Quantum phase transition and Resurgence: Lessons from 3d N=4 SQED

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[arXiv: 2103.13654]

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Hiyoshi Phys. Keio U.[A], YITP[B], NCBJ[C], RECNS Keio[D], Akita->Kindai U.[E]

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Motivations

It is important to determine phase structures of QFTs

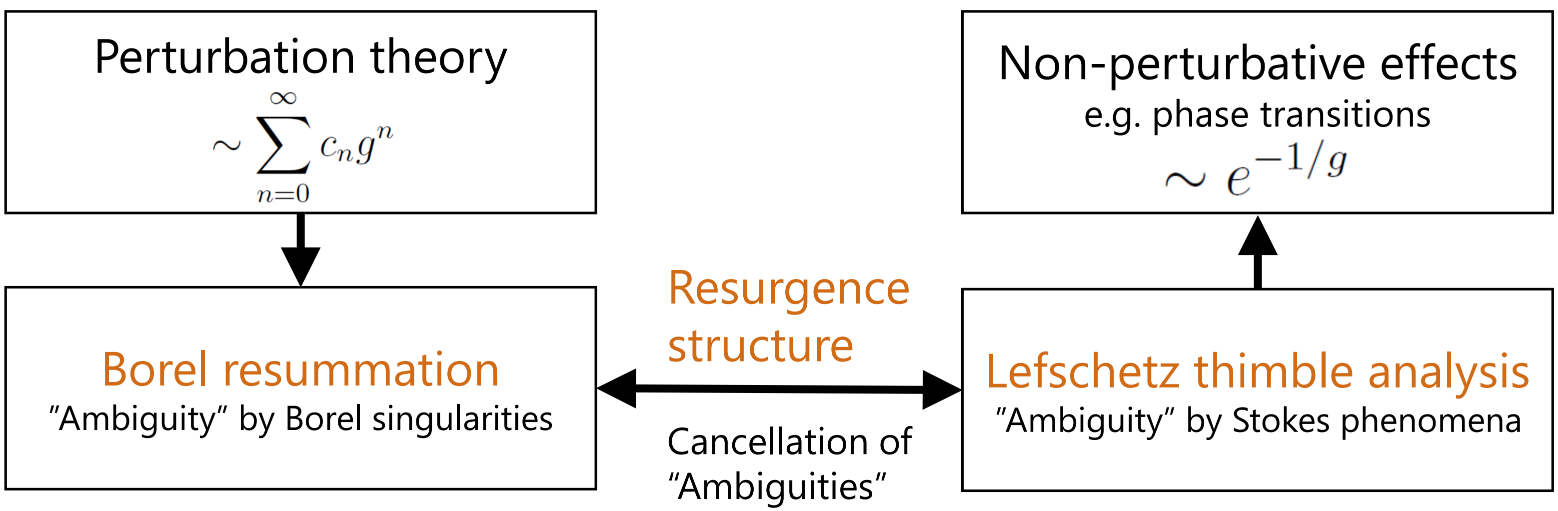
- Related to symmetry, renormalization grp, energy gap, topological order,…

[J. Ecalle, 81]

Lectures and reviews, e.g.
[M. Marino, 12]
[D. Dorigoni, 19]
[I. Aniceto, G. Basar, R. Schiappa, 19]

Description by resurgence theory

- One of the approaches to non-perturbative physics
- Decoding non-perturbative information from perturbation theory



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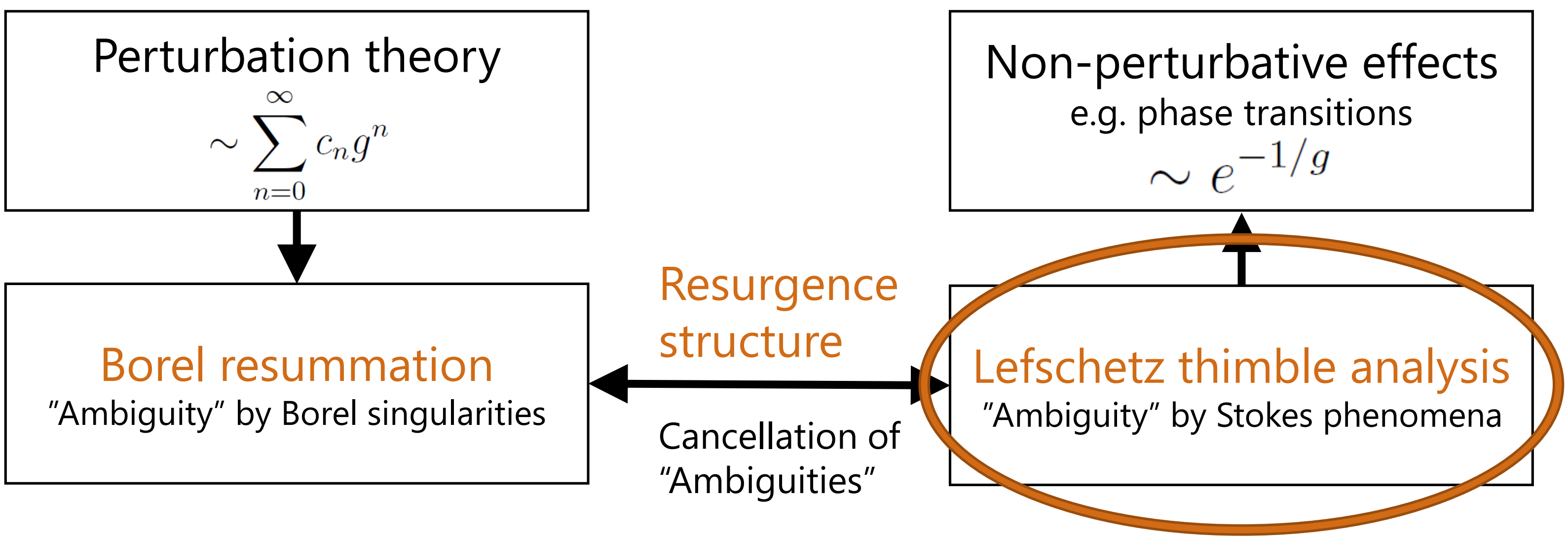

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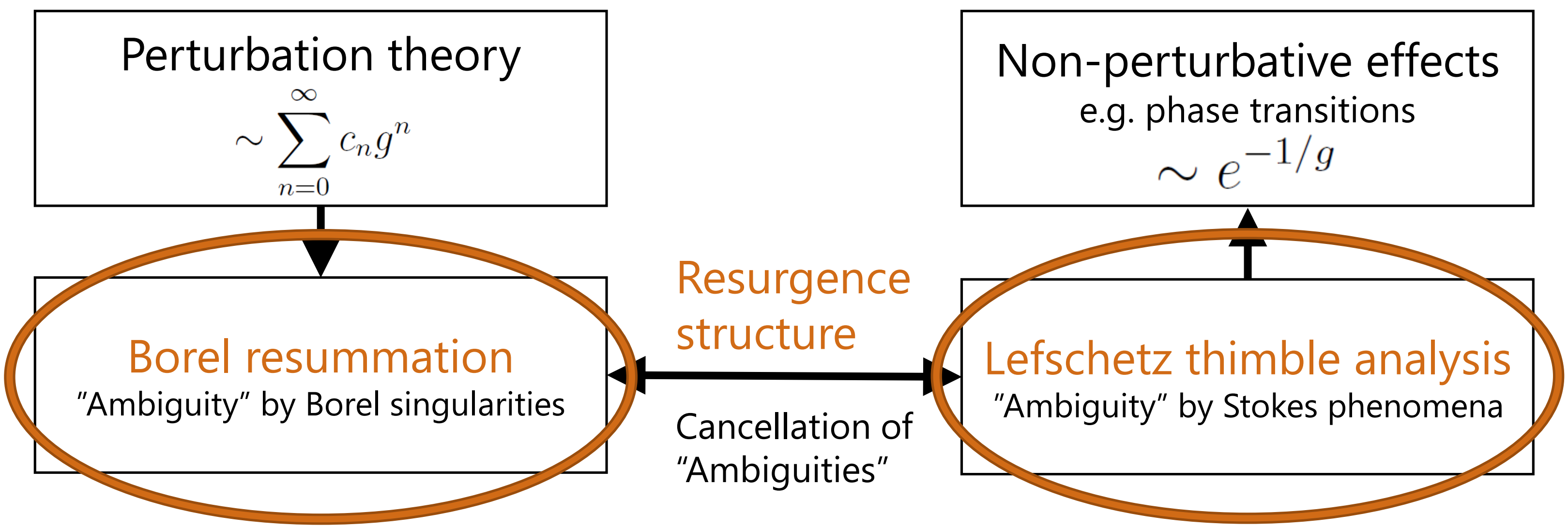
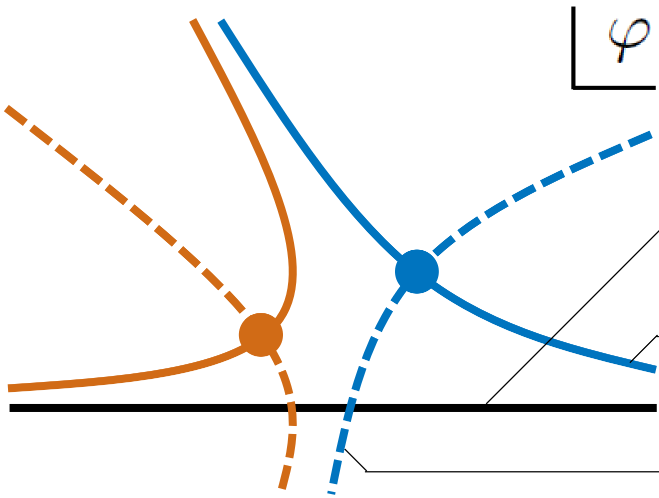


Lefschetz thimble analysis

Decomposition of path integral contour to Lefschetz thimbles

[E. Witten, 11]
[M. Cristoforetti et al., 12,13,14]
[H. Fujii et al., 13]
[G. Aarts, 13]
[A. Alexandru et al, 16]



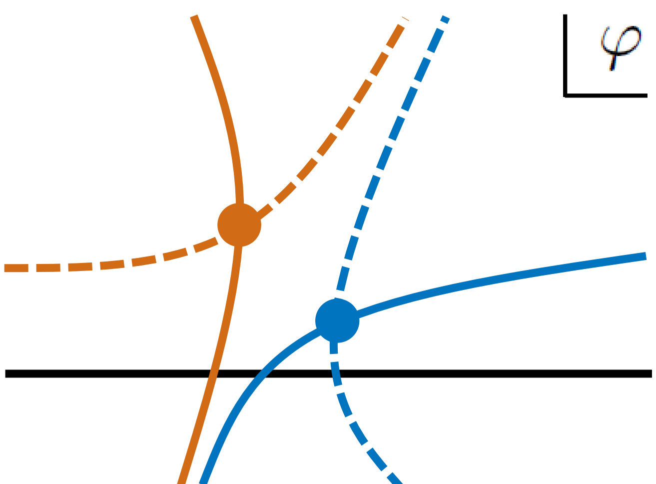
$$\left.\frac{\delta S[\varphi]}{\delta \varphi}\right|_{\varphi=\varphi_i} = 0$$

$$\mathcal{I} = \sum_i n_i \mathcal{J}_i$$

$$\mathcal{J}_i : \frac{\mathrm{d}\varphi(t)}{\mathrm{d}t} = +\overline{\frac{\delta S[\varphi]}{\delta \varphi}}, \quad \varphi(-\infty) = \varphi_i$$

$$\mathcal{K}_i : \frac{\mathrm{d}\varphi(t)}{\mathrm{d}t} = -\overline{\frac{\delta S[\varphi]}{\delta \varphi}}, \quad \varphi(-\infty) = \varphi_i$$

Changes of contributions to path integral



Stokes pheno. : Discrete jump of n_i

Anti-Stokes pheno. : Jump of $S[\varphi_i] \gtrless S[\varphi_j]$

⟷ 1st order phase transition

Borel resummation

Resuming a non-convergent formal series

[J. Ecalle, 81]

Lectures and reviews, e.g.
[M. Marino, 12]
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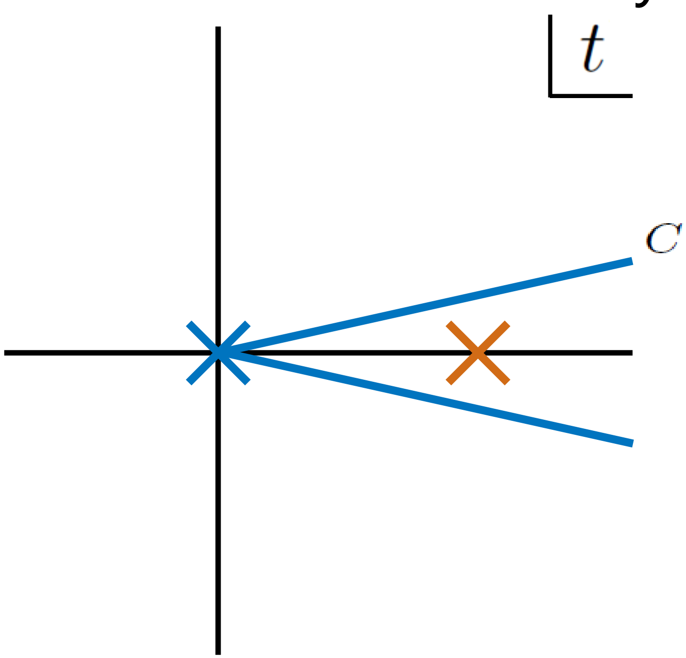
[I. Aniceto, G. Basar, R. Schiappa, 19]

$$Z(g) = \sum_{n=0}^{\infty} c_n g^{n+r}, \quad c_n \sim n!$$ Formal series

$$\mathcal{S}Z(g) = \int_C \mathrm{d}t\ e^{-t/g} \mathcal{B}Z(t)$$ Borel resummation

$$\mathcal{B}Z(t) = \sum_{n=0}^{\infty} \frac{c_n}{\Gamma(n+r)} t^{n+r-1}$$ Borel transformation

Borel transformation may have Borel singularities



Borel singularities are associated with saddles

[Lipatov, 77]

$$\mathcal{B}Z(t) \sim \sum_{n=0}^{\infty} \left(\frac{t}{S[\varphi_i]} \right)^n = \frac{1}{1 - \frac{t}{S[\varphi_i]}}$$

