Strong coupling dynamics of 3d SCFT:

properties and applications of superconformal index

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6th Asian Winter School on Strings, Particles and Cosmology

Kusatsu, Japan (10 – 20 January 2012)

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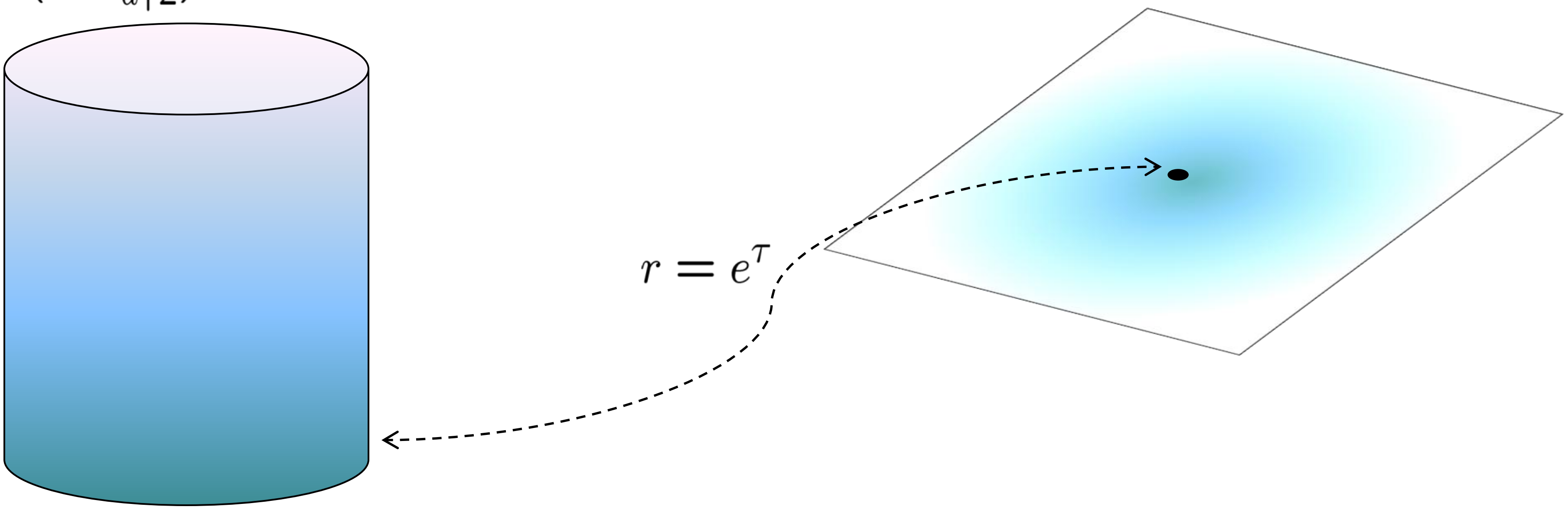
Motivation

- CFT on flat space: continuous spectrum
- If CFT can be put on (curved space) x R(time): discrete spectrum
- CFT on S^d x R: radial quantization

$\partial(AdS_{d+2}) = S^d \times R$ (boundary of global AdS)

CFT on R^{d+1}



- Local operators at origin (past) create states: operator-state map
- Dilatation ~ Hamiltonian

Motivation: radial quantization

- Example: free 3d scalar

$$\phi(x) = \frac{1}{r^{1/2}}\phi_S(x)$$

$$\begin{aligned}\int d^3x\partial_\mu\phi\partial^\mu\phi &= \int r^2 dr d\Omega_2 \left[\left(\frac{1}{r^{1/2}}\partial_r\phi_S - \frac{1}{2r^{3/2}}\phi_S\right)^2 + \frac{1}{r^2}g^{ij}\partial_i(r^{-1/2}\phi_S)\partial_j(r^{-1/2}\phi_S)\right] \\ &= \int d\tau d\Omega_2 \left[(\partial_\tau\phi_S)^2 + (\nabla_i\phi_S)^2 + \boxed{\frac{1}{4}(\phi_S)^2} - \cancel{\phi_S\partial_\tau\phi_S}\right]\end{aligned}$$

conformal mass term

- Works with conformal interactions
- Gauge fields, fermions can also be suitably transformed (guaranteed by conformal invariance): actually, possible in any conformally flat background
- More on S^3 CFT tomorrow…

