Higher symmetries and logical gates of Z2 toric code

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KEK Conference 2024

Thanks to

Maissam Barkeshli
(UMD)

Po-Shen Hsin
(UCLA)

Guanyu Zhu
(IBM)

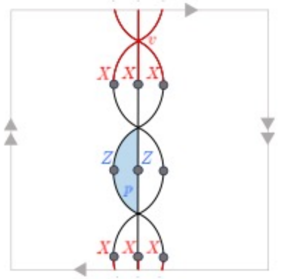
Maissam Barkeshli, Po-Shen Hsin, RK, arXiv: 2311.05674

“Higher-group symmetry of (3+1)D fermionic Z2 gauge theory: logical CCZ, CS, and T gates from higher symmetry”

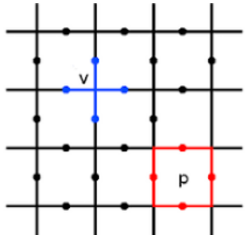
RK and Guanyu Zhu, arXiv: 2310.06917

“Fault-tolerant logical gates via constant depth circuits and emergent symmetries on non-orientable topological stabilizer and Floquet codes”

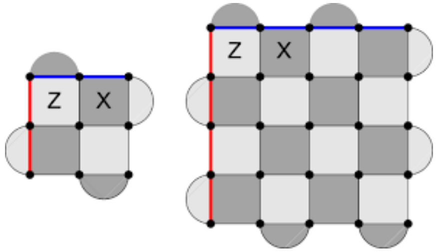
Perspective of stabilizer codes...



Shor's 9-digit code



Toric code



Surface code

Haah's code

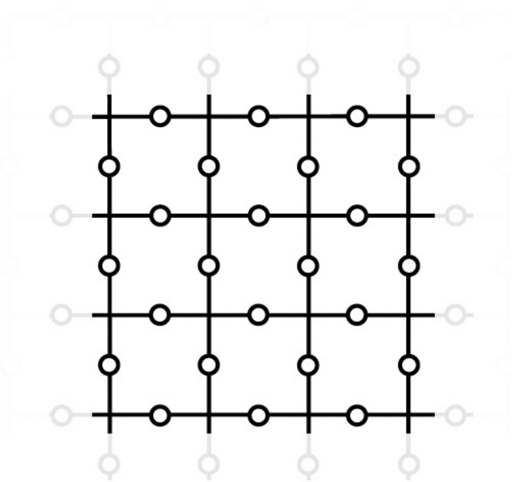
...

Quantum Information	Condensed Matter	High Energy
(Pauli) Stabilizer group $\mathcal{S} = \langle S_1, S_2, \dots \rangle$ $[S_i, S_j] = 0$	Stabilizer Hamiltonian $H = -\sum_j S_j$	Topological Quantum Field Theory (TQFT)
Code space $\mathcal{S} \lvert\psi\rangle = \lvert\psi\rangle$	Ground state subspace (topological ordered phase)	Hilbert space of TQFT
Logical operators (Normalizer of *S*)	(Emergent) symmetry of Hamiltonian	Symmetry of TQFT

Z2 toric code in 2+1D

$$H = -\sum_{v} \; X X X X \; - \sum_{p} \; Z Z Z Z$$

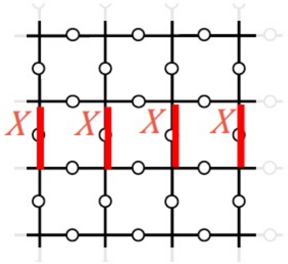
On a torus, ground state degeneracy is 2^2. Two logical qubits.



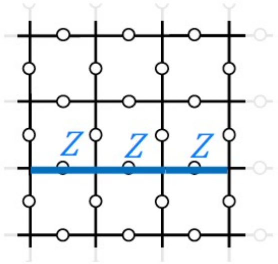
Qubits on edges

Quantum Information	Condensed Matter	High Energy
Two logical qubits	Z2 topological ordered phase	Z2 gauge theory
Pauli X, Z gate	e, m particle (anyons)	Wilson/’t Hooft operator

Anyons…



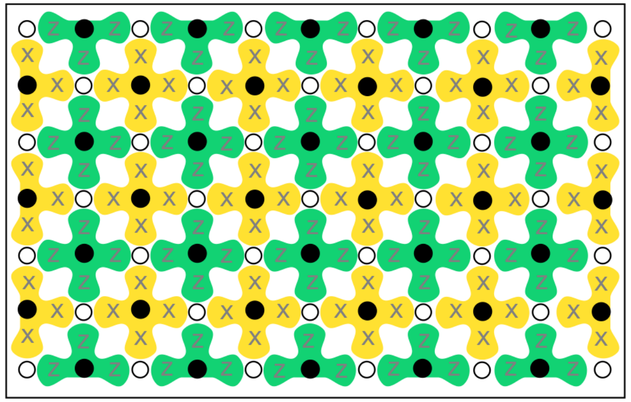
m particle



e particle

Z2 toric code and topological order in Lab

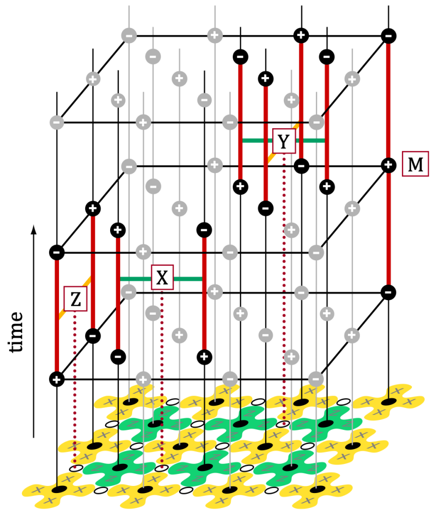
Rather than a Hamiltonian system, it is a sequence of measurements:



Step 1: measure X stabilizers to detect Z error ⟶ correct

Step 2: measure Z stabilizers to detect X error ⟶ correct

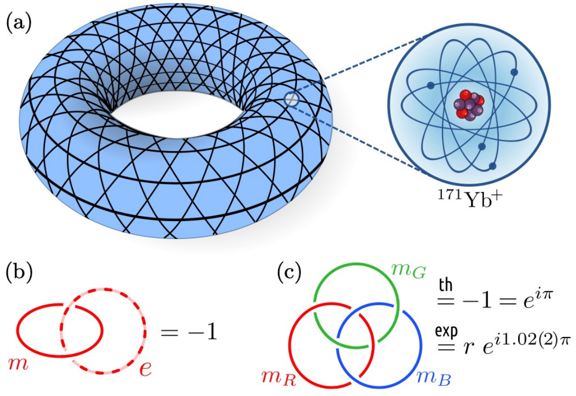
Repeat the above process



Picture: [Fowler=Mariantoni=Martinis=Cleland]

➢ Recent experiment on realizing non-Abelian topological order (D8 gauge theory)