Resurgence

Conjecture : "Ambiguities" cancel each other in QFTs

→ We can decode information of non-perturbative effects from perturbation theory

[J. Ecalle, 81]

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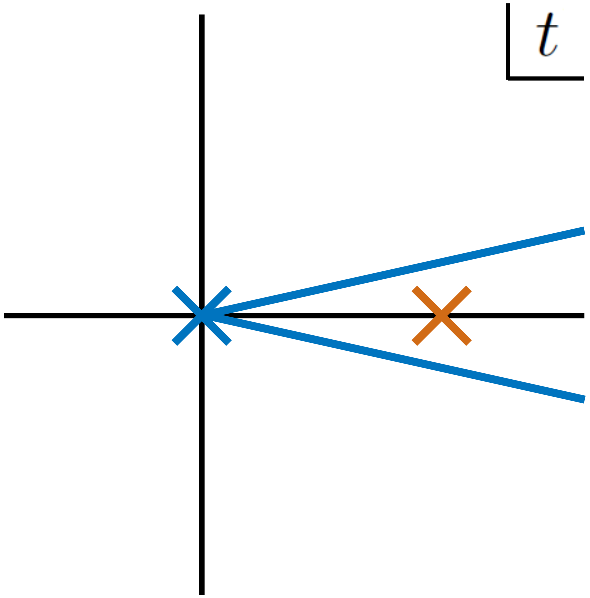
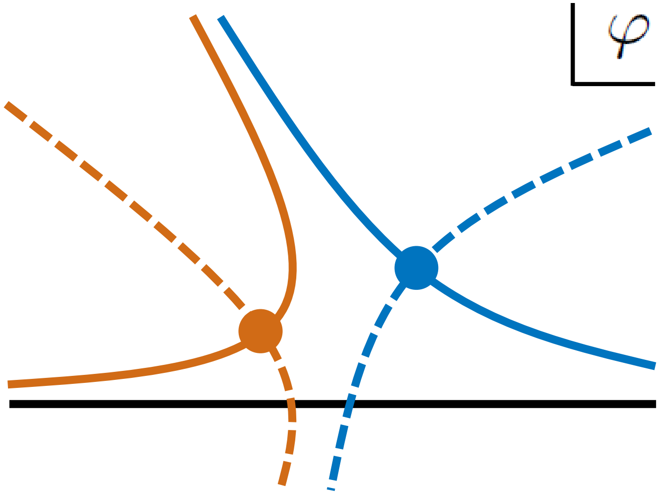
Borel resummation



"Ambiguity" of part. func. $= -ie^{-S[\varphi_i]/g}$

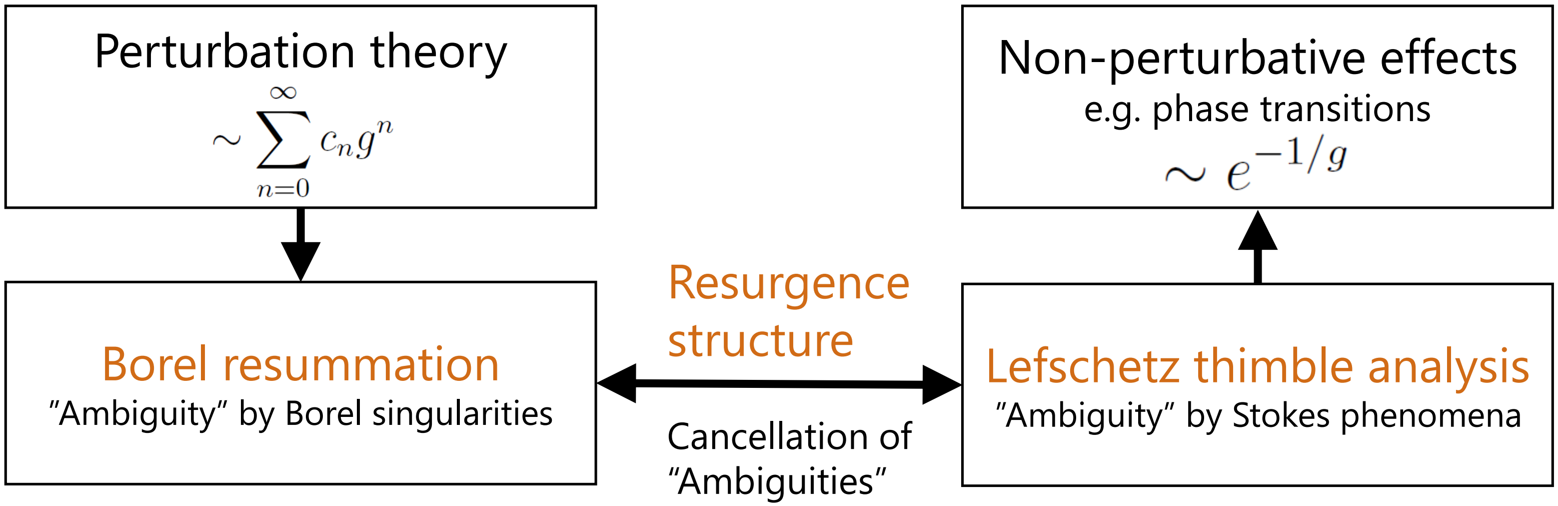
Lefschetz thimble analysis



"Ambiguity" of part. func. $= +ie^{-S[\varphi_i]/g}$

Resurgence

Scenario



Applications to phase transitions

- (Generically) 1st order phase transition is an Anti-Stokes phenomenon
- 0次元Gross-Neveu, Nambu-Jona-Lasinio like model [T. Kanazawa, Y. Tanizaki, 15]
- Massive fermion + Chern-Simons [T. Kanazawa, Y. Tanizaki, 15]
- 2dim Yang-Mills on lattice (reduced to Gross-Witten-Wadia) [G. Dunne et al., 16, 17, 18]

etc.

→ Is resurgence applicable to 2nd order phase transitions or more realistic QFTs?

Outline of our work

Model

[Russo, Tierz, 17]

- $3d$ $\mathcal{N}=4$ $U(1)$ SUSY gauge theory + $2N$ hypermultiplets (charge 1)
- Fayet-Illiopoulos parameter η, flavor mass m

→ 2nd order quantum phase transition at the large-flavor limit

Result: resurgence is applicable!

[T. Fujimori, M. Honda, S. Kamata, T. Misumi, N. Sakai, TY, 21]

TAKE-HOME MESSAGES

- Lefschetz thimble analysis
  - Two phases are distinguished by Stokes phenomena
  - The order of the phase transition is determined by "a collision of saddles."
- Borel resummation
  - Two phases are distinguished by Stokes phenomena
  - The order of the phase transition is determined by "a collision of Borel singularities."

→
- ❑ 2nd order phase trans. is the simultaneous Stokes and anti-Stokes pheno.
- ❑ The order of phase transitions can be decoded from a perturbative series
- ❑ Generalized to other systems

Contents

✓ Introduction
- Lefschetz thimble analysis
- Borel resummation
- Conclusion and future works

3dim SQED and quantum phase transition

- Setup

[Russo, Tierz, 17]

Model: *3d $\mathcal{N}=4$ $U(1)$* SUSY gauge + *$2N$* hypermultiplets(charge *1*)
Parameters: Fayet-Illiopoulos parameter η, flavor mass m

'tHooft like parameter: $\lambda = \eta/N$

- Exact expression for the part. func. on S^3 (by SUSY localization technique)

[Pestun, 12]
[A. Kapustin, B. Willett, I. Yaakov, 10]
[N. Hama, K. Hosomichi, S. Lee, 11]
[D. L. Jafferis, 12]

$$Z = \int_{-\infty}^{\infty} \mathrm{d}\sigma \, \frac{e^{-i\eta\sigma}}{\left[2\cosh\frac{\sigma+m}{2} \cdot 2\cosh\frac{\sigma-m}{2}\right]^N}$$

- 2nd order quantum phase transition at the 't Hooft like limit (λ =fix. $N \to \infty$)

[Russo, Tierz, 17]

$$\frac{\mathrm{d}^2 F}{\mathrm{d}\lambda^2} = \begin{cases} \frac{N}{1+\lambda^2}\left(1 + \frac{\cosh m}{\sqrt{1-\lambda^2\sinh^2 m}}\right) & \lambda < \lambda_\mathrm{c} \\ \frac{N}{1+\lambda^2} & \lambda \geq \lambda_\mathrm{c} \end{cases} \qquad \lambda_\mathrm{c} \equiv \frac{1}{\sinh m}$$

Lefschetz thimble structure

Brief summary

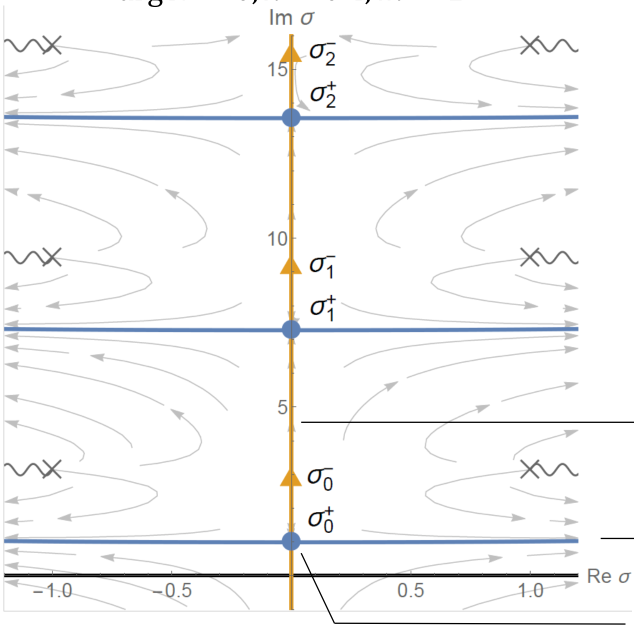
[T. Fujimori, M. Honda, S. Kamata, T. Misumi, N. Sakai, TY, 21]

$\lambda < \lambda_c$	$\longleftrightarrow$	$\lambda \geq \lambda_c$
Single trivial saddle		Infinite number of saddles (Two of them survive the large-flavor limit)
No Stokes phenomenon		Infinite number of Stokes phenomena

Discrete jump of the intersection numbers

$\lambda < \lambda_c$

$\arg N = 0, \lambda = 0.4, m = 1$



$\lambda \geq \lambda_c$

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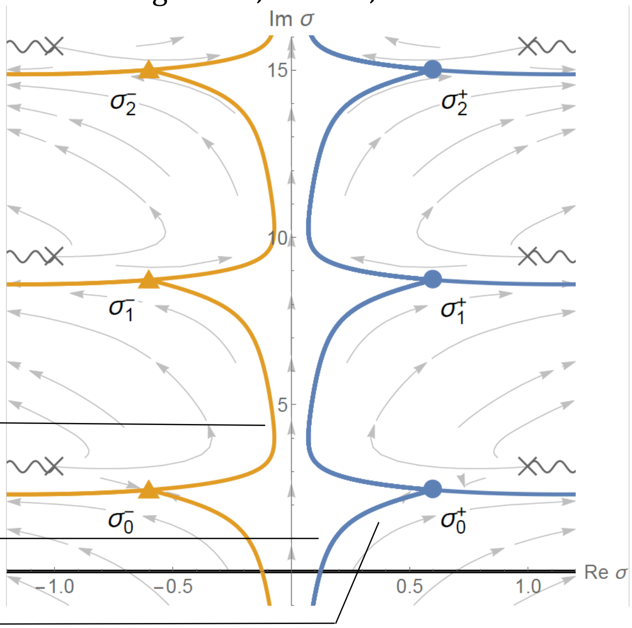

branch cut

thimble

thimble

Trivial saddle

Lefschetz thimble structure

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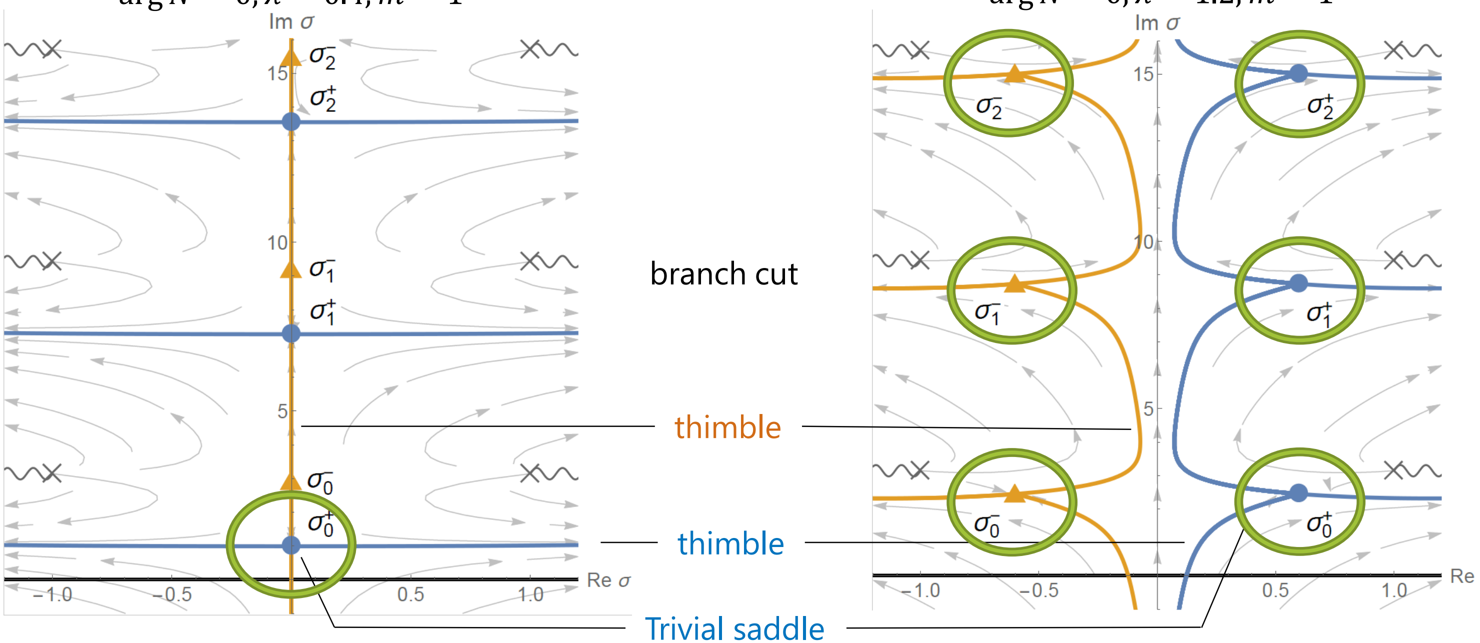
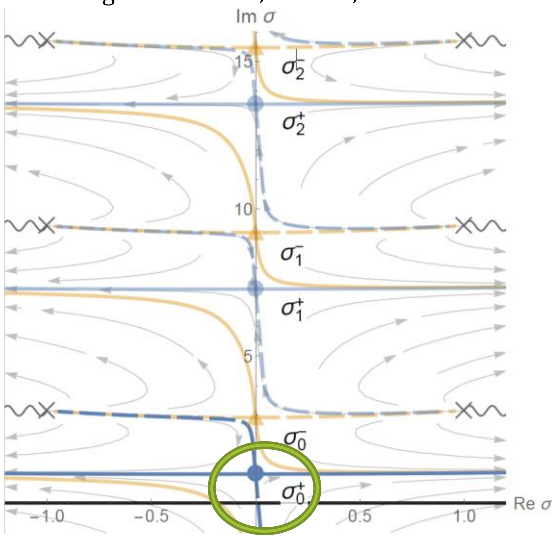