Motivation

- Partition function: $Z = \mathrm{Tr}[\, e^{-H/T}]$ depends on the detailed dynamics of the theory, depends sensitively on the coupling constants

- "Index" in supersymmetric theories: $I = \mathrm{Tr}[\, (-1)^F e^{-(H + \dots)/T}]$

$$\text{non BPS}: \ |B\rangle \overset{Q,Q^\dagger}{\longleftrightarrow} |F\rangle \ : \ \text{doesn't contribute to index}$$

- When BPS states cease to be so as continuous couplings are changed, it always happens in boson/fermion pairs: index is insensitive to it.

- Depends less sensitively on coupling constants: no continuous coupling

- May depend on discrete couplings: still, one can calculate strong coupling spectrum relying on SUSY (example: CS coupling k)

Superconformal algebra

- 3d superconformal algebra (important for our purpose):

$$2\{Q^i_\alpha, S^{j\beta}\} = \delta^{ij}\delta^\beta_\alpha D + R^{ij}\delta^\beta_\alpha + \delta^{ij} J_\alpha{}^\beta$$

mutually Hermitian conjugate
(after radial quantization)

- Pick an N=2 subalgebra, & a pair of Hermitian-conjugate supercharges

$$Q_\alpha \equiv Q^1 + iQ^2 \ , \quad \bar{Q}_\alpha \ , \quad S_\alpha \ , \quad \bar{S}_\alpha : \ \{Q_\alpha, S^\beta\} = \delta^\beta_\alpha D - R\delta^\beta_\alpha + J_\alpha{}^\beta$$

$$Q \equiv Q_- \ , \quad S \equiv S^- : \ \{Q, S\} = D - R - J_3$$

- BPS bound:

$$D \geq R + J_3 \quad (\text{check}: D(Q) = 1/2,\ R(Q) = 1,\ J_3(Q) = -1/2)$$

Definition of the index

- Cartans of the algebra (& flavor symmetry) commuting with Q:

$$[D+J_3,Q]=0,\ \ [D-R-J_3,Q]=2[\{Q,S\},Q]=0,\ \ [\text{flavor},Q]=0$$

- Index: counts "states on sphere" or "operators on flat space"

$$I(x)=\mathrm{Tr}\left[(-1)^F(x')^{\{Q,S\}}x^{D+j_3}(\text{other flavors})\right]$$

- Pairs of states NOT annihilated by Q,S cancel out: Witten index

$$Q|B\rangle=|F\rangle\neq 0\ ,\quad S|F\rangle=\frac{1}{D-R-j_3}|B\rangle:\ \text{same } H+j_3 \text{ and flavor charges}$$

- chemical potential analogous to inverse-temperature: β with $x=e^{-\beta}$

Definition of the index

- Path integral formulation on $S^2 \times S^1$: periodic b.c.

$$\text{operator insertions}: \; x^{H+j_3}(x')^{H-R-j_3} = e^{-(\beta+\beta')H} e^{\beta' R} e^{(\beta'-\beta)j_3}$$

$$I(x) = \int_{\tau \sim \tau + (\beta+\beta')} D(\text{fields}) e^{-S_{Eucl}}$$

- Insertion of $(-1)^F$: twists the anti-periodic b.c. for fermions to periodic b.c.
- Twistings: other operator insertions

$$\partial_\tau \to \partial_\tau + \frac{(\beta'-\beta)j_3 + \beta' R}{\beta+\beta'}$$

- Integrand preserves Q, same b.c. for B/F: integral invariant under Q
- Q is **nilpotent**: $Q^2 = 0$ (off-shell if one uses 3d N=2 superfields)

Localization

- Addition of Q-exact term to the action does not change the integral

$$\frac{d}{dt}\int e^{-S_{Eucl}-tQV} = \int (QV)e^{-S_{Eucl}-tQV} = \int Q\left(Ve^{-S_{Eucl}-tQV}\right) = 0$$