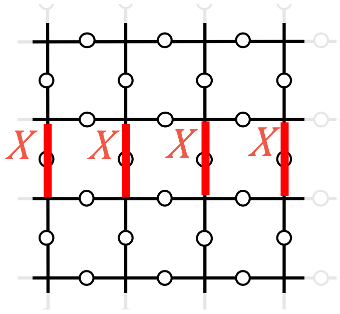
[Iqbal et al (Quantinuum)]



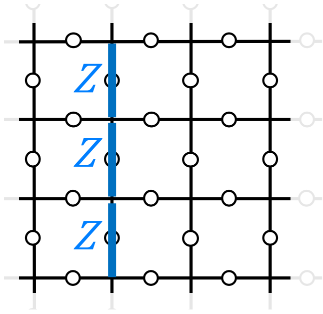
Logical gate of 2+1D Z2 toric code: symmetry of quantum code

Logical gates:

"wrapping an anyon"

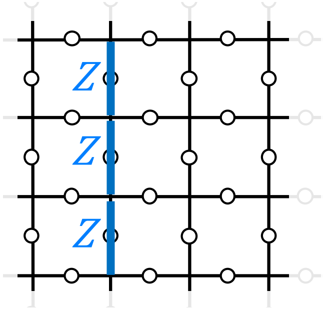


Pauli X gate:
m particle

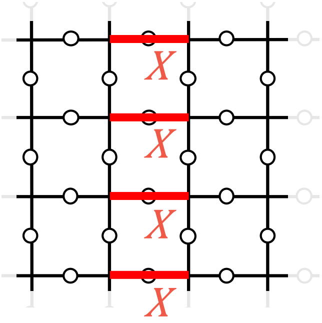


Pauli Z gate:
e particle

In addition, the 2d Z2 toric code (on torus) has a Z2 symmetry exchanging e ⟷ m:



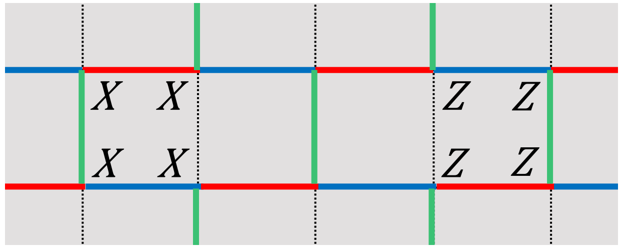
⟷



The e ⟷ m symmetry gives rise to Hadamard-like* logical gate.

(*technically it's H1H2 SWAP)

Logical gate of 2+1D Z2 toric code

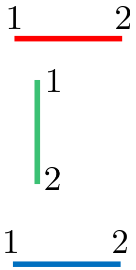


Qubits on vertices

$U_{\mathrm{R}}: \quad \exp\left(\frac{i\pi}{4} Z_1 X_2\right)$

$U_{\mathrm{G}}: \quad \exp\left(\frac{i\pi}{4} Y_1 Y_2\right)$

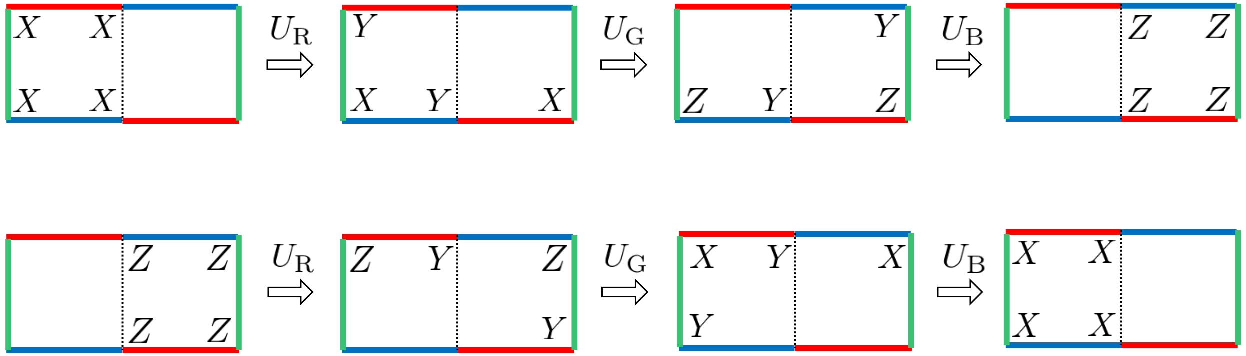
$U_{\mathrm{B}}: \quad \exp\left(\frac{i\pi}{4} Z_1 X_2\right)$



$U = U_B \, U_G \, U_R$

Local constant-depth circuit

Permutes X and Z stabilizers: e ⟷ m exchange

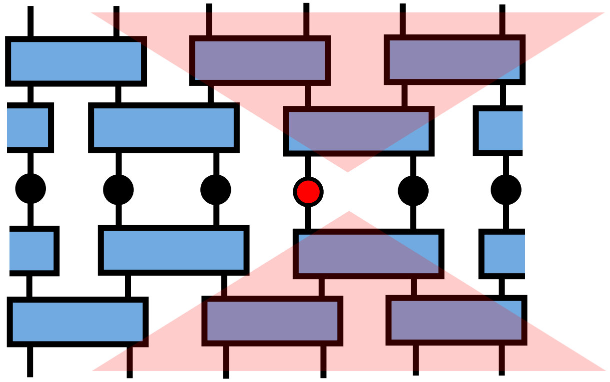


Clifford gates in (2+1)D by local constant-depth circuit

- Fault-tolerant logical gate is realized by a local constant-depth circuit.

  Error can propagate along the shallow light cone, which is local

  (transversal gate = onsite symmetry)



There's a constraint that local constant-depth circuit in 2d can only implement logical Clifford gate [Bravyi=Koenig]

(for local stabilizer codes)

$U(\text{Pauli})U^{-1} = \text{Pauli}$

But Clifford gates alone cannot do universal quantum computation.

(Clifford quantum circuit can be simulated by classical computer)

[Gottesman=Knill]

Non-Clifford gates

Gapped boundaries, group cohomology and fault-tolerant logical gates

Beni Yoshida
Walter Burke Institute for Theoretical Physics, California Institute of Technology, Pasadena, California 91125, USA
Perimeter Institute for Theoretical Physics, Waterloo, Ontario N2L 2Y5, Canada and
Kavli Institute for Theoretical Physics, University of California, Santa Barbara, CA 93106, USA
(Dated: September 14, 2015)

- Non-Clifford gate via local constant depth circuit: 3+1D code

  Need 3d to realize fault-tolerant non-Clifford gate, which is necessary for universal QC

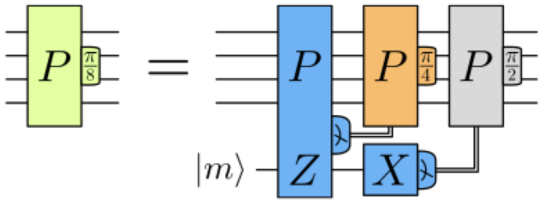
  Clifford + a single non-Clifford = Universal.