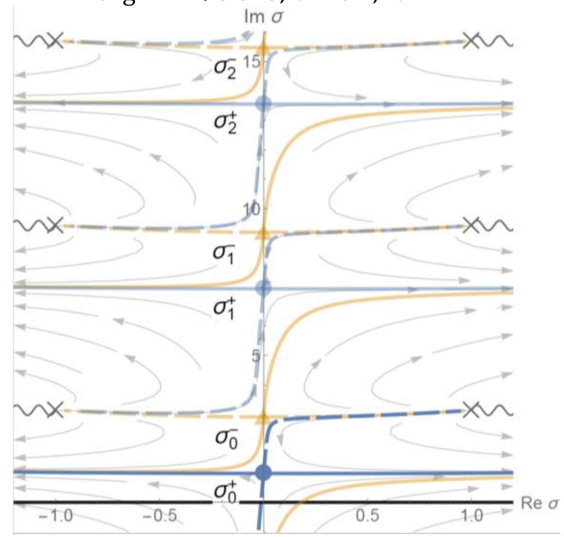
Lefschetz thimble structure

No Stokes phenomenon for $\lambda < \lambda_c$

$\arg N = -0.025, \lambda = 0.4, m = 1$



$\arg N = +0.025, \lambda = 0.4, m = 1$



Infinite number of Stokes phenomena for $\lambda > \lambda_c$

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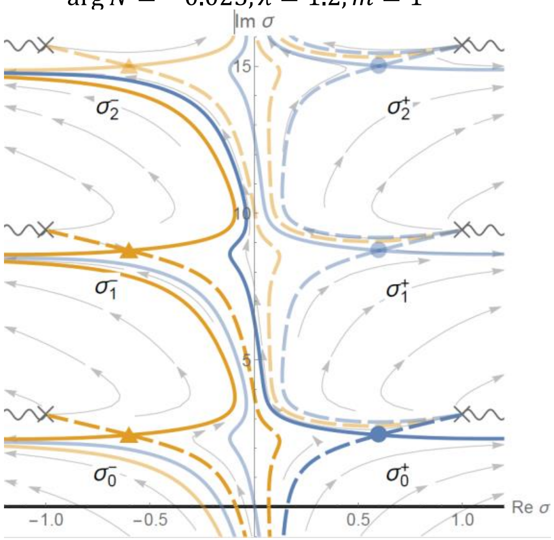
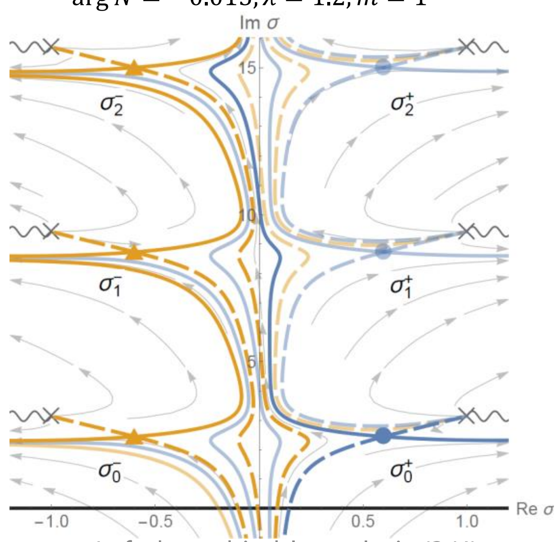


$\arg N = -0.015, \lambda = 1.2, m = 1$



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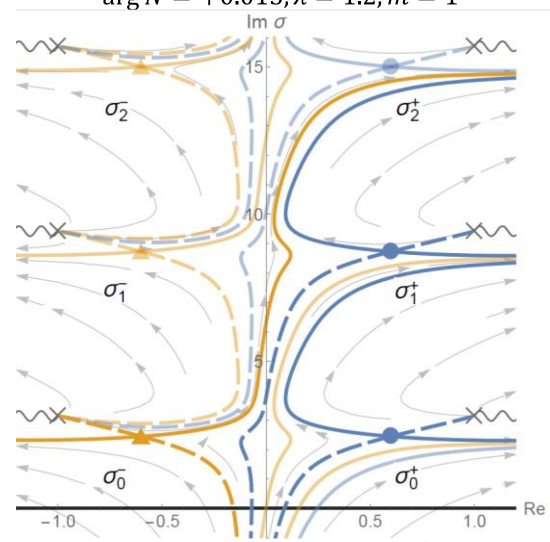

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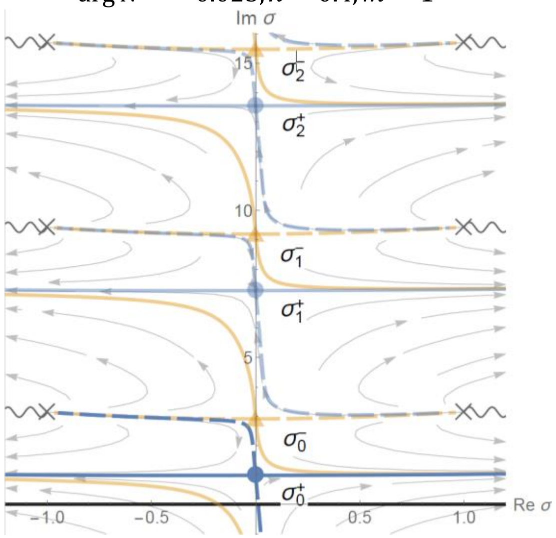

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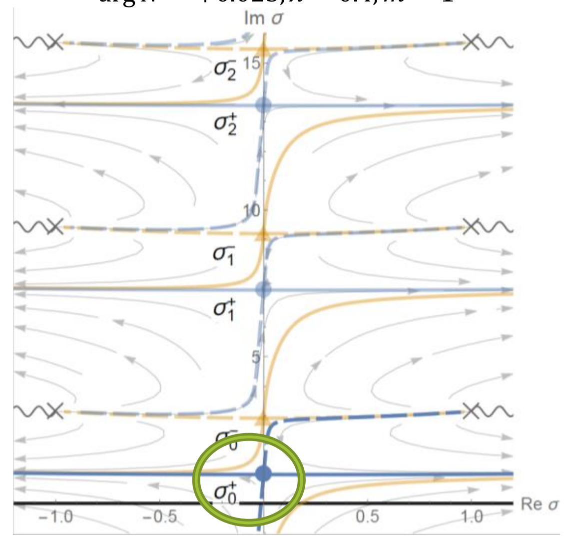

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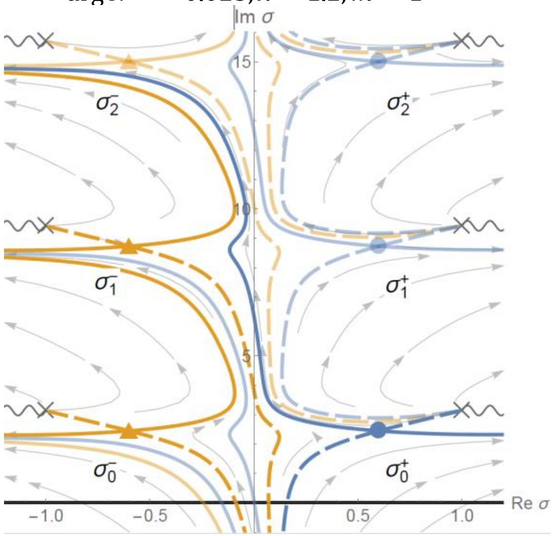

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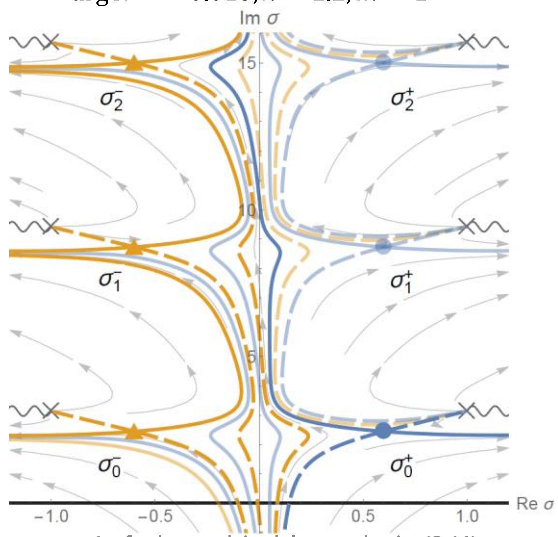

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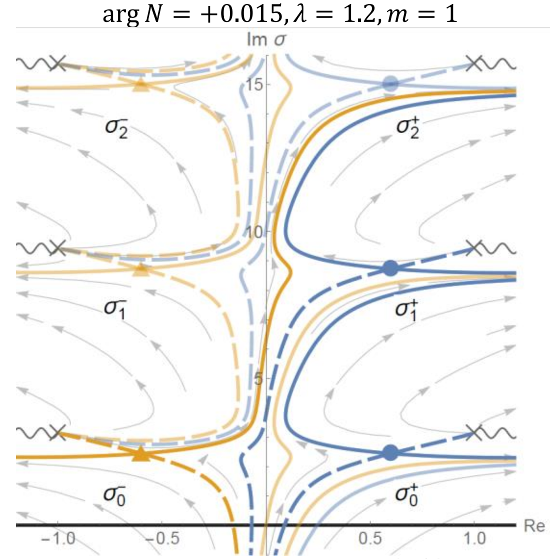

Lefschetz thimble structure

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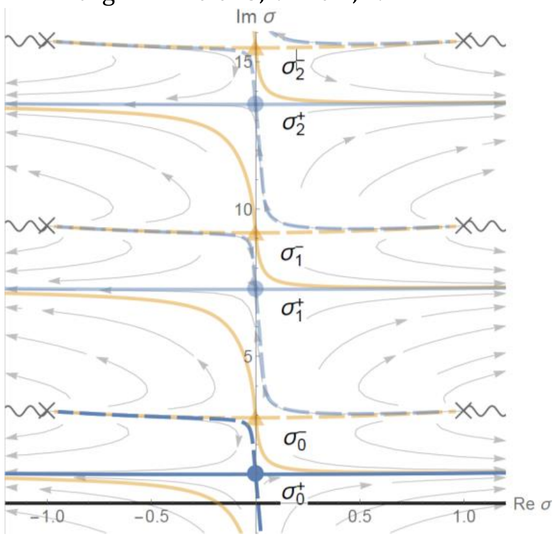

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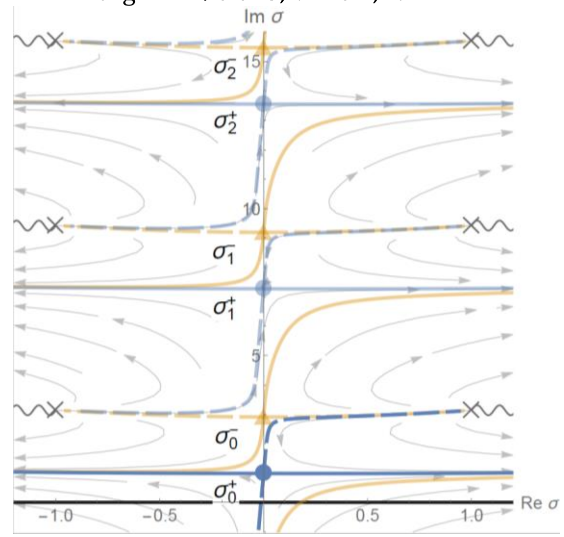

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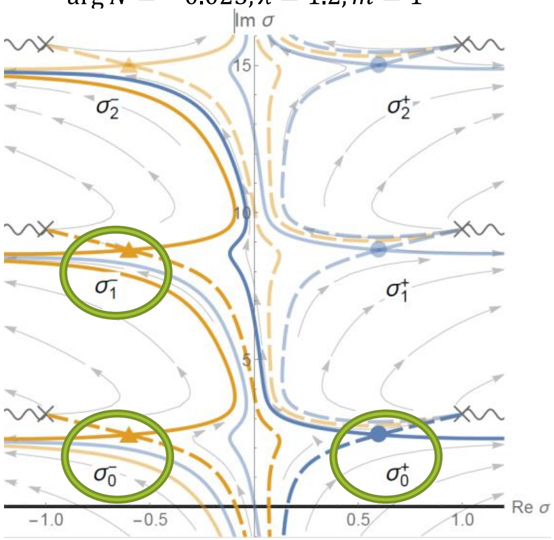

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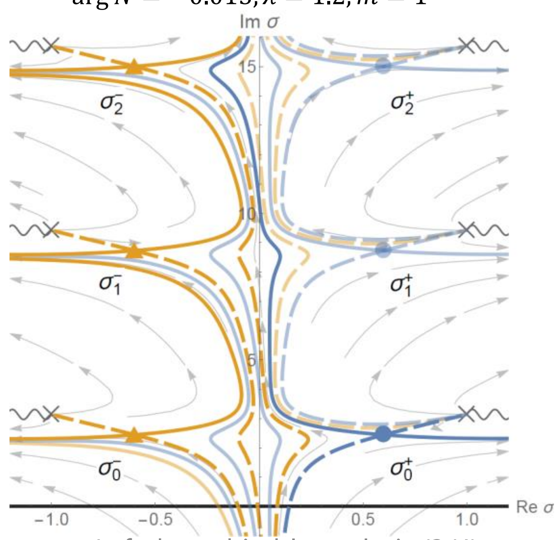