- Can choose a suitable t QV to make calculation easy in large t limit
- Gaussian "approximation" : quadratic fluctuations around saddle points
- Choice:

$$t\{Q,V\} = \frac{1}{g^2}\int d^3x\; r\,\boxed{\mathcal{W}^\alpha \mathcal{W}_\alpha\Big|_{\theta^2\bar{\theta}^0}}$$

dimension = 4

$$\mathcal{W}_\alpha(y) \sim -\sqrt{2}i\lambda_\alpha(y) + 2D(y)\theta_\alpha + (\gamma^\mu\theta)_\alpha\left(D_\mu\sigma - \star F_\mu\right)(y) + \sqrt{2}\theta^2(\gamma^\mu D_\mu\bar{\lambda}(y))_\alpha$$

$$\Delta S = t\{Q,V\} = \frac{1}{2g^2}\int_{1\le r\le e^{\beta+\beta'}} d^3x\; r\left[(\star F_\mu - D_\mu\sigma)^2 - D^2 + \lambda^\alpha(\sigma^\mu)_{\alpha\beta}D_\mu\bar{\lambda}^\beta\right]$$

$$0 \le \tau \le \beta+\beta'$$

Some calculations: saddle points

- Monopole operators

$$A = \frac{H}{2}(\pm 1 - \cos\theta d\phi) \ , \quad H \in (\text{Cartan subalgebra})$$

$$\exp(2\pi i H) = 1 \ , \quad U(N): \ H = \text{diag}(n_1, n_2, \cdots, n_N) \ \ n_i \in \mathbb{Z}$$

- BPS solution: $\sigma = -\frac{H}{2}$, or on $\mathbb{R}^3 : \sigma = -\frac{H}{2r}$
- Holonomies: (or… r A_r = A_t)

$$A_\tau = \text{const} \ , \quad [A_\tau, H] = 0 \ \rightarrow \quad U(N) : A_\tau = \frac{1}{\beta + \beta'} \ \text{diag}(\alpha_1, \alpha_2, \cdots, \alpha_N)$$

- Large gauge transformation: zero modes

$$U(N): \ A_\tau \rightarrow U A_\tau U^\dagger + i U \partial_\tau U^\dagger \ , \quad U = \exp\left[\frac{2\pi i \tau}{\beta + \beta'} \text{diag}(m_1, m_2, \cdots, m_N)\right]$$

$$\alpha_i \sim \alpha_i + 2\pi$$

$$\text{for general } G : A_\tau \cdot \alpha \sim A_\tau \cdot \alpha + 2\pi \ \ \text{with all simple roots } \alpha$$

- All matter fields = 0 at the saddle point

Some calculations

- Gaussian integration around these backgrounds
- Holonomy integration should be done exactly (no Gaussian suppression)
- Also, one should sum over monopole charges

- Some “1-loop” piece (of order t^0) appears by inserting saddle point solution to the classical action

$$\frac{1}{4\pi}\int_{S^2\times S^1}\mathrm{tr}A\wedge F=\frac{1}{4\pi}\int_{\mathcal{M}_4}\mathrm{tr}F\wedge F$$

$\mathrm{tr}\,(H\alpha)$

U(N)

$$\frac{1}{4\pi}\mathrm{tr}\int_{S^2\times D_2}F\wedge F=\frac{1}{2\pi}\mathrm{tr}\int_{D_2}F\cdot\int_{S^2}F=\frac{1}{2\pi}\int_{S^1}A\cdot\int_{S^2}F=\sum_{i=1}^{N}n_i\alpha_i$$

- For U(N) Chern-Simons gauge fields with level k:

$$e^{iS_{CS}}\rightarrow e^{ik\sum_{i=1}^{N}n_i\alpha_i}$$

Calculations: determinants

- Scalar coupled to n units of Dirac monopole, quadratic fluctuation action:

$$\mathcal{L}_{\rm bos} = -\phi^\dagger D^\mu D_\mu \phi + \frac{1}{4}\phi^\dagger\phi + \phi^\dagger\sigma^2\phi = \phi^\dagger\left(-(D_\tau)^2 - D_{S^2}^2 + \frac{1}{4} + \frac{n^2}{4}\right)\phi$$