  Example: CCZ gate in 3d Z2^3 toric code. Use 0-form symmetry in (3+1)D Z2^3 gauge theory. [Yoshida]

In 2+1D...

- Non-Clifford gate in 2+1D via magic state distillation

  Prepare an ancilla state 'magic state' to implement non-Clifford gate on the code



[Bravyi=Kitaev]

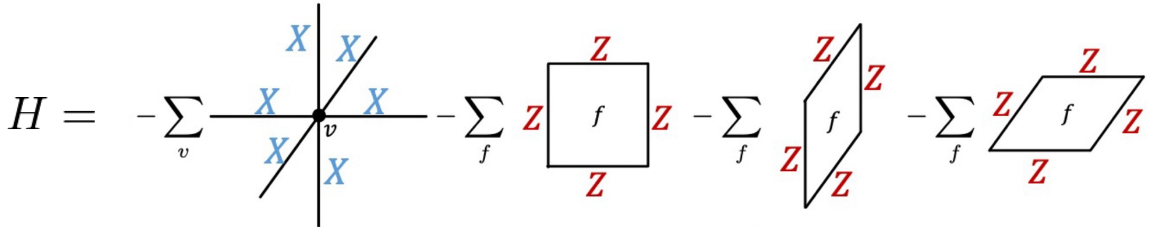
- Non-Clifford gate in 2+1D via entangling it with 3+1D stabilizer code (code switching)

[Bombin]

[Beverland=Kubica=Svore]

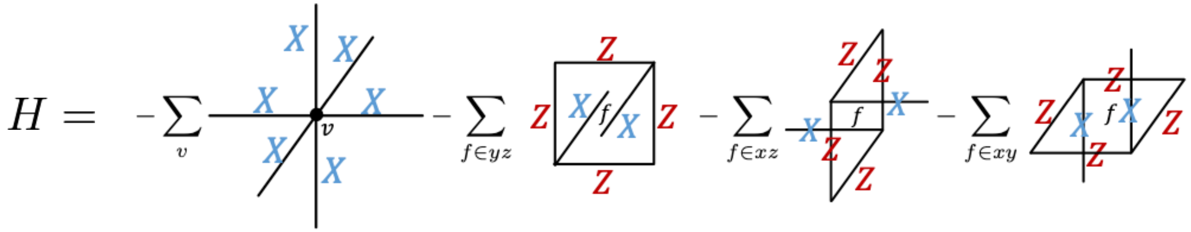
3+1D Z2 toric code with emergent fermion

- 3+1D Z2 toric code with emergent boson particle (Qubits on edges of cubic lattice)



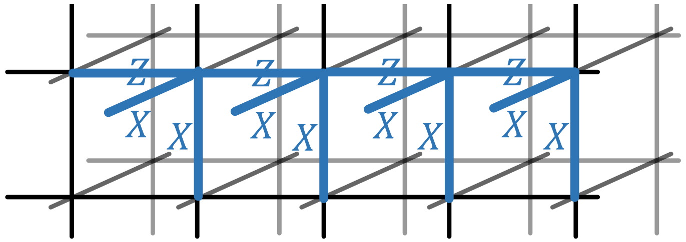
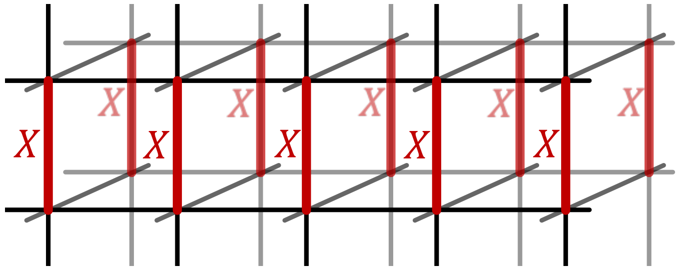
(3+1)D Z2 gauge theory. Action of effective Z2 gauge theory... $\pi \int da \cup b$

- 3+1D Z2 toric code with emergent fermion particle



(3+1)D Z2 gauge theory with emergent fermion. Action of effective Z2 gauge theory... $\pi \int da \cup b + b \cup b$

Pauli gates of 3+1D Z2 toric code w/ fermion

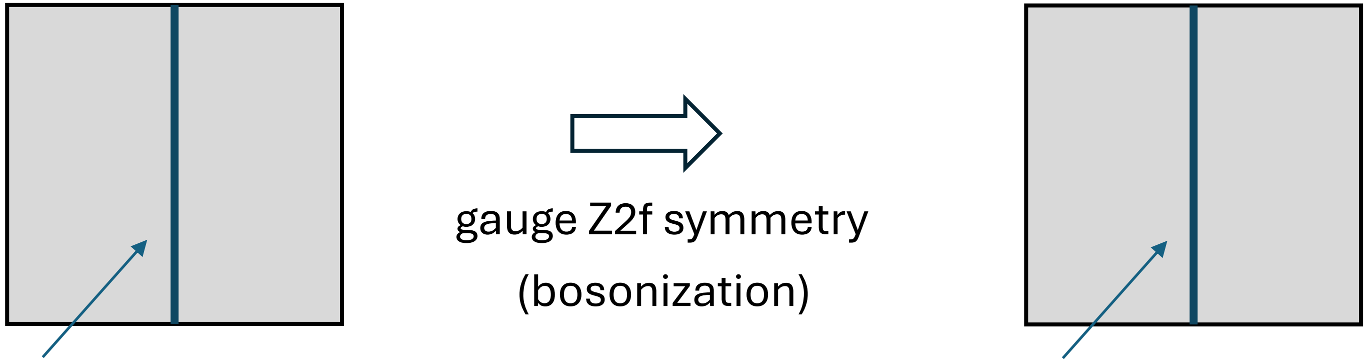
Quantum Information	Condensed Matter	Global symmetry
Pauli Z gate 	Emergent fermion	Z2 2-form symmetry
Pauli X gate 	Magnetic flux	Z2 1-form symmetry

What else?

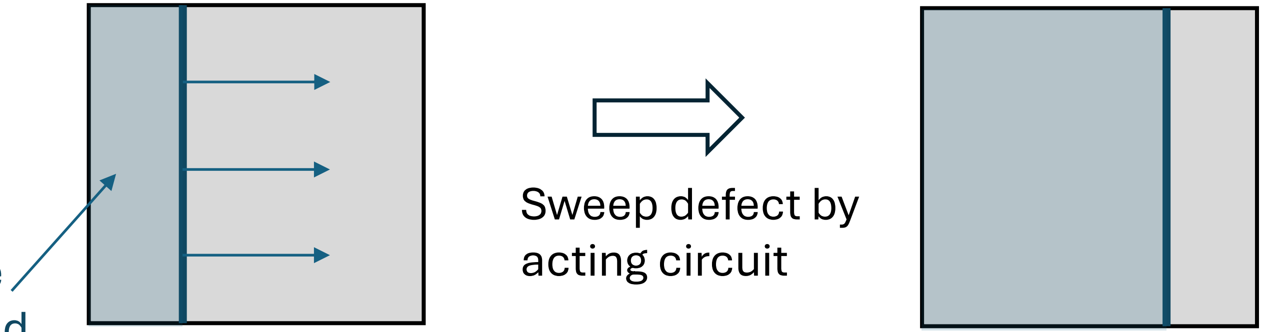
Other logical gates come from generalized symmetry of Z2 gauge theory

Fault-tolerant logical gate, via pumping topological phase

- Z2 toric code w/ emergent fermion is regarded as “Z2f gauge theory”

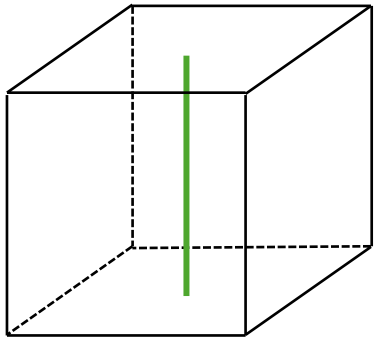


- One can sweep the defect by (quasi-) local constant-depth circuit, which results in a logical gate.