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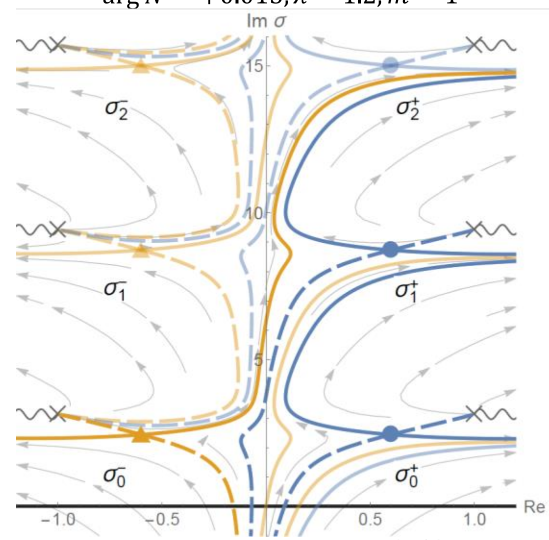

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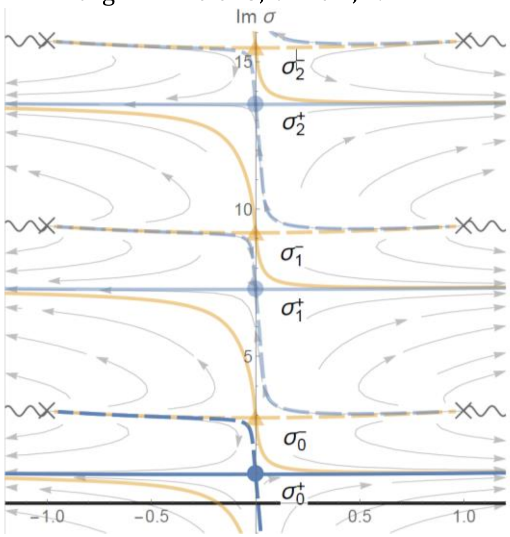

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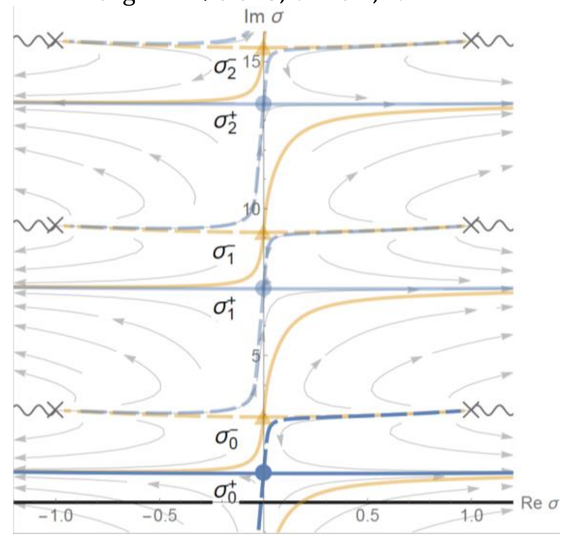

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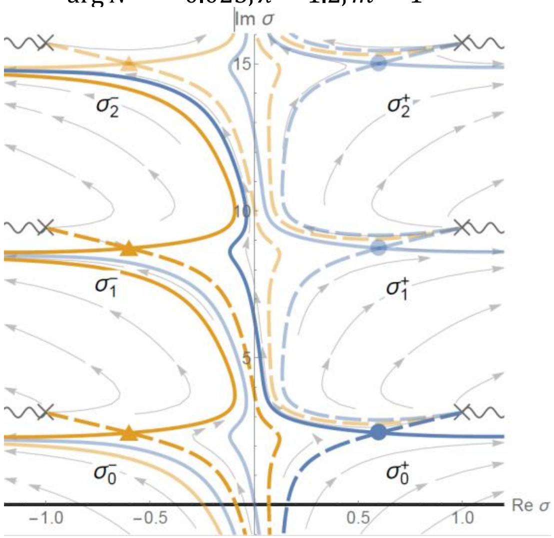

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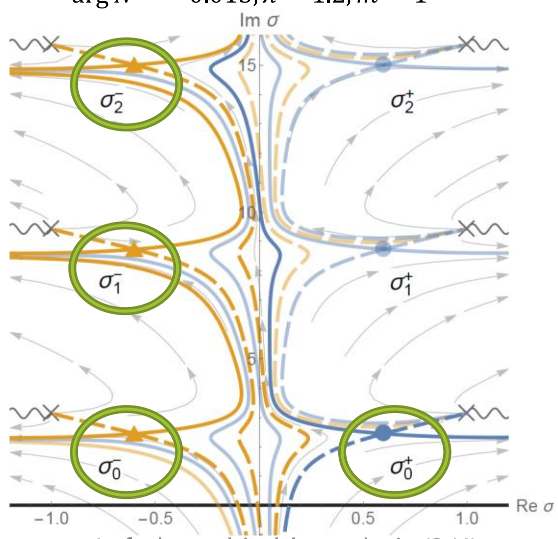

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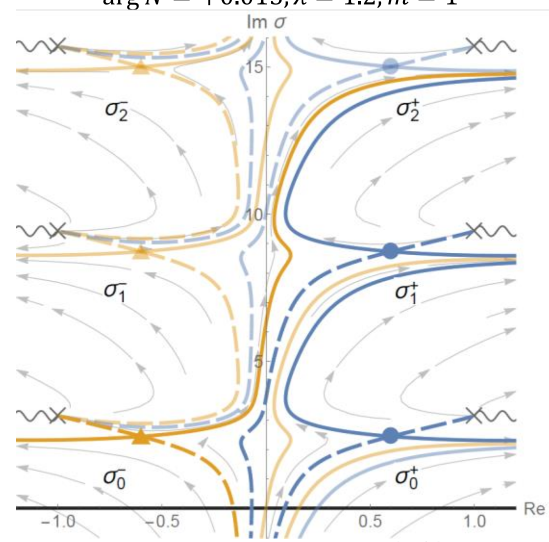

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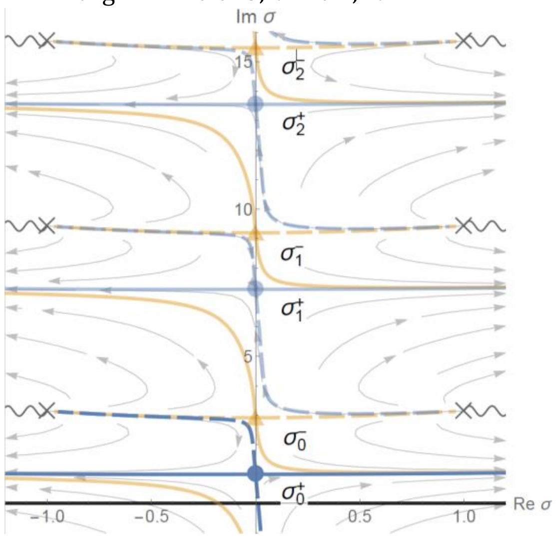

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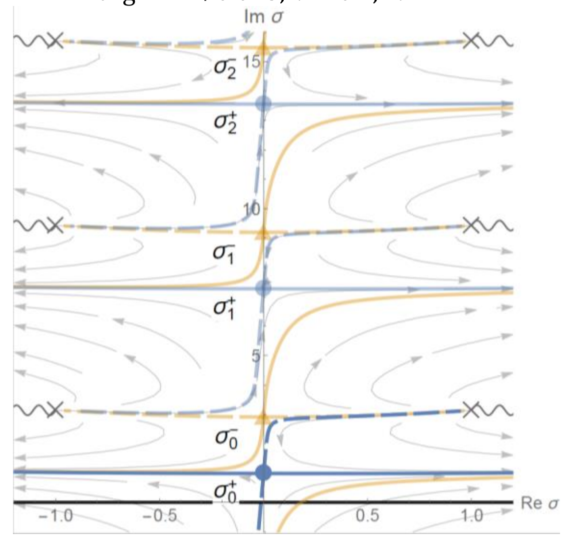

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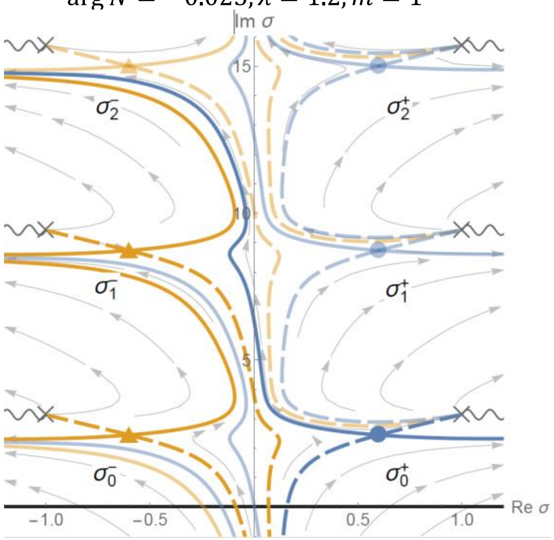

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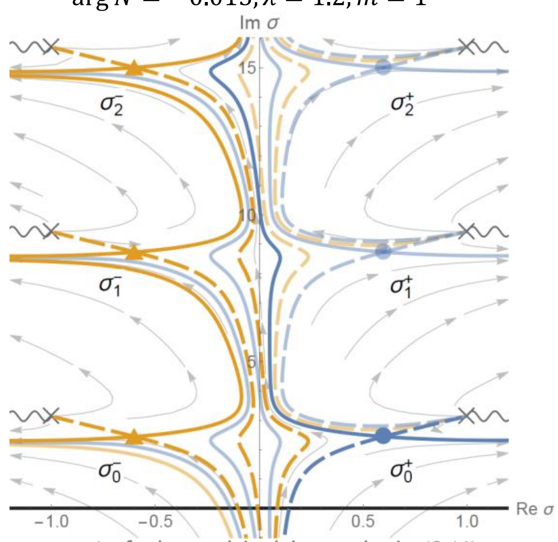

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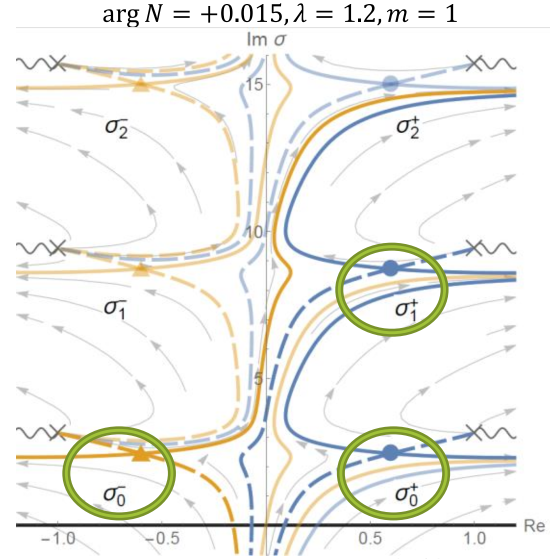
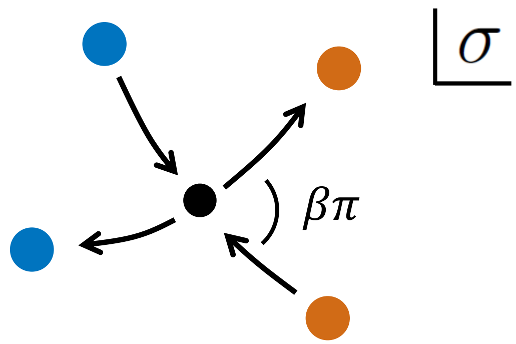


The order of phase trans. and collision of saddles

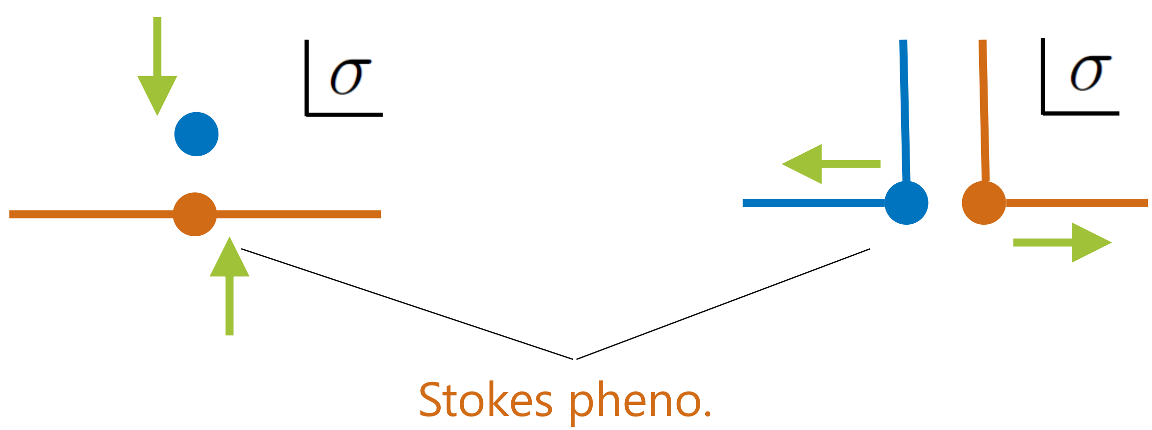
The order of phase transitions is determined
by the scattering angle of collided saddles



$\lceil (n+1)\beta \rceil$ -th order phase transition

Particularly $n = 2,\ \alpha = 1/2$, then 2nd order phase transition

$\lambda < \lambda_c$ ⟷ $\lambda \geq \lambda_c$



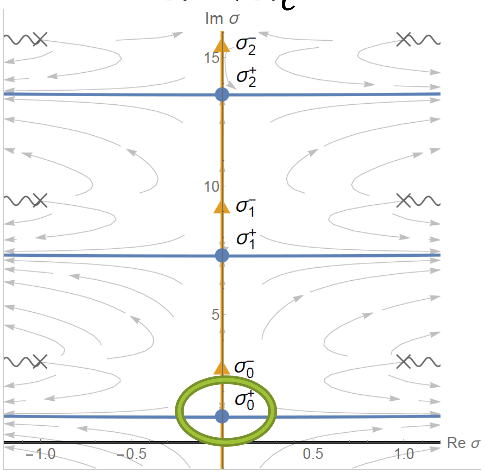
Contents

Large-flavor expansion and Borel resummation

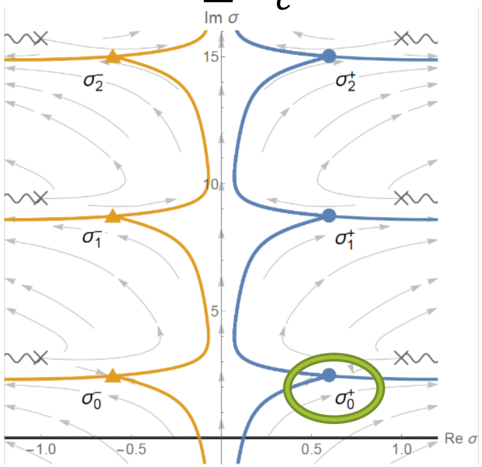
Decoding information of other saddles and the phase transition from expansion around the trivial saddle