- (scalar) monopole spherical harmonics [Wu-Yang (1976)]:

$$SU(2) \text{ rep. with } j = \frac{|n|}{2}, \frac{|n|+1}{2}, \cdots \text{ (no s waves if } n\neq 0) \ : \ -D_a^2 Y_{jm} = \left(j(j+1) - \frac{n^2}{4}\right)Y_{jm}$$

- determinant:

$$\begin{aligned}
\text{Det}_{\rm bos} &= \prod_{j,j_3}\det\left[-\left(\partial_\tau + \frac{i\rho(\alpha)-(\beta-\beta')j_3+\beta' R}{\beta+\beta'}\right)^2 + \left(j+\frac{1}{2}\right)^2\right] \\
&= \prod_{n=\infty}^{\infty}\prod_{j,j_3}\det\left[\left(\frac{2\pi n - \rho(\alpha) - i(\beta-\beta')m + i\beta' R}{\beta+\beta'}\right)^2 + \left(j+\frac{1}{2}\right)^2\right] \\
&\sim \prod_{j,j_3}(-2i)\sin\left[\frac{\rho(\alpha)+i\beta(\epsilon_j+j_3)+i\beta'(\epsilon_j-R-j_3)}{2}\right]\times(\text{sign flip } \alpha, j_3, R)
\end{aligned}$$

- Sum over scalar modes: “letter index” $\qquad \epsilon_j \equiv j + \frac{1}{2} : \ \text{energy}$

$$\text{Det}^{-1} = \prod_{\text{all scalars}}\prod_\rho\prod_{j,j_3}\frac{e^{\frac{1}{2}\rho(\alpha)}x^{-\frac{\epsilon_j+j_3}{2}}(x')^{-\frac{\epsilon_j-R-j_3}{2}}}{1-e^{i\rho(\alpha)}x^{\epsilon_j+j_3}(x')^{\epsilon_j-R-j_3}} \ \to \ x^{\epsilon_0}\exp\left[\sum_{n=1}^\infty \frac{1}{n} f(x^n, (x')^n, e^{in\rho(\alpha)})\right]$$

letter index: $f(x) = \sum_{\text{all scalars}}\sum_\rho\sum_{j,j_3} e^{i\rho(\alpha)}x^{\epsilon_j+j_3}(x')^{\epsilon_j-R-j_3} = \text{tr}_{\rm modes}\left(e^{i\alpha}x^{E+J_3}(x')^{E-R-J_3}\right)$

Calculations: fermion determinants

- Spinor monopole harmonics:

$$\bar{\psi}\,[\bar{\sigma}^\mu D_\mu]_{4d}\,\psi \to \bar{\psi}\left(D_\tau - i[\sigma^a D_a]_{S^2} - \sigma^3\sigma\right)\psi \to D_\tau - i\sigma^a D_a + \frac{n}{2}\sigma^3$$

$$\left(i\sigma^a D_a + \frac{n}{2}\sigma^3\right)\Psi = \lambda\Psi$$

$$j \geq \frac{|\rho(n)|+1}{2} \quad \begin{pmatrix} \Psi_+ \\ \Psi_- \end{pmatrix} = \begin{pmatrix} Y_{jm} \\ \pm Y_{jm} \end{pmatrix} \qquad \lambda = \pm\left(j + \frac{1}{2}\right)$$

$$j = \frac{|\rho(n)|-1}{2} \quad \begin{pmatrix} Y_{jm} \\ 0 \end{pmatrix} \text{ if } \rho(n) > 0 \; ; \; \begin{pmatrix} 0 \\ Y_{jm} \end{pmatrix} \text{ if } \rho(n) < 0 \qquad \lambda = -\left(j + \frac{1}{2}\right)$$

- Index from matter supermultiplets: many cancelations

- Expectation: only some modes with j = j_3 will be BPS

- Result:

$$f(x) = f_{\text{bos}}(x) + f_{\text{ferm}}(x) = \sum_\rho \sum_{\ell=0}^{\infty} e^{i\rho(\alpha)} x^{|\rho(n)|+\frac{1}{2}+2\ell} - e^{-i\rho(\alpha)} x^{|\rho(n)+\frac{3}{2}+2\ell} = \sum_\rho x^{|\rho(n)|} \frac{e^{i\rho(\alpha)} x^{1/2} - e^{-i\rho(\alpha)} x^{3/2}}{1 - x^2}$$

