations calculated in broken phase with the ones obtained in the unbroken phase. Let us remind that at the tree-level the vacuum expectation value v^2 is given by $v^2 \equiv \frac{m^2}{\lambda}$, where m^2 and λ are the parameters of the symmetric scalar potential

$$V = -m^2 \phi^+ \phi + \lambda \left(\phi^+ \phi\right)^2$$

$$\gamma_W - 2\frac{\beta_g}{g} = \gamma_{m^2} - \frac{\beta_\lambda}{\lambda} \ ,$$

$$\gamma_Z - \gamma_{m^2} + \frac{\beta_\lambda}{\lambda} = 2\left(\cos^2\theta_W \frac{\beta_g}{g} + \sin^2\theta_W \frac{\beta_{g'}}{g'}\right) ,$$

where the 2-loop RG functions $\beta_g, \beta_{g'}, \beta_\lambda, \gamma_{m^2}$ have been calculated in the unbroken phase

D.R.T. Jones '82

M.E. Machacek & M.T. Vaughn '83

C. Ford, I. Jack and D.R. Jones, '92

We have verified in the $\overline{MS}$ scheme, that these relations are valid up to 2-loop order in the broken phase with the

same RG functions. Thus the RG equations for the $\overline{MS}$ masses in the broken theory can be written

$$m_W^2(\mu^2) = \frac{1}{4}\frac{g^2(\mu^2)}{\lambda(\mu^2)} m^2(\mu^2) \ ,$$

$$m_Z^2(\mu^2) = \frac{1}{4}\frac{g^2(\mu^2)+g'(\mu^2)}{\lambda(\mu^2)} m^2(\mu^2) \ ,$$

$$m_H^2(\mu^2) = 2m^2(\mu^2) \ ,$$

$$m_t^2(\mu^2) = \frac{1}{2}\frac{y_t^2(\mu^2)}{\lambda(\mu^2)} m^2(\mu^2) \ .$$

Together with

Charge renormalization:

$$\frac{\delta e}{e} = Z_e^{-1} = \left\{ (Z_{\gamma\gamma})^{1/2} + \frac{s_W^0}{c_W^0}\frac{1}{2} Z_{Z\gamma} \right\}.$$

$\Rightarrow$ Only $\Pi_{\gamma\gamma}(0)$ and $\Pi_{Z\gamma}(0)$ needed (bubble diagrams)

(Degrassi& Vicini 03)

all renormalization counterterms in physical sector available!

⑥ First complete two–loop calculation of $2 \to 2$:

Fermi constant G_μ in terms of α M_Z and M_W (low energy expansion excellent approximation):

(Awramik&Czakon 02, Onishchenko&Veretin 02)

(Awramik, Czakon, Freitas & Weiglein 03)

m_H(GeV)	$\Delta r^{(\alpha)}$	$\Delta r^{(\alpha\alpha_s)}$	$\Delta r^{(\alpha\alpha_s^2)}$	$\Delta r^{(\alpha^2)}_{\rm ferm}$	$\Delta r^{(\alpha^2)}_{\rm bos}$	$\Delta r^{(G_F^2\alpha_s m_t^4)}$	$\Delta r^{(G_F^3 m_t^6)}$
100	283.41	35.89	7.23	28.56	0.64	−1.27	−0.16
200	307.35	35.89	7.23	30.02	0.35	−2.11	−0.09
300	323.27	35.89	7.23	31.10	0.23	−2.77	−0.03
600	353.01	35.89	7.23	32.68	0.05	−4.10	−0.09
1000	376.27	35.89	7.23	32.36	−0.41	−5.04	−1.04

Table 1: The numerical values ($\times 10^4$) of the different contributions to Δr specified in the table are given for different values of m_H and $M_W = 80.426$ GeV (the W and Z masses have been transformed so as to correspond to the real part of the complex pole).

The table shows that the two-loop QCD correction, $\Delta r^{(\alpha\alpha_s)}$, and the fermionic electroweak two-loop correction, $\Delta r^{(\alpha^2)}_{\rm ferm}$ are of similar size. They both amount to about 10% of the one-loop contribution, $\Delta r^{(\alpha)}$, entering with the same sign. The most important correction beyond these contributions is the three-loop QCD correction, $\Delta r^{(\alpha\alpha_s^2)}$,

which leads to a shift in M_W of about -11 MeV. For large values of m_H also the contribution $\Delta r^{(G_F^2 \alpha_s m_t^4)}$ becomes sizable. The purely bosonic two-loop contribution, $\Delta r_{\rm bos}^{(\alpha^2)}$, and the leading electroweak three-loop correction, $\Delta r^{(G_F^3 m_t^6)}$, give rise to shifts in MW which are significantly smaller than the experimental error envisaged for a future Linear Collider, $\delta M_W^{\rm exp,LC} = 7$ MeV.

⑦ Conclusion and perspectives

SM on–shell self–energies calculated; emphasis on analytical approach as far as feasible so far

- W **and** Z $O(\alpha^2)$**:**
  - **bosonic: two–scale integrals expansion in** $sin^2\theta_W = 1 - M_W^2/M_Z^2$, $M_V^2/m_H^2 \quad (V = W, Z)$**, six expansion coefficient analytically or analytic in terms of master–integrals, which for more than one scale in general are available as 1–dimensional integral representations only which can be calculated numerically.**
  - **fermionic (two–loop diagrams incl. one fermion loop): massless fermions exact; one heavy quark (top) expansion in** $sin^2\theta_W = 1 - M_W^2/M_Z^2$, M_V^2/m_H^2 **and** $M_V^2/m_t^2 \quad (V = W, Z)$**. Six expansion coefficients in each expansion parameter or in terms of master–integrals.**
- t $O(\alpha\alpha_s)$**:**
  - **assuming diagonal CKM matrix, exact analytic for the heavy quark (**t**) with a massless quark (**b**) in the doublet.**
  - **as a byproduct: tree and** $O(\alpha_s)$ **partial decay width** $t \to bW$ **(diagrams with** $W, \phi^\pm$ **lines)**

Next steps:

- **off–shell self–energies** $\Rightarrow$ **one more scale can be expressed in terms of master–integrals which may be represented as yb 1–dimensional integral representations.**
- **the same for vertex functions**
- **goal full 2–loop** $2 \to 2$ **calculation of observables (requires also the box diagrams)**
- **e.g. for 2–loop Bhabha project Gluza, Fleischer, Riemann, Tarasov**