[T. Fujimori, M. Honda, S. Kamata, T. Misumi, N. Sakai, TY, 21]

$\lambda < \lambda_c$



$\lambda \geq \lambda_c$



$$Z(\lambda;N) = \frac{1}{2^N}\int d\sigma\ e^{-NS(\lambda;\sigma)}, \quad S(\lambda;\sigma) = -i\lambda\sigma + \ln(\cosh\sigma + \cosh m)$$

$$\overset{\text{around } \sigma_0^+}{=} \frac{1}{2^N}\sqrt{\frac{2\pi}{NS''(\lambda;\sigma_0^+)}}e^{-NS(\lambda;\sigma_0^+)}\sum_{l=0}^{\infty}\frac{a_l(\lambda)}{N^l}$$

Formal expansion

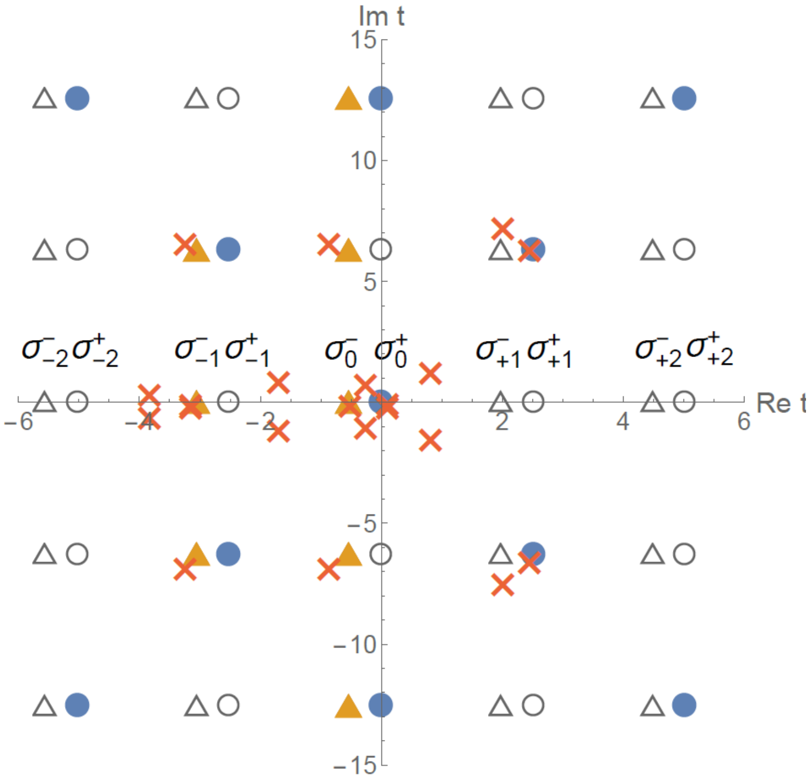
↓

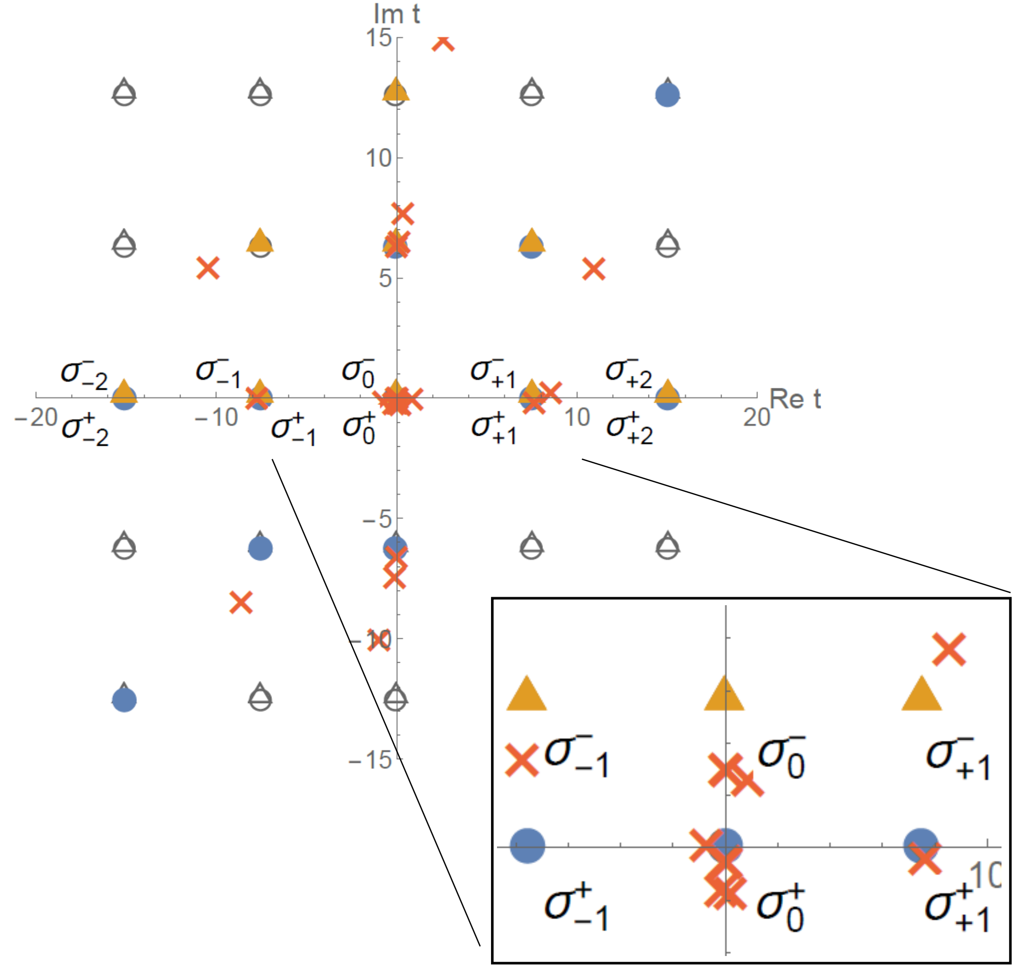
Borel resummation

$$\mathcal{S}Z(\lambda;N) = \frac{1}{2^N}\sqrt{\frac{2\pi}{NS''(\lambda;\sigma_0^+)}}e^{-NS(\lambda;\sigma_0^+)}\cdot N\int_C dt\ e^{-Nt}\sum_{l=0}^{\infty}\frac{a_l(\lambda)}{\Gamma(l+1)}t^l$$

Borel resummation

Lefschetz thimble structure is encoded in Borel plane structure



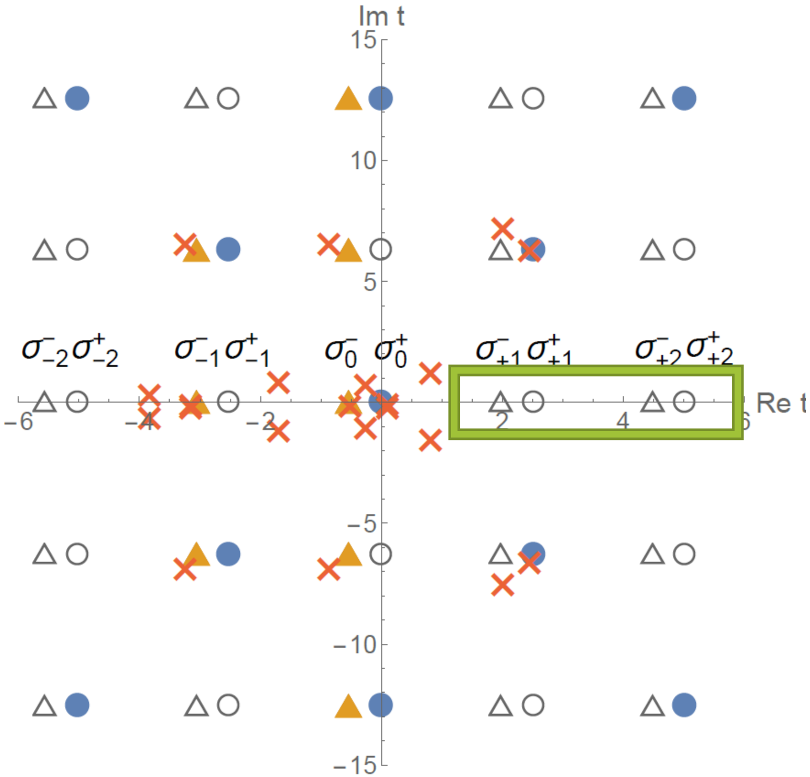


Borel resummation

Lefschetz thimble structure is encoded in Borel plane structure

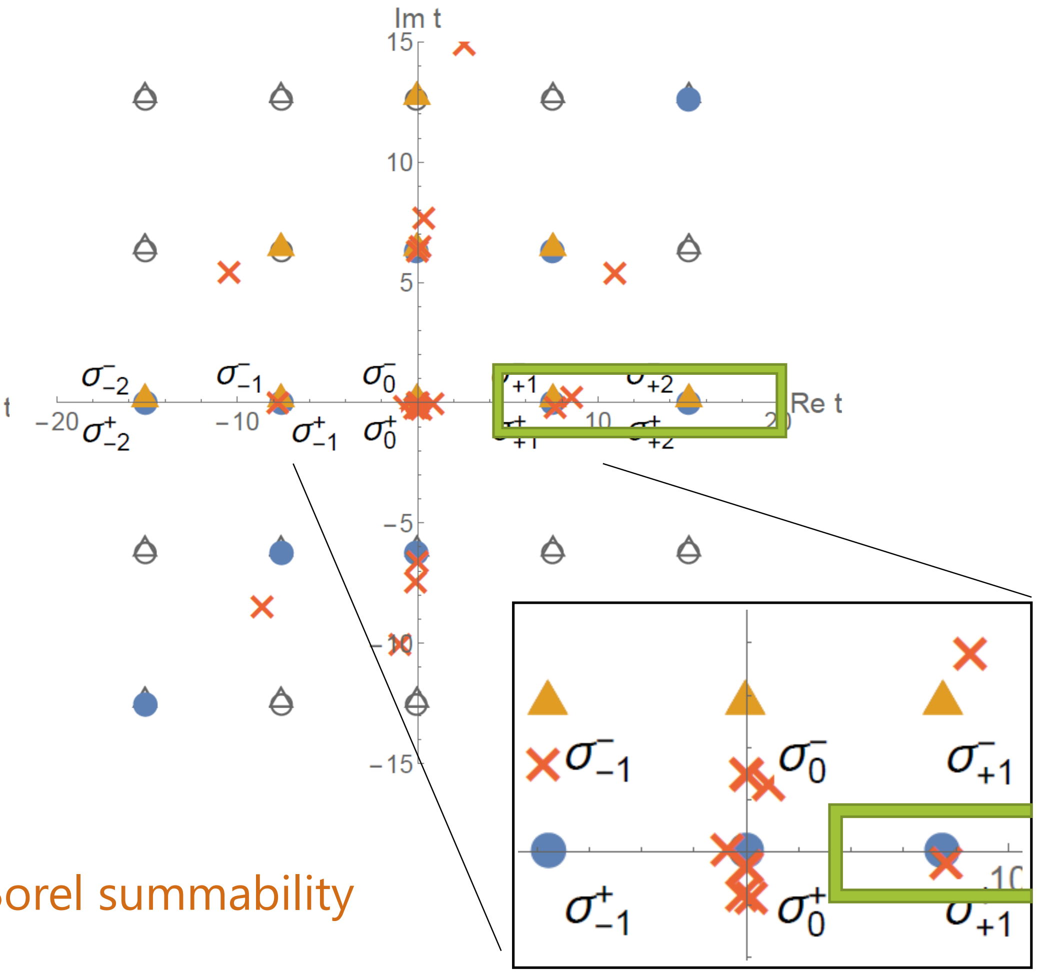
$\lambda < \lambda_c$

$\lambda = 0.4, m = 1$, Padé-Uniformized (25,25)



$\lambda \geq \lambda_c$

$\lambda = 1.2, m = 1$, Padé-Uniformized (25,25)