A digression: combinatoric understanding

- Consider a sector from trivial background: H=0
- Index independent of k: free theory (large k) [Bhattacharya-Minwalla]
- SUSY & charges:

charges	D	R	J_3	$E-R-J_3$
ϕ	$1/2$	$-1/2$	0	1
$\phi^\dagger$	$1/2$	$1/2$	0	0
ψ_+	1	$1/2$	$1/2$	0
∂_{++}	1	0	1	0

- Local bosonic & fermionic BPS operators: $(\partial_{++})^\ell \phi^\dagger \,, \quad (\partial_{++})^\ell \psi$
- Letter index: $f(x) = \sum_\rho \frac{e^{i\rho(\alpha)} x^{1/2} - e^{-i\rho(\alpha)} x^{3/2}}{1-x^2}$ color chemical potential
- Full index:

$$I(x) = \boxed{\frac{1}{N!}\int_0^{2\pi}\prod_{i=1}^N \frac{d\alpha_i}{2\pi}\prod_{i<j}\left(2\sin\frac{\alpha_i-\alpha_j}{2}\right)^2} \exp\left[\sum_{n=1}^\infty \frac{1}{n} f(x^n, e^{in\alpha})\right]$$

integration over color chemical potential with
Haar measure: gauge-invariant operators

U(N)

root valued : $\alpha(\alpha_i)$

- Holonomy integral will provide gauge invariance projection.

Calculations: vector multiplet determinent

- Contribution from "vector multiplet" fields: one might naively think this would not appear (no propagating degrees)
- A more careful study of determinants from Q-exact term:
- Letter index for vector supermultiplet:

$$f_{\rm vec}(x, e^{i\alpha}) = -\sum_{i\neq j} x^{|n_i-n_j|} e^{i(\alpha_i-\alpha_j)}\ , \quad \det{}_{\rm vec} = \prod_{i\neq j}\left(1 - x^{|n_i-n_j|} e^{i(\alpha_i-\alpha_j)}\right)$$

- With H=0, this is simply the Haar measure:

$$\det{}_{\rm vec} = \prod_{i\neq j}\left(1 - e^{i(\alpha_i-\alpha_j)}\right) = \prod_{i<j}\left(2\sin\frac{\alpha_i-\alpha_j}{2}\right)^2$$

- Only finitely many degrees: coming from matters in nontrivial monopole background [K. Madhu, S. Kim] via a mechanism similar to the "Higgs mechanism" in CSm theories [Mukhi-Papageorgakis] [Mukhi]

Miscellaneous pieces & final result

- Part of vector determinant: Faddeev-Popov determinant
- FP det: Haar measure for subgroup unbroken by monopoles
- Weyl group factor is guessed (boldly trusting combinatoric interpretation)

$$\text{E.g. } H = (3,3,2,1,1,0,0,0) : U(8) \to U(2)\times U(1)\times U(2)\times U(3) \to |\text{Weyl}| = 2!1!2!3!$$

- Zero-point "energy":

$$\epsilon_0 \equiv \mathrm{tr}_{\text{mode}} \frac{\epsilon + j_3}{2} = \frac{1}{2} \lim_{x\to 1^-} x \frac{d}{dx} [f(x) + f_{\text{vec}}(x)] = \frac{1}{4} \sum_{\text{chiral}} \sum_{\rho} |\rho(n)| - \sum_{i<j} |n_i - n_j|$$

- Result:

$$I(x) = \frac{x^{\epsilon_0}}{|\text{Weyl}|} \int \prod_i \frac{d\alpha_i}{2\pi} e^{ik\sum_i n_i \alpha_i} \exp\left[\sum_{n=1}^{\infty} \frac{1}{n} \left(f(x^n, e^{in\alpha}) + f_{\text{vec}}(x^n, e^{in\alpha})\right)\right]$$

- At this stage, applies to 3d SCFT with known Lagrangian descriptions.
- Also, can add chemical potentials for flavor symmetries.