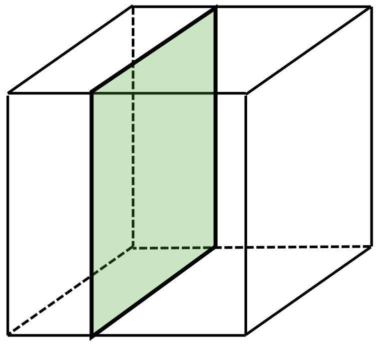
More symmetry of 3d Z2 toric code with emergent fermion

- Sweeping invertible topological phase w/ Z2f symmetry defines emergent invertible symmetry.

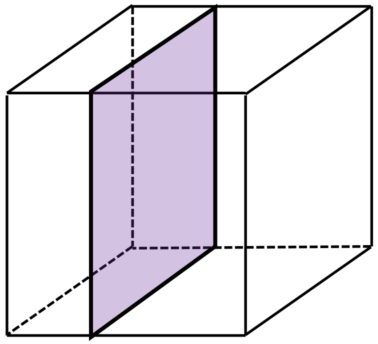


Kitaev chain defect

Sweep defect by acting circuit



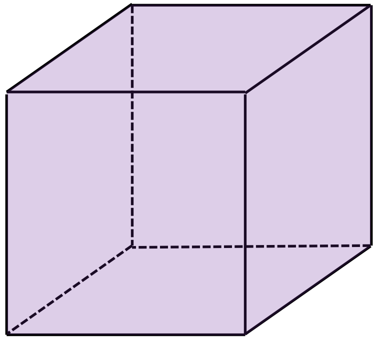
Z2 (1-form) symmetry



p+ip superconductor defect

chiral phase with c- = 1/2

Sweep defect by acting circuit



Z8 (0-form) symmetry

[Barkeshli=Hsin=RK]

Symmetry of Z2 gauge theory in (3+1)D: when electric particle is a fermion

Action: $\pi \int da \cup b + w_2 \cup b$ "dynamical spin structure" $da = w_2$

Let's consider invertible symmetry of (3+1)D Z2 gauge theory w/ fermion:

- Z2 2-form symmetry generated by electric line operator. $\pi \int_\gamma a$
- Z2 1-form symmetry generated by magnetic surface operator. $\pi \int_\Sigma b$
- Z2 1-form symmetry generated by Kitaev's Majorana chain (1+1D spin invertible phase):

  $\mathrm{Arf}(\Sigma)$ on a surface Σ

- Z8 0-form symmetry generated by gravitational CS theory (p+ip superconductor, 2+1D spin invertible phase):

  "p+ip symmetry defect" $\mathrm{CS}_{\mathrm{grav}}(M^3)$

0-form symmetry of (3+1)D fermionic Z2 gauge theory

Let’s look more at 0-form symmetry of (3+1)D fermionic Z2 gauge theory; why Z8?

- Fusion rule of 0-form symmetry follows the stacking law of (2+1)D spin invertible phase, classified by $\Omega^3_{\mathrm{Spin}}(\mathrm{pt}) = \mathbb{Z}$

  So most naively, the 0-form symmetry is Z. But some of them doesn’t act faithfully on Hilbert space.

- Reduction from Z to Z16 happens, since 16 copies of p+ip superconductors defines bosonic E8 phase.

  So, symmetry operator for 16Z is decoupled from dynamical spin structure, i.e., trivial operator on Hilbert space

- Further reduction to Z8 happens.

  8 copies of p+ip superconductors does not depend on spin structure, though it needs spin structure to be defined.

  i.e., trivial operator on Hilbert space.

Z4 subgroup of 0-form symmetry: 3-group symmetry

Let's consider the Z4 subgroup of the Z8 0-form symmetry.

Z4 0-form symmetry, together with 1, 2-form symmetry, forms 3-group structure of invertible symmetries

- Z4 (subgroup) 0-form symmetry. background C_1
- Z2 1-form symmetry generated by magnetic surface. background B_2
- Z2 1-form symmetry generated by Kitaev's Majorana chain. background C_2
- Z2 2-form symmetry generated by electric Wilson line. background C_3

3-group equation:

$$dC_3 = Sq^2(C_2) + B_2 \cup C_2 + \left(\frac{d\tilde{B}_2}{2} + w_3\right) \cup C_1 + (B_2 + w_2) \cup \frac{d\tilde{C}_1}{4}$$

3-group involving Kitaev chain 1-form symmetry

3-group involving p+ip 0-form symmetry

[Kapustin=Thorngren, Wang=Gu, Barkeshli=Chen=Hsin=RK]

Whole structure of 0,1,2-form symmetry becomes non-invertible

Let's study the whole algebraic structure of the symmetry.

First, the 0-form symmetry (p+ip) induces permutation of 1-form symmetry generators:

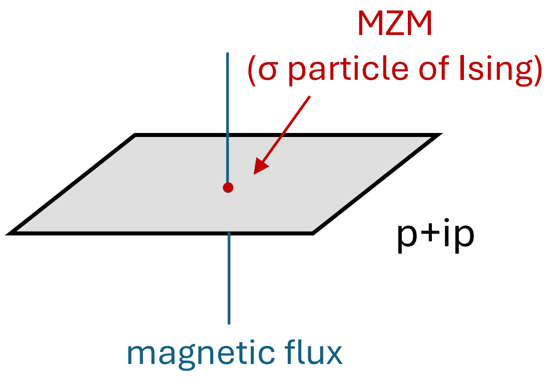
$$U^{(2)} \to U^{(2)}, \quad V^{(2)} \to V^{(2)} U^{(2)}$$

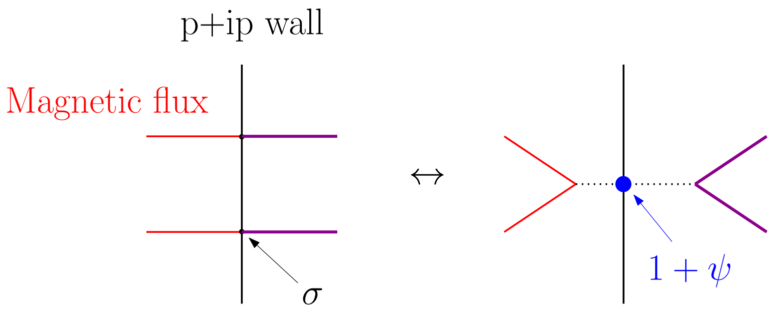
[Johnson-Freyd, Yang=Cheng]

$V^{(2)}$: magnetic surface operator $\qquad$ $U^{(2)}$: Kitaev chain surface operator

Intersection between magnetic surface and p+ip defect bounds a Majorana zero mode

Fusion rule σ x σ = 1 + ψ, σ x ψ = σ implies that total symmetry structure is non-invertible.