- Stokes pheno. correspond to Borel summability

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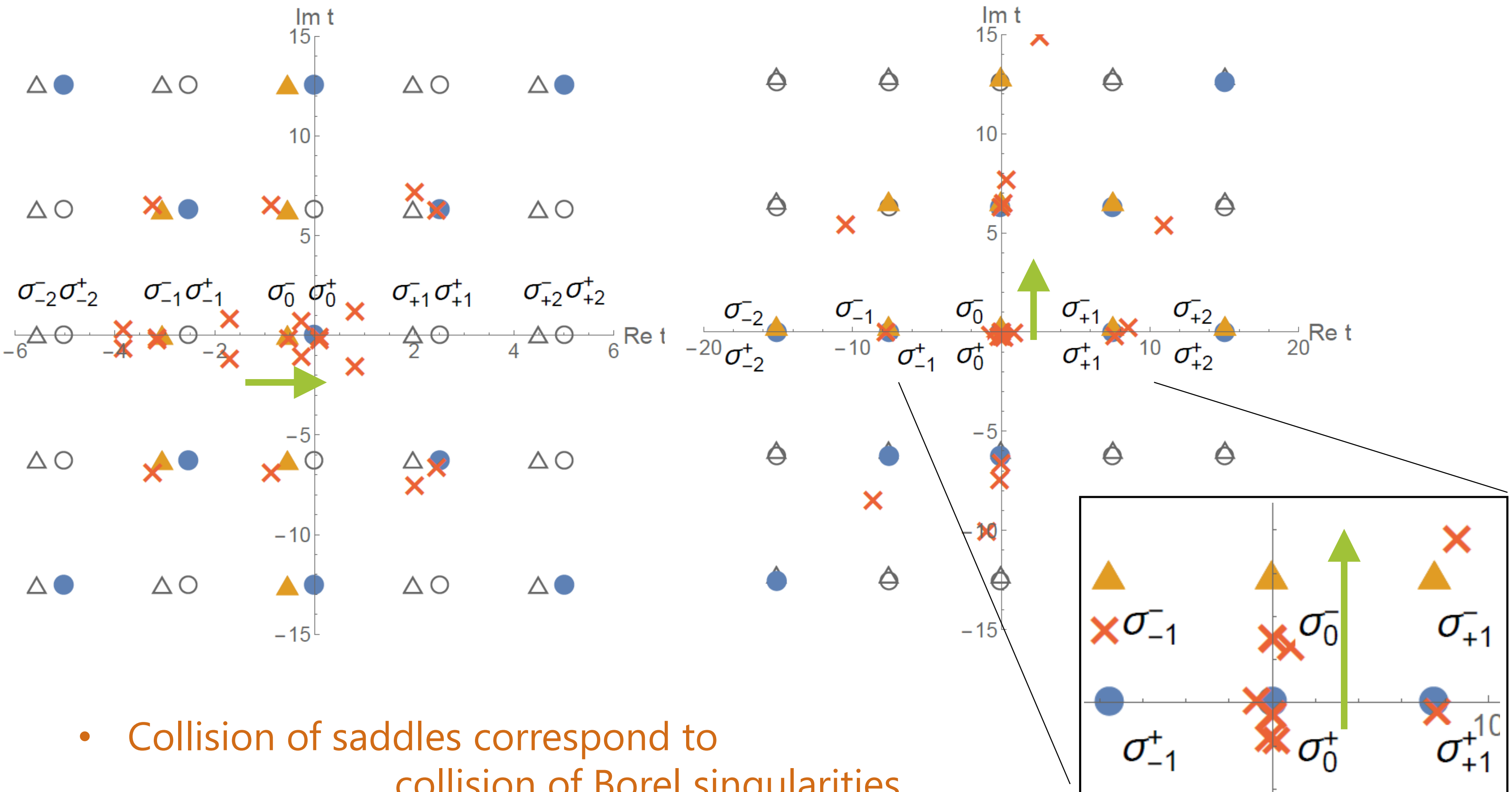


- Collision of saddles correspond to collision of Borel singularities
- An anti-Stokes pheno. occurs

Contents

- ✓ Introduction
- ✓ Lefschetz thimble analysis
- ✓ Borel resummation
- Conclusion and future works

Conclusion and future works

- Resurgence is an approach to non-perturbative physics from perturbation theory
- Q: Is resurgence applicable to 2nd order phase transitions or more realistic QFTs?

TAKE-HOME MESSAGES

A: resurgence is applicable!

- Lefschetz thimble analysis
  - Two phases are distinguished by Stokes pheno.
  - The order of the phase transition is determined by "a collision of saddles."
- Borel resummation
  - Two phases are distinguished by Stokes pheno.
  - The order of the phase transition is determined by "a collision of Borel singularities."

❑ 2nd order phase trans. is the simultaneous Stokes and anti-Stokes pheno.
→ ❑ The order of phase transitions can be decoded from a perturbative series
❑ Generalized to other systems

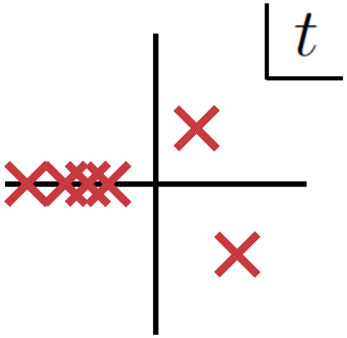
- Relation to Lee-Yang zeros ?
- Expansion with respect to other parameters ?
- Physical meaning of the phase transition ?

Backups

Borel transformation and Padé approximation

Search Borel singularities by the Padé approximation

$$\sum_{l=0}^{l_{\max}} \frac{a_l(\lambda)}{\Gamma(l+1)} t^l = \frac{P_{l_{\max}/2}(\lambda;t)}{Q_{l_{\max}/2}(\lambda;t)} + \mathcal{O}(t^{l_{\max}+1})$$



Typically, the Padé approximation becomes worse outside the closest singularity

[Costin, Dunne, 20]

The Padé approximation is improved by a uniformization map

$$\mathcal{B}F(t) \mapsto \mathcal{B}F(\phi_n(u))$$

$$\phi_n(u) = t_n(1-e^{-u}), \quad \phi_n^{-1}(t) = -\ln(1-t/t_n)$$

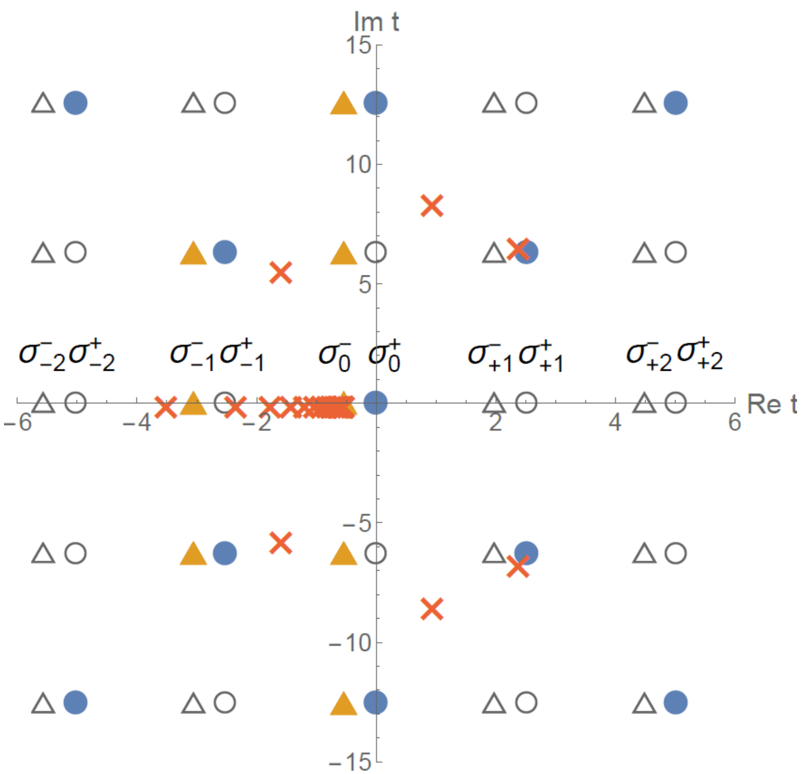
The closest singularity is sent to ∞

Borel singularities

- The Borel plane structure is consistent with the Lefschetz thimble structure
- Still there are artifacts, and missing singularities far from the origin

$\lambda < \lambda_c$

$\lambda = 0.4, m = 1$, Padé (25,25)



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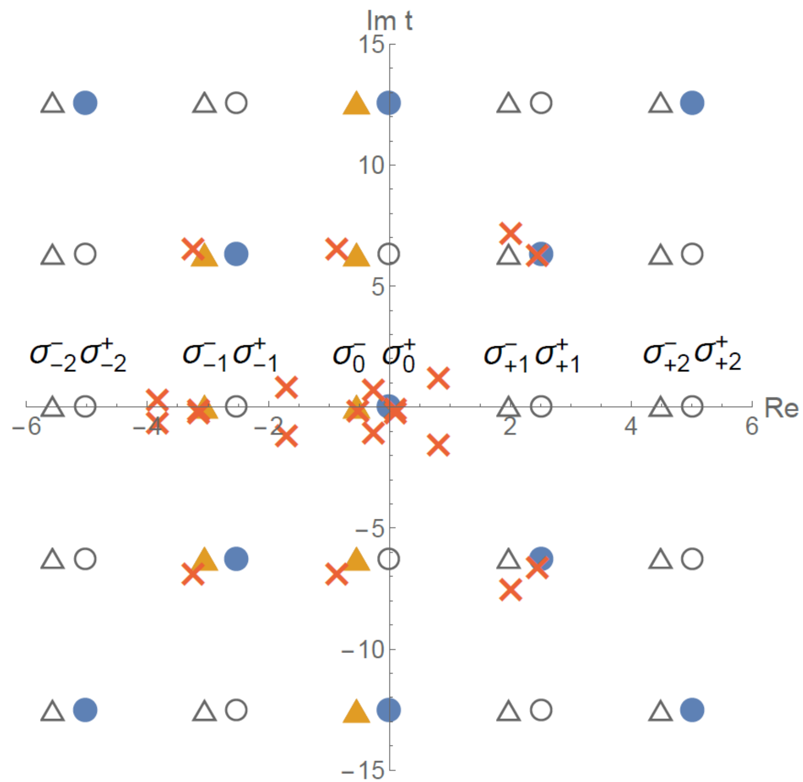

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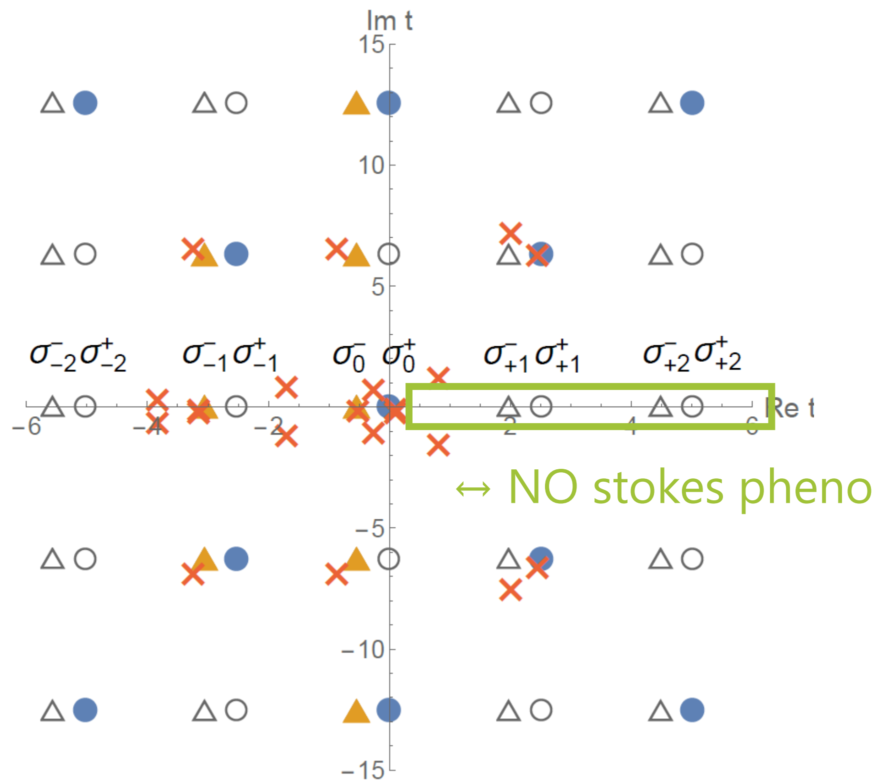

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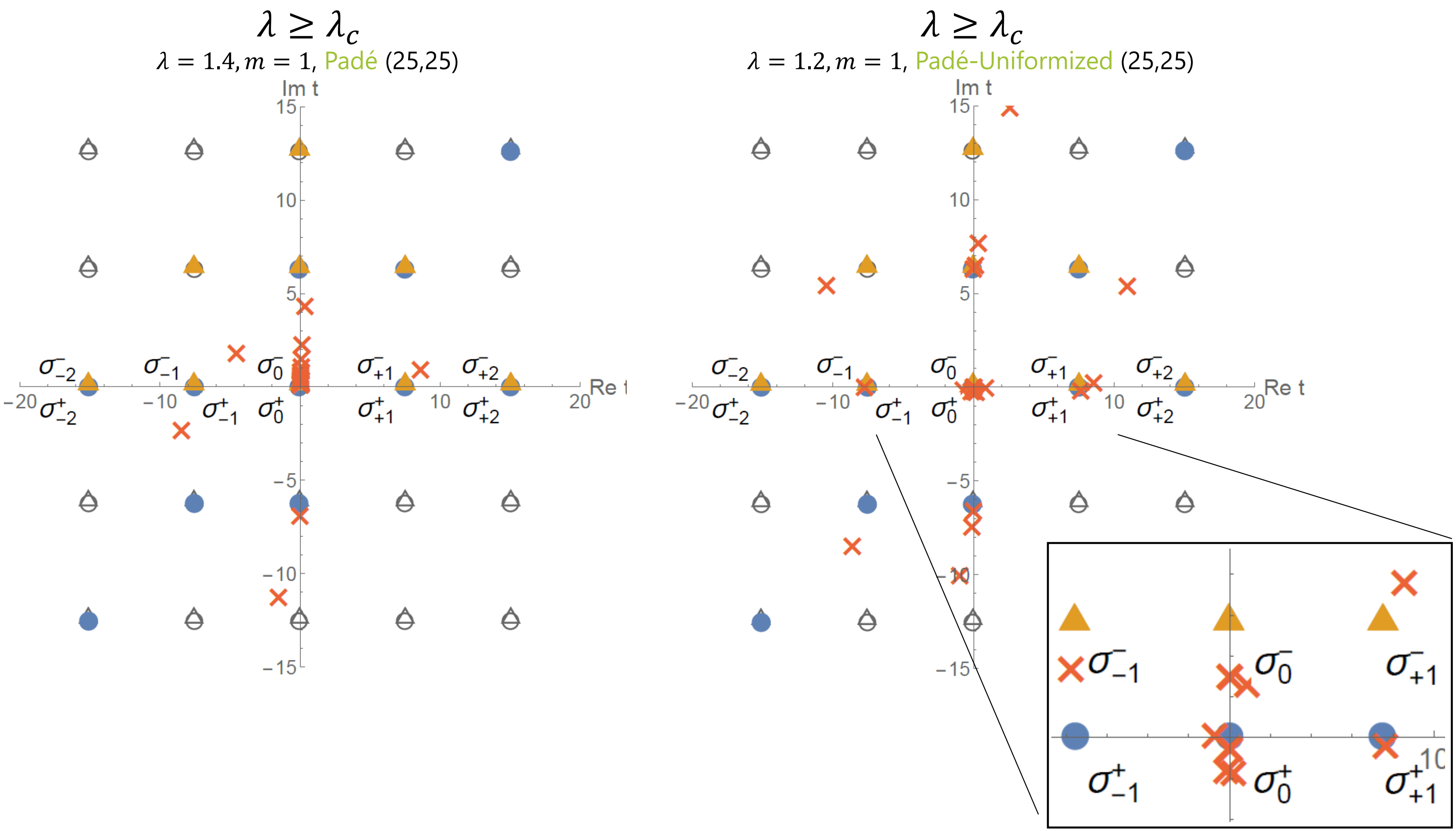