Caveats & generalizations

- Can compute from “UV theories” if the U(1) R-symmetry used for index is visible in UV: index does not change under RG flow
- Simpler for N=3 or more SUSY: For N=2, should know U(1) R-charges
- Tricky cases:
- N=8 SYM: R-symmetry enhances from SO(7) to SO(8) in IR
- (-1) factors for monopoles [Dimofte-Gaiotto-Gukov (2011)]: irrelevant for simple theories like ABJM

ABJM index

- Formulae:

$$H = \mathrm{diag}(n_1, n_2, \cdots, n_N), \ \tilde{H} = \mathrm{diag}(\tilde{n}_1, \tilde{n}_2, \cdots, \tilde{n}_N)$$

$$\epsilon_0 = \sum_{i,j} |n_i - \tilde{n}_j| - \sum_{i<j} |n_i - n_j| - \sum_{i<j} |\tilde{n}_i - \tilde{n}_j|$$

$$e^{iS_{CS}} \to e^{ik\sum_{i=1}^{N}(n_i\alpha_i - \tilde{n}_i\tilde{\alpha}_i)}$$

- At large N, one can compare with SUGRA index on AdS_4 x S^7/Z_k
- Assumes low energy: $E \sim \mathcal{O}(1)$ or $\beta \sim \mathcal{O}(1)$ as $N \to \infty$

- The latter is manifestly invariant under N=8 SUSY at k=1,2
- Need to carry out a large N approximation of the index

The gravity index: $AdS_4 \times S^7$: KK spectrum

KK fields in AdS_4 preserving our Q,S

range of n	$\epsilon_0[SO(2)]$	$SO(3)$	$SO(8)$[orth.(Qs in vector)]	Δ	contribution
$n \geq 1$	$\frac{n}{2}$	0	$(\frac{n}{2}, \frac{n}{2}, \frac{n}{2}, \frac{-n}{2})$	0	$+$
$n \geq 1$	$\frac{n+1}{2}$	$\frac{1}{2}$	$(\frac{n}{2}, \frac{n}{2}, \frac{n}{2}, \frac{-(n-2)}{2})$	0	$+$
$n \geq 2$	$\frac{n+2}{2}$	1	$(\frac{n}{2}, \frac{n}{2}, \frac{(n-2)}{2}, \frac{-(n-2)}{2})$	0	$+$
$n \geq 2$	$\frac{n+3}{2}$	$\frac{3}{2}$	$(\frac{n}{2}, \frac{(n-2)}{2}, \frac{(n-2)}{2}, \frac{-(n-2)}{2})$	0	$+$
$n \geq 2$	$\frac{n+4}{2}$	2	$(\frac{(n-2)}{2}, \frac{(n-2)}{2}, \frac{(n-2)}{2}, \frac{-(n-2)}{2})$	1	$+$
$n \geq 2$	$\frac{n+2}{2}$	0	$(\frac{n}{2}, \frac{n}{2}, \frac{n}{2}, \frac{-(n-4)}{2})$	1	$+$
$n \geq 3$	$\frac{n+3}{2}$	$\frac{1}{2}$	$(\frac{n}{2}, \frac{n}{2}, \frac{(n-2)}{2}, \frac{-(n-4)}{2})$	1	$+$
$n \geq 3$	$\frac{n+4}{2}$	1	$(\frac{n}{2}, \frac{(n-2)}{2}, \frac{(n-2)}{2}, \frac{-(n-4)}{2})$	1	$+$
$n \geq 3$	$\frac{n+5}{2}$	$\frac{3}{2}$	$(\frac{(n-2)}{2}, \frac{(n-2)}{2}, \frac{(n-2)}{2}, \frac{-(n-4)}{2})$	2	$+$
$n \geq 4$	$\frac{n+5}{2}$	$\frac{1}{2}$	$(\frac{n}{2}, \frac{(n-2)}{2}, \frac{(n-4)}{2}, \frac{-(n-4)}{2})$	2	$+$
$n \geq 4$	$\frac{n+7}{2}$	$\frac{1}{2}$	$(\frac{(n-2)}{2}, \frac{(n-4)}{2}, \frac{(n-4)}{2}, \frac{-(n-4)}{2})$	4	$+$
$n \geq 4$	$\frac{n+6}{2}$	1	$(\frac{(n-2)}{2}, \frac{(n-2)}{2}, \frac{(n-4)}{2}, \frac{-(n-4)}{2})$	3	$+$
$n \geq 4$	$\frac{n+4}{2}$	0	$(\frac{n}{2}, \frac{n}{2}, \frac{(n-4)}{2}, \frac{-(n-4)}{2})$	2	$+$
$n \geq 4$	$\frac{n+6}{2}$	0	$(\frac{n}{2}, \frac{(n-4)}{2}, \frac{(n-4)}{2}, \frac{-(n-4)}{2})$	3	$+$
$n \geq 4$	$\frac{n+8}{2}$	0	$(\frac{(n-4)}{2}, \frac{(n-4)}{2}, \frac{(n-4)}{2}, \frac{-(n-4)}{2})$	6	$+$
$n = 1$	2	$\frac{1}{2}$	$(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$	1	$-$
$n = 1$	$\frac{5}{2}$	0	$(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2})$	2	$-$
$n = 2$	3	0	$(1, 1, 0, 0)$	2	$-$
$n = 2$	$\frac{7}{2}$	$\frac{1}{2}$	$(1, 0, 0, 0)$	2	$-$
$n = 2$	4	1	$(0, 0, 0, 0)$	3	$-$