”defect-valued associator”

<table>
<tr><th>Quantum Information</th><th>Condensed Matter</th><th>Global symmetry</th></tr>
<tr><td>?</td><td>Decorate Kitaev chain, and sweep it</td><td>Z2 1-form symmetry</td></tr>
<tr><td>?</td><td>Decorate p+ip superconductor, and sweep it</td><td>Z8 0-form symmetry</td></tr>
<tr><td colspan="2">?</td><td>Higher-group/Non-invertible symmetry</td></tr>
</table>

1-form symmetry generated by Kitaev chain: logical Clifford CZ or S gate

- Kitaev chain defect:

  Fermion is created at the intersection between m surface and defect

  This effect is translated into a commutation relation of operators

$$(\text{Kitaev})X(\text{Kitaev})^{-1}X^{-1} \propto Z$$

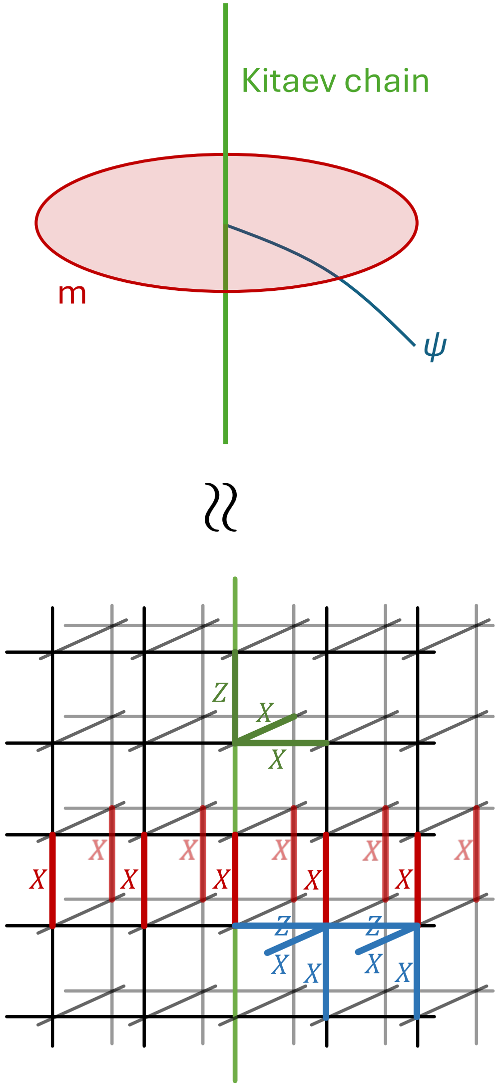
Kitaev chain surface, Clifford gate

magnetic surface, Pauli X

fermion line, Pauli Z

This implies that the Kitaev chain can do CZ or S like Clifford gate.

[RK=Zhu]



- Non-trivial algebraic mixture between 1-form and 2-form symmetry: called a higher-group (3-group)

1-form symmetry generated by Kitaev chain: logical Clifford CZ or S gate

- Generic expression for logical gate (space is generic 3-manifold)

$$\mathcal{W}_{\mathrm{K}}(\Sigma_j) = \prod_{\substack{k,l \\ k<l, j\neq k, j\neq l}} CZ_{k,l}^{\int_{M^3} \sigma_j \sigma_k \sigma_l} \cdot \prod_{\substack{k \\ j\neq k}} CZ_{j,k}^{\int_{M^3} \sigma_j \sigma_k \sigma_k} (S_k^\dagger)^{\int_{M^3} \sigma_j \sigma_k \sigma_k} \cdot (e^{\frac{2\pi i}{8}} S_j^\dagger)^{\int_{M^3} \sigma_j \sigma_j \sigma_j}.$$

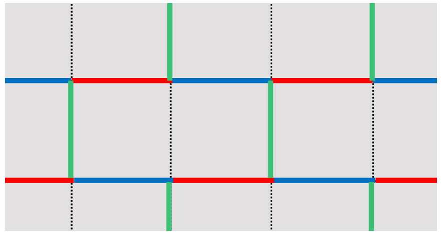
Kitaev chain logical gate

Indices of logical qubits

σ_j : Poincare dual of Σ_j

S = diag(1,i)

- Kitaev chain operator can be expressed by a local constant-depth circuit, so fault-tolerant



Qubits on vertices

$U_{\mathrm{R}}: \quad \exp\left(\frac{i\pi}{4} Z_1 X_2\right)$

$U_{\mathrm{G}}: \quad \exp\left(\frac{i\pi}{4} Y_1 Y_2\right)$

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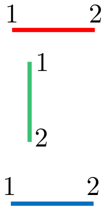


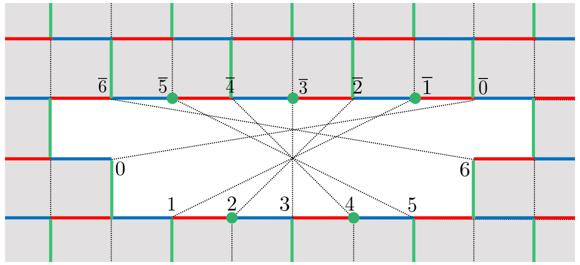
$U = U_B U_G U_R$

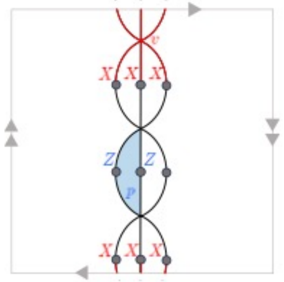
circuit for Kitaev chain operator

- Kitaev chain circuit can also be defined on an unoriented surface: S gate

  e.g., Z2 toric code on a surface w/ cross-cap [RK=Zhu]

  (FYI: Shor's 9-digit code = Z2 toric code on real projective plane)





Shor's code

Non-Clifford gate of 3d Z2 toric code w/ fermion: pumping a p+ip superconductor

Commutation relation of operators: $(p+ip)X(p+ip)^{-1}X^{-1} \propto \mathrm{Kitaev}$ ... Non-Clifford gate