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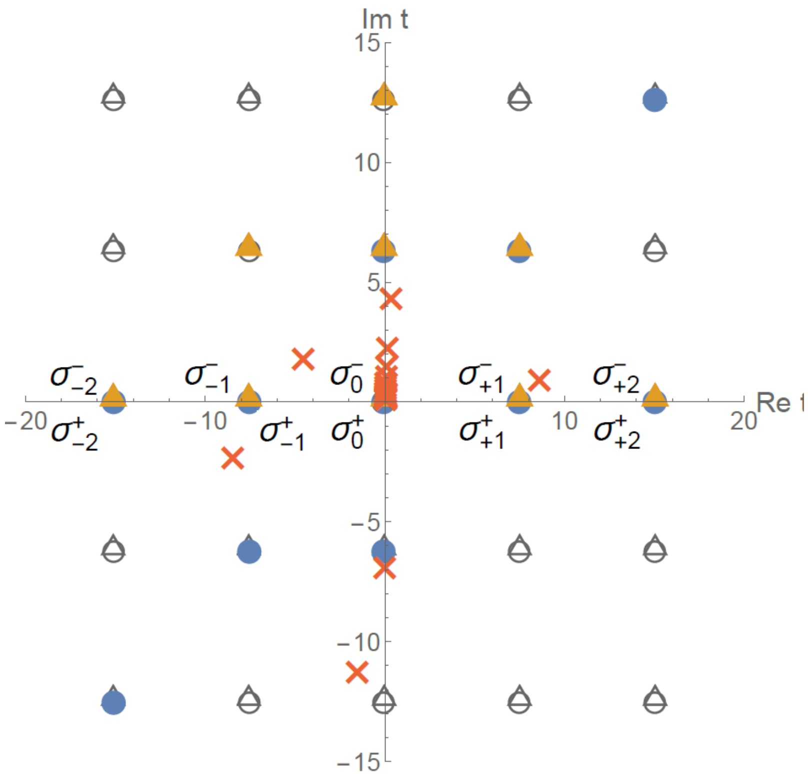

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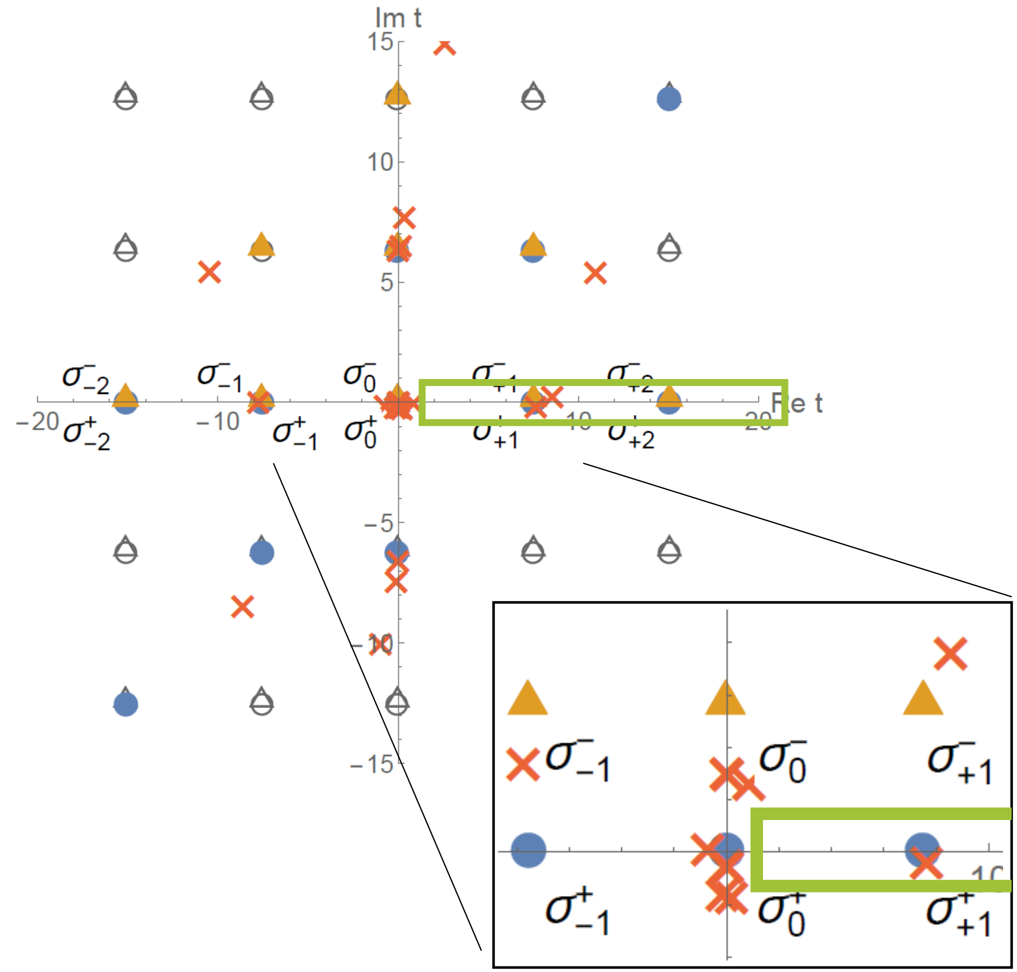
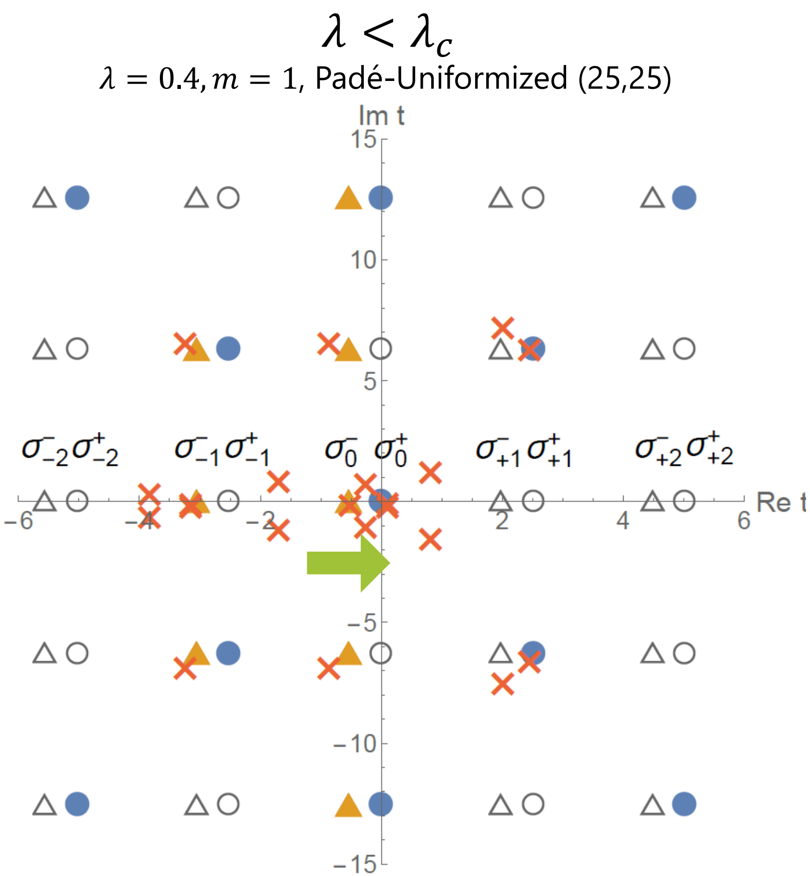
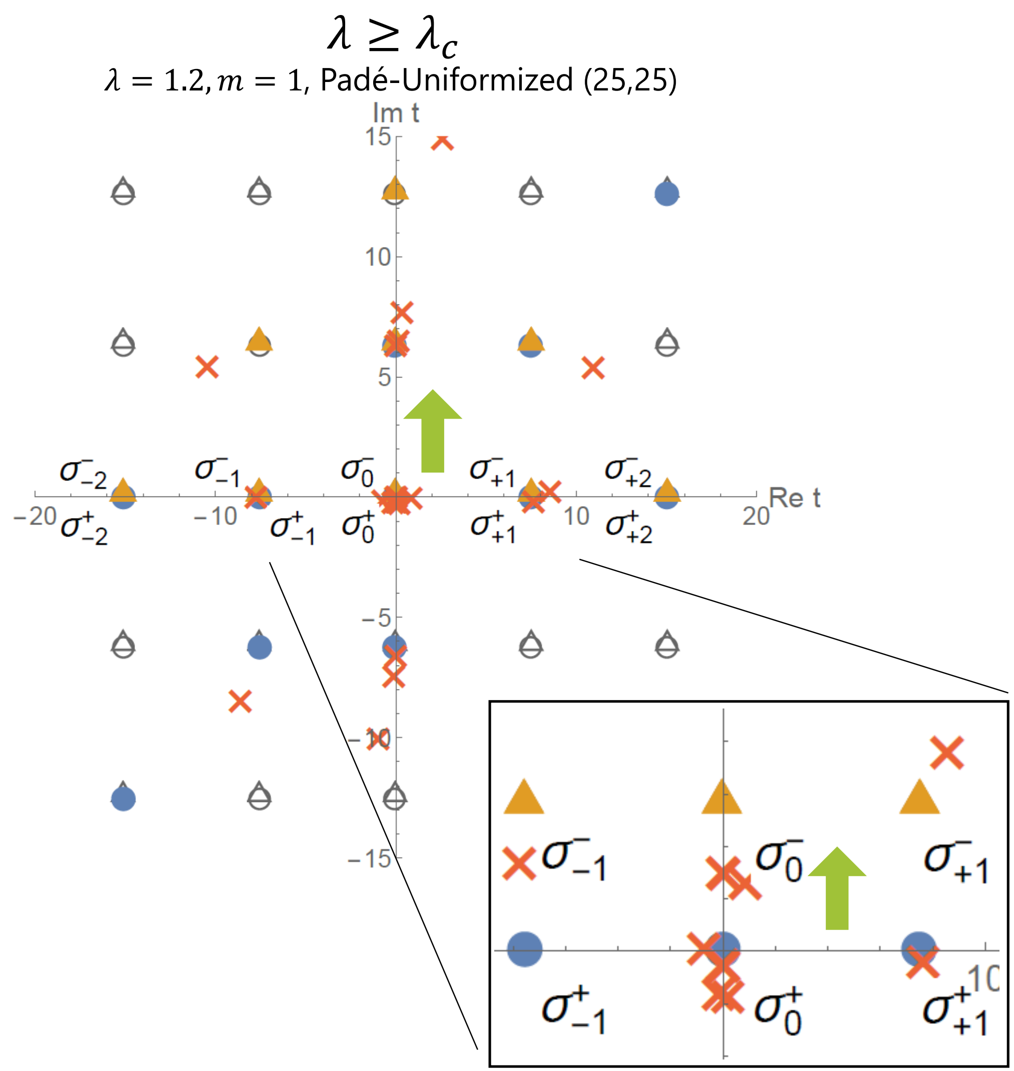


The order of the phase transition

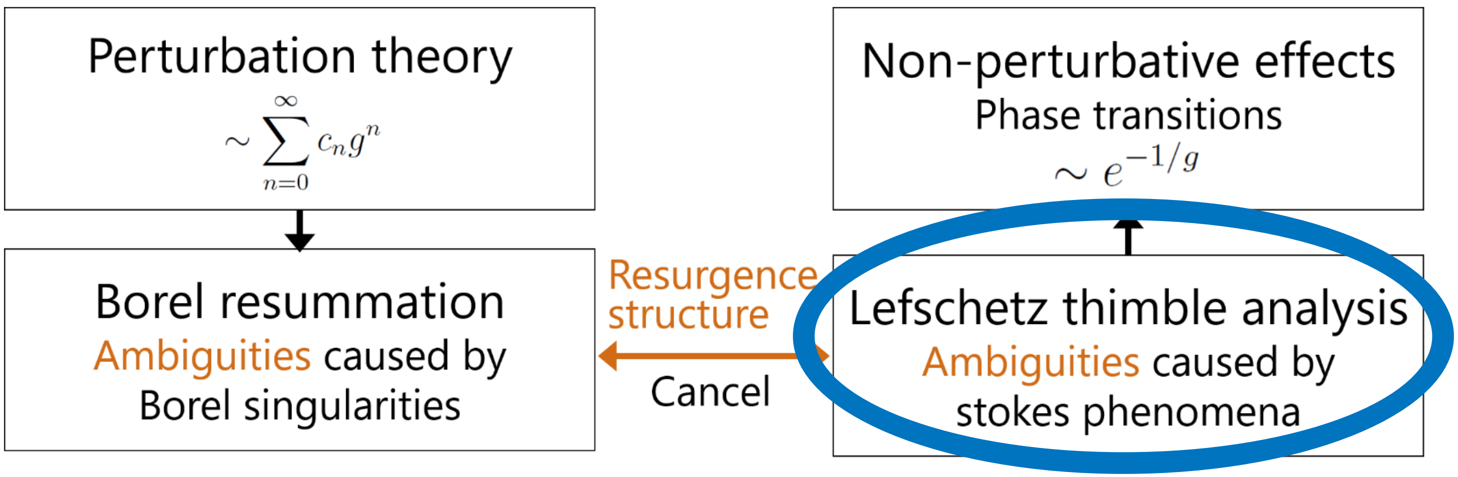
2nd order phase transition corresponds to

collision of two saddles with the reflection angle $\pi/2$





Lefschetz thimble analysis



Lefschetz thimble analysis

0dim Sine-Gordon model

[Cherman, Dorigoni, Unsal, 14]
[Cherman, Koroteev, Unsal, 14]

$$Z(g) = \frac{1}{(2\pi g)^{1/2}} \int_{-\pi/2}^{\pi/2} \mathrm{d}\phi \; e^{-S(\phi)/g}, \qquad S(\phi) = \frac{1}{2}\sin^2\phi$$

Saddles and Lefschetz thimbles

$$0 = \frac{\mathrm{d}S(\phi)}{\mathrm{d}\phi} \Rightarrow \phi = 0, \pm\frac{\pi}{2}$$

Trivial saddle and non-trivial saddles

$$\mathcal{J}_i : \frac{\mathrm{d}\phi(t)}{\mathrm{d}t} = \overline{\frac{\mathrm{d}S}{\mathrm{d}\phi}}, \quad \phi(-\infty) = \phi_i \qquad \mathrm{Im}\, S(\phi(t)) = \mathrm{const.}, \quad \mathrm{Re}\, S(\phi_i) \le \mathrm{Re}\, S(\phi(t))$$