- Generic expression of p+ip logical gate on any oriented 3-manifold:

$$U = \prod_{j<k<l} (CCZ_{j,k,l})^{\int_{M_3} \sigma_j\sigma_k\sigma_l} \cdot \prod_{j<k} (CS^{\dagger}_{j,k})^{\int_{M_3} \sigma_j\sigma_k\sigma_k} \cdot \prod_{j} (T_j)^{\int_{M_3} \sigma_j\sigma_j\sigma_j}$$

Indices of logical qubits

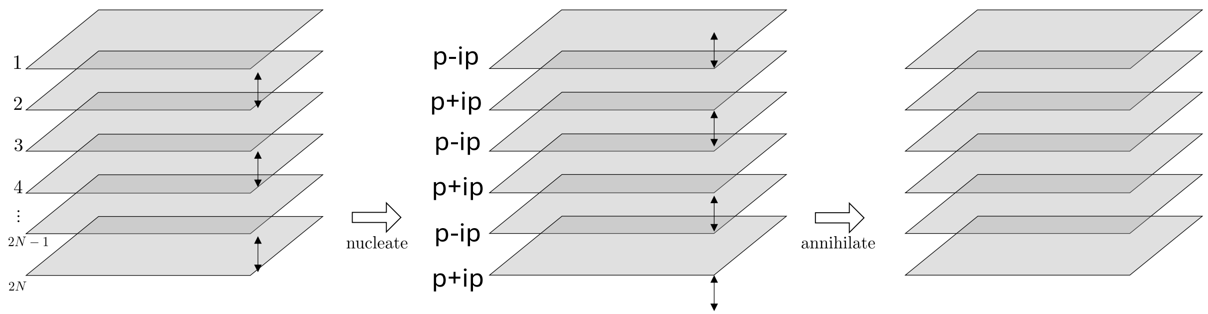
σ_j : Poincare dual of Σ_j

S = diag(1,i)

T = diag(1,exp(2πi/8))

It can implement non-Clifford CCZ, Controlled-S, T gate.

- Pumping p+ip superconductor in layered system with periodic boundary condition can do CCZ gate



[Fidkowski=Hastings]

$$U = U_{\mathrm{annih}} U_{\mathrm{nucl}}$$

This can give rise to an exact symmetry of the (3+1)D Z2 toric code w/ fermion (after bosonization).

The unitary U is a finite time evolution by local Hamiltonian w/ exponentially decaying tails

-> fault tolerance is expected

Non-Clifford gate of 3d Z2 toric code w/ fermion: pumping a p+ip superconductor

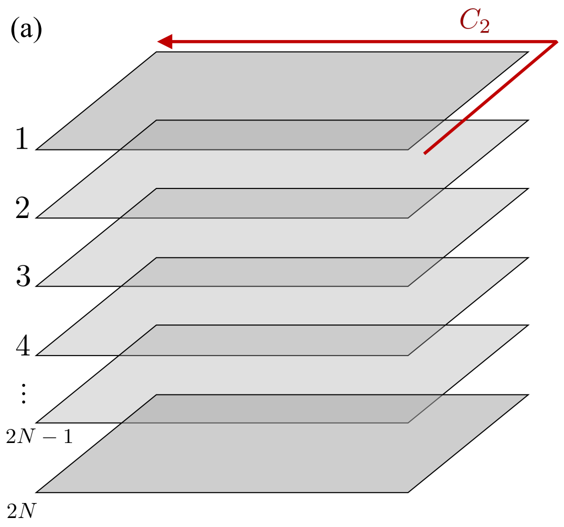
- Generic expression of p+ip logical gate on any oriented 3-manifold:

$$U = \prod_{j<k<l} (CCZ_{j,k,l})^{\int_{M_3} \sigma_j \sigma_k \sigma_l} \cdot \prod_{j<k} (CS^{\dagger}_{j,k})^{\int_{M_3} \sigma_j \sigma_k \sigma_k} \cdot \prod_{j} (T_j)^{\int_{M_3} \sigma_j \sigma_j \sigma_j}$$

It can implement non-Clifford CCZ, Controlled-S, T gate.

- Pumping p+ip superconductor in layered system with "C2 rotation-twisted" boundary condition can do CS gate

This can make a space $T^2 \rtimes S^1$



$$U = U_{\text{annih}} U_{\text{nucl}}$$

yz plane is non-orientable, so S gate

$$\text{Kitaev}_{y,z} = CZ_{y,z} CZ_{x,y} S^{\dagger}_y$$

$$\text{p} + \text{ip} = CCZ_{x,y,z} CS^{\dagger}_{x,y}$$

Non-Clifford gate of 3d Z2 toric code w/ fermion: pumping a p+ip superconductor

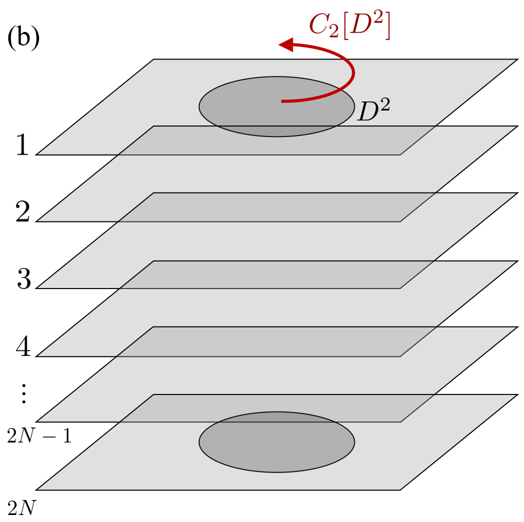
- Generic expression of p+ip logical gate on any oriented 3-manifold:

$$U = \prod_{j<k<l} (CCZ_{j,k,l})^{\int_{M_3} \sigma_j \sigma_k \sigma_l} \cdot \prod_{j<k} (CS^\dagger_{j,k})^{\int_{M_3} \sigma_j \sigma_k \sigma_k} \cdot \prod_j (T_j)^{\int_{M_3} \sigma_j \sigma_j \sigma_j}$$

It can implement non-Clifford CCZ, Controlled-S, T gate.

- It also suggests possibility of logical T gate by pumping p+ip superconductor through a 3-manifold RP3

RP3 topology is realized by a ‘‘partial C2 rotation’’, which can make a space $T^3 \# \mathbb{RP}^3$



$$U = U_{\text{annih}} U_{\text{nucl}}$$

non-unitary on ground state code space,
But defines non-Clifford logical gate after projecting onto code space.