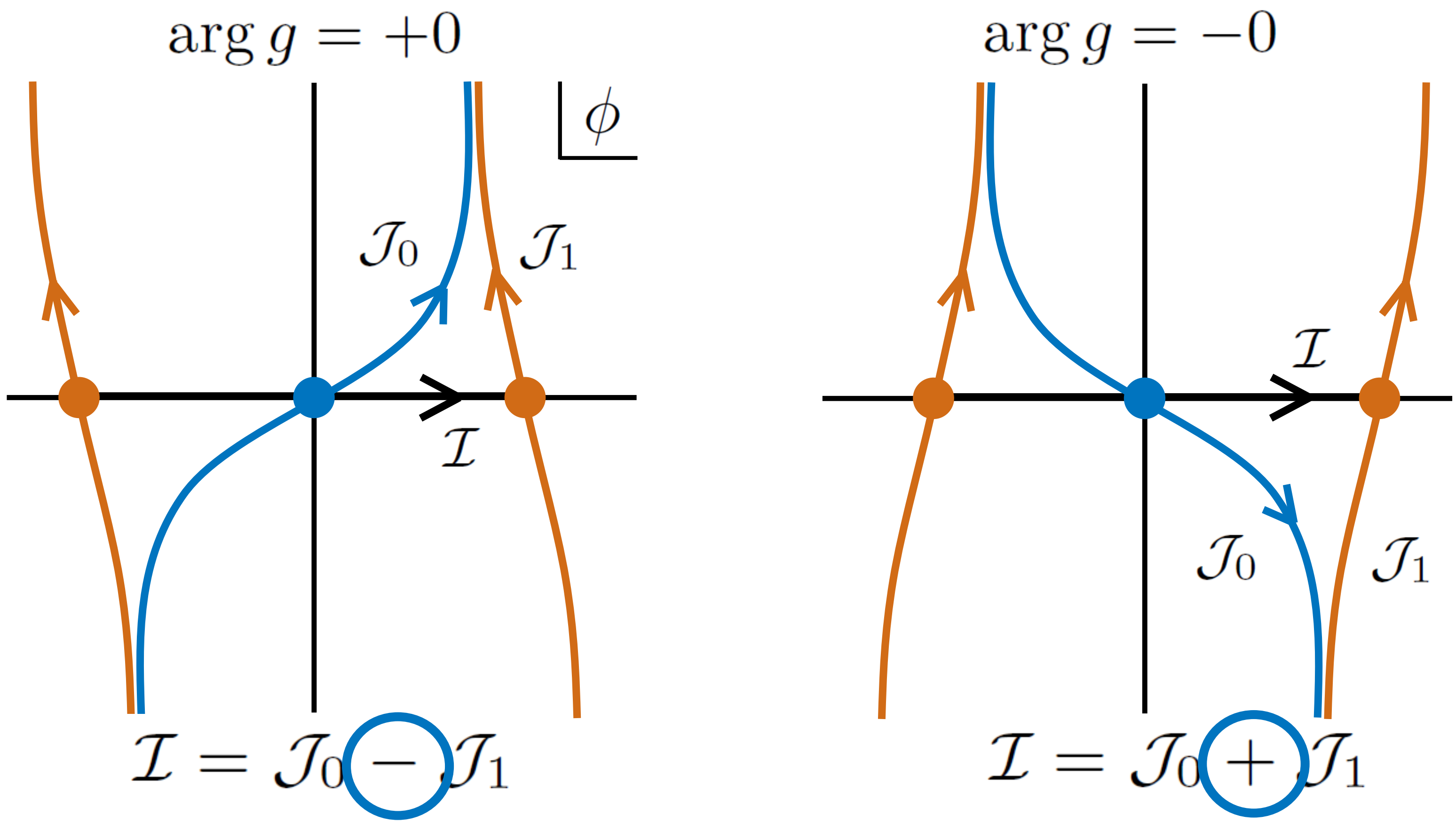
$$\mathcal{K}_i : \frac{\mathrm{d}\phi(t)}{\mathrm{d}t} = -\overline{\frac{\mathrm{d}S}{\mathrm{d}\phi}}, \quad \phi(-\infty) = \phi_i \qquad \mathrm{Im}\, S(\phi(t)) = \mathrm{const.}, \quad \mathrm{Re}\, S(\phi_i) \ge \mathrm{Re}\, S(\phi(t))$$

Lefschetz thimble analysis

Around $\arg g = 0$,

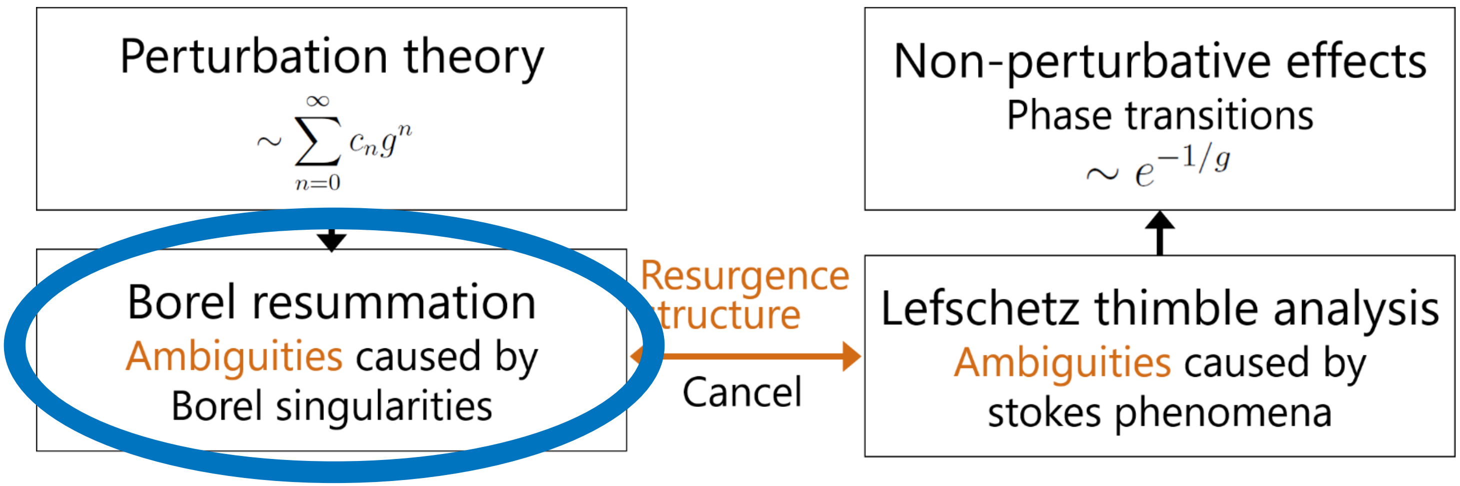
[Cherman, Dorigoni, Unsal, 14]
[Cherman, Koroteev, Unsal, 14]

Stokes phenomenon associated with the trivial saddle



NO Stokes phenomenon associated with the non-trivial saddles

Borel resummation



Borel resummation

[Cherman, Dorigoni, Unsal, 14]
[Cherman, Koroteev, Unsal, 14]

Perturbation theory around the trivial saddle diverges

$$Z(g) = \frac{1}{(2\pi g)^{1/2}} \int_{-\pi/2}^{\pi/2} \mathrm{d}\phi \ e^{-\frac{1}{2g}\sin^2\phi}$$

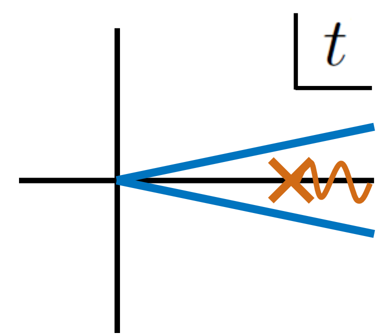
$$\overset{\text{around } \phi=0}{=} e^{-S(0)/g} \cdot \frac{1}{g} \sum_{n=0}^{\infty} \frac{(+2)^n \Gamma(n+1/2)^2}{\Gamma(1/2)^2 \Gamma(n+1)} g^{n+1}$$

There is a Borel singularity (and a branch cut) around $\arg g = 0$

$$\mathcal{S}Z(g) = \int_C \mathrm{d}t \ e^{-t/g} \mathcal{B}Z(t)$$

$$= e^{-S(0)/g} \cdot \frac{1}{g} \int_C \mathrm{d}t \ e^{-t/g} \sum_{n=0} \frac{2^n \Gamma(n+1/2)^2}{\Gamma(1/2)^2 \Gamma(n+1)^2} (+t)^n$$

$$= e^{-S(0)/g} \cdot \frac{1}{g} \int_C \mathrm{d}t \ e^{-t/g} {}_2F_1\left(\frac{1}{2}, \frac{1}{2}, 1; +2t\right)$$



Borel resummation

[Cherman, Dorigoni, Unsal, 14]
[Cherman, Koroteev, Unsal, 14]

Perturbation theory around a non-trivial saddle diverges

$$Z(g) = \frac{1}{(2\pi g)^{1/2}} \int_{-\pi/2}^{\pi/2} \mathrm{d}\phi \; e^{-\frac{1}{2g}\sin^2\phi}$$

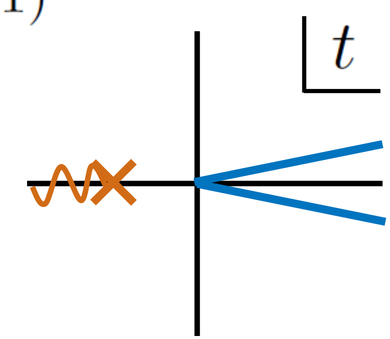
$$\overset{\text{around } \phi=\pi/2}{=} i e^{-S(\pi/2)/g} \cdot \frac{1}{g} \sum_{n=0}^{\infty} \frac{(-2)^n \Gamma(n+1/2)^2}{\Gamma(1/2)^2 \Gamma(n+1)} g^{n+1}$$

There is NO Borel singularity (nor branch cut) around $\arg g = 0$

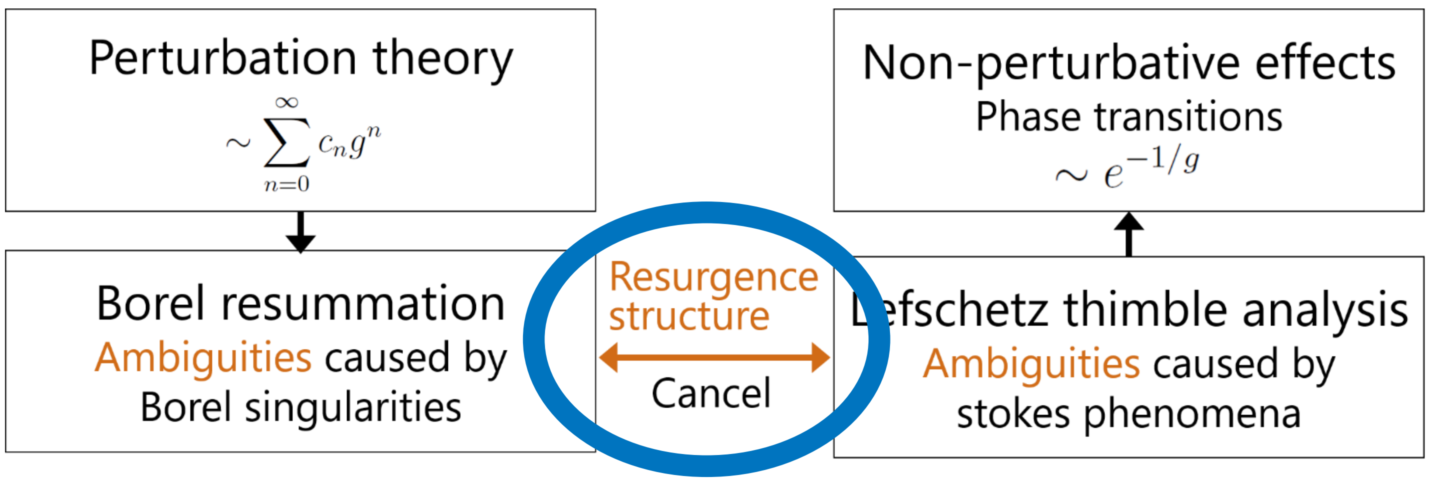
$$\mathcal{S}Z(g) = \int_C \mathrm{d}t \; e^{-t/g} \mathcal{B}Z(t)$$

$$= i e^{-S(\pi/2)/g} \cdot \frac{1}{g} \int_C \mathrm{d}t \; e^{-t/g} \sum_{n=0} \frac{2^n \Gamma(n+1/2)^2}{\Gamma(1/2)^2 \Gamma(n+1)^2} (-t)^n$$

$$= i e^{-S(\pi/2)/g} \cdot \frac{1}{g} \int_C \mathrm{d}t \; e^{-t/g} {}_2F_1\left(\frac{1}{2}, \frac{1}{2}, 1; -2t\right)$$



Resurgence



Perturbation theory
$\sim \sum_{n=0}^{\infty} c_n g^n$
Non-perturbative effects
Phase transitions
$\sim e^{-1/g}$
Borel resummation
Ambiguities caused by
Borel singularities
Resurgence
structure
Cancel
Lefschetz thimble analysis
Ambiguities caused by
stokes phenomena

Resurgence structure

[Cherman, Dorigoni, Unsal, 14]
[Cherman, Koroteev, Unsal, 14]

The two types of ambiguities cancel

and the location of the Borel singularity agrees with $S\left(\frac{\pi}{2}\right) = 1/2$

$$\mathcal{S}Z(g) = \mathcal{S}_{\pm}Z(g)|_{\text{around }\phi=0} \mp \mathcal{S}Z(g)|_{\text{around }\phi=\pi/2}$$

$$= e^{-S(0)/g} \cdot \frac{1}{g}\int_{C\pm} \mathrm{d}t\ e^{-t/g}\,{}_2F_1\left(\frac{1}{2},\frac{1}{2},1;+2t\right)$$

$$\mp ie^{-S(\pi/2)/g} \cdot \frac{1}{g}\int_{C} \mathrm{d}t\ e^{-t/g}\,{}_2F_1\left(\frac{1}{2},\frac{1}{2},1;-2t\right)$$

$$= \mathrm{Re}\ \mathcal{S}_{\pm}Z(g)|_{\text{around }\phi=0}$$

Information of non-trivial saddles is
encoded in perturbation theory around the trivial saddle