On a logical qubit implemented by RP3, we have $\text{Kitaev} = e^{\frac{2\pi i}{8}} S^\dagger$

$$\text{p} + \text{ip} = T$$

Quantum Information	Condensed Matter	Global symmetry
Control-Z or S gate (depending on choice of surface)	Decorate Kitaev chain, and sweep it	Z2 1-form symmetry
Non-Clifford CCZ, CS, or T gate (depending on choice of 3-manifold)	Decorate p+ip superconductor, and sweep it	Z8 0-form symmetry
Commutation relation of logicals	$(\text{Kitaev})X(\text{Kitaev})^{-1}X^{-1} \propto Z$ $(p+ip)X(p+ip)^{-1}X^{-1} \propto \text{Kitaev}$	Higher-group/Non-invertible symmetry

Future work

- Developing the more complete algebraic description of non-invertible symmetry of toric code
- Symmetry of general finite gauge theory with emergent fermions
- Fault tolerance of T gate with partial rotation?
- Role of non-invertible symmetry in error correction context?
- New logical gates of other codes from symmetry, e.g., Floquet codes, quantum LDPC codes, etc

Backup slides

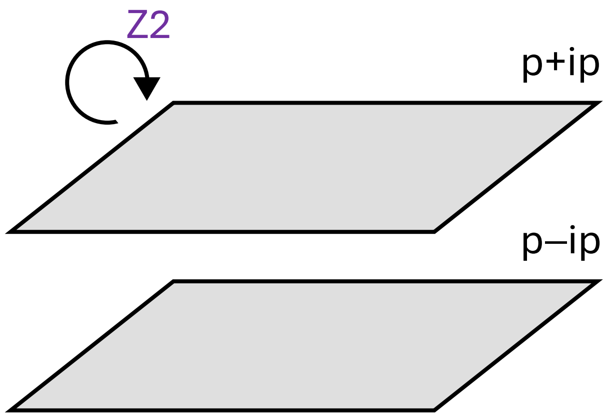
0-form symmetry of (3+1)D fermionic Z2 gauge theory

Let's look more at 0-form symmetry of (3+1)D fermionic Z2 gauge theory; why Z8?

- One (indirect) way to see Z8 reduction is consider (p+ip) x (p-ip), which is spin SPT phase w/ Z2 symmetry

  Classified by torsion of $\Omega^3_{\mathrm{Spin}}(B\mathbb{Z}_2) = \mathbb{Z}_8 \times \mathbb{Z}$

  This implies that 8 copies of (p+ip) x (p-ip) doesn't depend on spin structure of each layer.



Z2 background gauge field: "difference in spin structure of two layers"

0-form symmetry of (3+1)D fermionic Z2 gauge theory

Let's look more at 0-form symmetry of (3+1)D fermionic Z2 gauge theory; why Z8?

- One can directly study spin-structure dependence for 8 copies of p+ip phase.

2 copies of p+ip: U(1) (spin) Chern-Simons theory

$$\frac{1}{4\pi}udu + \pi\frac{du}{2\pi}a = \frac{1}{4\pi}u'du' - \frac{\pi}{4}ada$$

$da = w_2$: spin structure

$u' = u + \pi a$

Spin structure dependence of 2 copies: $-\frac{\pi}{4}\int ada$

Spin structure dependence of 8 copies: $-\pi\int ada$

Effect of shifting spin structure: $\pi\int -(a+B)d(a+B) + ada = -\pi\int 2Bda + BdB = 0$ mod 2π

Z8 reduction of 0-form symmetry v.s. mixed gravitational anomaly

The 0-form symmetry has a mixed gravitational anomaly.

$$\frac{2\pi}{16}\int \tilde{C}_1 \cup (p_1/3)$$

This originates from framing anomaly of gravitational CS theory carrying $c_- = 1/2$.

One puzzle is that this is not a well-defined action of Z8 gauge field, since it's not invariant under $C_1 \to C_1 + 8\lambda_1$

while it makes sense when C1 were Z16 gauge field.

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Solution is to turn on background for 2-form symmetry generated by Wilson line

$$\frac{2\pi}{16}\int \tilde{C}_1 \cup (p_1/3) + \pi \int C_3 \cup w_2$$

Invariant under the gauge transformation $C_1 \to C_1 + 8\lambda_1, \quad C_3 \to C_3 + \lambda_1 \cup w_2$ ($p_1/3 = w_2^2$ mod 2)

anomaly of Z8 symmetry is well-defined, with non-trivial mixture with 2-form symmetry: symptom of 3-group

Whole structure of 0,1,2-form symmetry becomes non-invertible

Let's study the whole algebraic structure of the symmetry.

First, the 0-form symmetry (p+ip) induces permutation of 1-form symmetry generators:

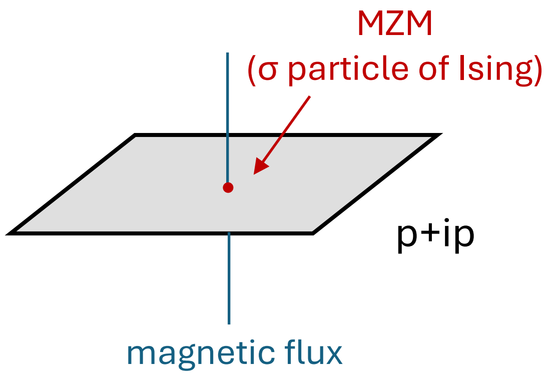
$$U^{(2)} \to U^{(2)}, \quad V^{(2)} \to V^{(2)}U^{(2)}$$

[Johnson-Freyd, Yang=Cheng]

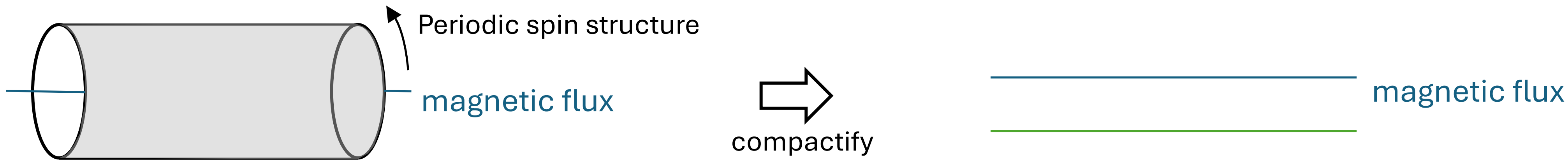
$V^{(2)}$: magnetic surface operator

$U^{(2)}$: Kitaev chain surface operator

Intersection between magnetic surface and p+ip defect bounds a Majorana zero mode, which is a termination of the (1+1)D spin invertible phase, i.e., Kitaev's Majorana chain



- this can be seen from compactifying Majorana fermion:



(2+1)D Majorana fermion with negative mass

(1+1)D Majorana fermion with negative mass

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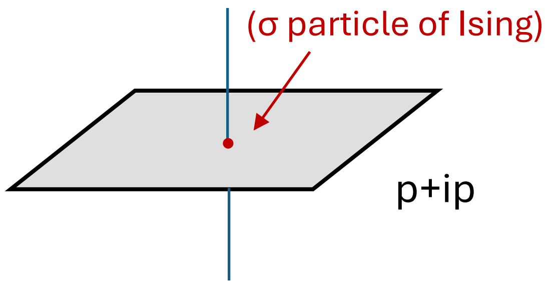
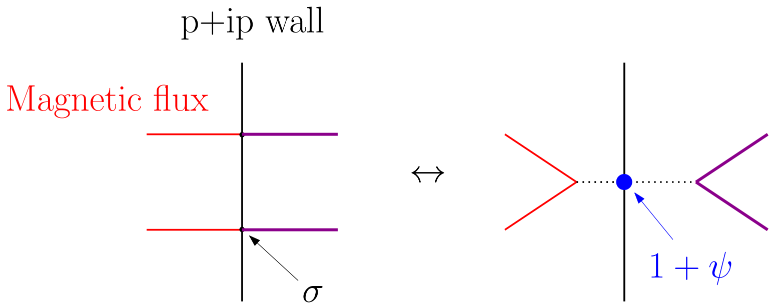


Fusion rule σ x σ = 1 + ψ, σ x ψ = σ implies that total symmetry structure is non-invertible.



"defect-valued associator"

Z4 subgroup of 0-form symmetry: 3-group symmetry

Let's consider the Z4 subgroup of the Z8 0-form symmetry.

Z4 0-form symmetry, together with 1, 2-form symmetry, forms 3-group structure of invertible symmetries

- Z4 (subgroup) 0-form symmetry. background C_1
- Z2 1-form symmetry generated by magnetic surface. background B_2
- Z2 1-form symmetry generated by Kitaev's Majorana chain. background C_2
- Z2 2-form symmetry generated by electric Wilson line. background C_3

3-group equation:

$$dC_3 = Sq^2(C_2) + B_2 \cup C_2 + \left(\frac{d\tilde{B}_2}{2} + w_3\right) \cup C_1 + (B_2 + w_2) \cup \frac{d\tilde{C}_1}{4}$$

3-group involving Kitaev chain 1-form symmetry

3-group involving p+ip 0-form symmetry

[Kapustin=Thorngren, Wang=Gu, Barkeshli=Chen=Hsin=RK]

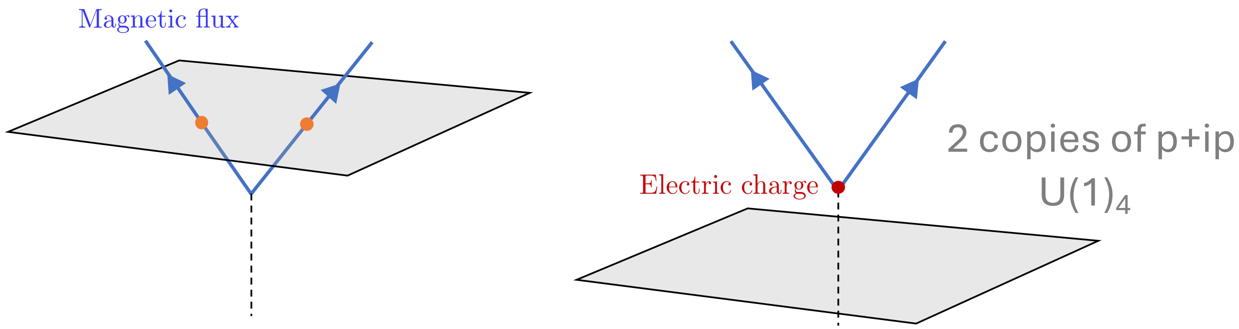
Z4 subgroup of 0-form symmetry: 3-group symmetry

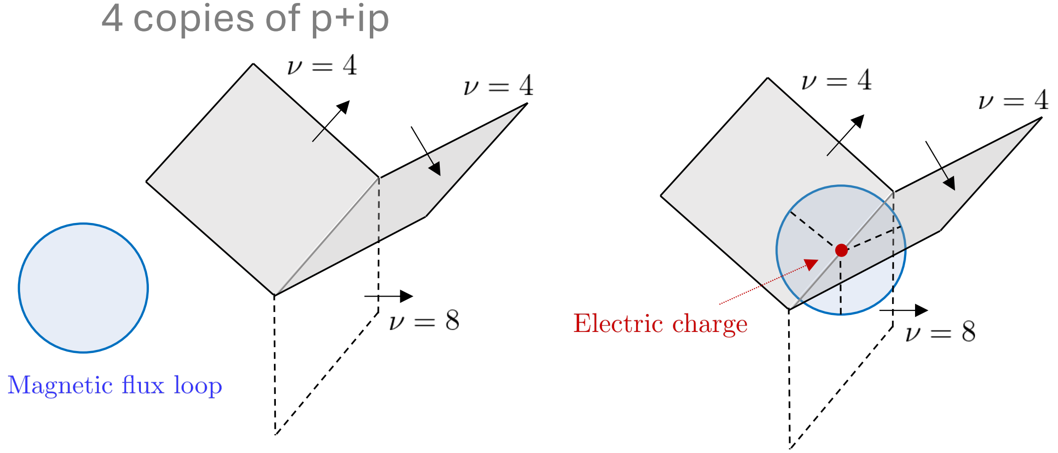
Let us comment on physical interpretation of the 3-group involving 0-form symmetry:

$$dC_3 = \left(\frac{d\tilde{B}_2}{2} + w_3\right) \cup C_1 + (B_2 + w_2) \cup \frac{d\tilde{C}_1}{4}$$

electric — magnetic — Z4 0-form

- first term $\left(\frac{d\tilde{B}_2}{2} + w_3\right) \cup C_1$
- second term $(B_2 + w_2) \cup \frac{d\tilde{C}_1}{4}$





Junction of magnetic defects go through 2 copies of p+ip.

Fusion rule v x v = ψ produces an electric particle

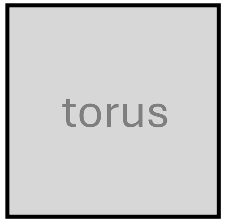
Magnetic defect intersecting with junction of 0-form symmetry produces a fermion (in 8 copies of p+ip, flux is fermion)

1-form symmetry generated by Kitaev chain: logical Clifford CZ or S gate

$$\mathcal{W}_{\mathrm{K}}(\Sigma_j) = \prod_{\substack{k,l \\ k<l, j\neq k, j\neq l}} CZ_{k,l}^{\int_{M^3} \sigma_j \sigma_k \sigma_l} \cdot \prod_{\substack{k \\ j\neq k}} CZ_{j,k}^{\int_{M^3} \sigma_j \sigma_k \sigma_k} (S_k^{\dagger})^{\int_{M^3} \sigma_j \sigma_k \sigma_k} \cdot (e^{\frac{2\pi i}{8}} S_j^{\dagger})^{\int_{M^3} \sigma_j \sigma_j \sigma_j}.$$

Kitaev chain logical gate

- CZ gate originates from Arf invariant on oriented surface $\mathcal{W}_{\mathrm{K}}(\Sigma_j) = \mathrm{Arf}(\Sigma_j)$



AP/P (= 0/1)

AP/P (= 0/1)

spin structure	Arf
$\|00\rangle$	1
$\|01\rangle$	1
$\|10\rangle$	1
$\|11\rangle$	–1

diag(1,1,1,-1) = CZ

- S gate originates from Arf-Brown-Kervaire invariant on unoriented surface $\mathcal{W}_{\mathrm{K}}(\Sigma_j) = \mathrm{ABK}(\Sigma_j)$

8^{th} root of unity