The gravity index

- Index over single particle states:

$$I^{\rm sp} = \frac{\left(1-x\sqrt{xy_1y_2y_3}\right)\left(1-x\sqrt{\frac{xy_3}{y_1y_2}}\right)\left(1-x\sqrt{\frac{xy_1}{y_2y_3}}\right)\left(1-x\sqrt{\frac{xy_2}{y_1y_3}}\right)}{\left(1-\sqrt{\frac{xy_1y_3}{y_2}}\right)\left(1-\sqrt{\frac{xy_2y_3}{y_1}}\right)\left(1-\sqrt{\frac{xy_1y_2}{y_3}}\right)\left(1-\sqrt{\frac{x}{y_1y_2y_3}}\right)(1-x^2)^2} - \frac{1-x^2+x^4}{(1-x^2)^2}$$

- With Z_k, mod out

$$I^{\rm sp} = \sum_{n=-\infty}^{\infty} y_3^{\frac{n}{2}} I^{\rm sp}_n(x,y_1,y_2) \qquad I^{\rm sp}_{\mathbb{Z}_k}(x,y_1,y_2,y_3) \equiv \sum_{n=-\infty}^{\infty} y_3^{\frac{kn}{2}} I^{\rm sp}_{kn}(x,y_1,y_2)$$

- Full gravity index:

$$I_{\rm mp}(x,y_1,y_2,y_3) = \exp\left[\sum_{n=1}^{\infty}\frac{1}{n} I^{\rm sp}_{\mathbb{Z}_k}(x^n,y_1^n,y_2^n,y_3^n)\right]$$

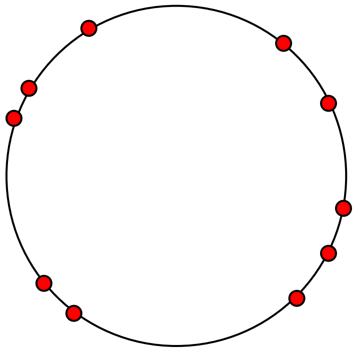
- k=1: gravity BPS states are all connected by 32 SUSY to chiral ones.
- From CFT, the nontrivial issue is whether SUSY enhances well, as N=6 (24 SUSY) is not enough to generate all states from chiral states.

The large N index

- Graviton index factorized: $I_{\mathrm{IIA\ grav.}} I^{+}_{KK} I^{-}_{KK}$ or $I_{\mathrm{IIA\ grav.}} I_{D0} I_{\overline{D0}}$

- Large N, low energy: most of the U(1)^N do not support Dirac monopoles
- `Large' unbroken gauge groups U(N – O(1))

- identical holonomy variables $\longrightarrow$ distributions [Brezin et.al. ('78)]



$$\to \rho(\theta) \ ; \quad \int_0^{2\pi} d\theta \rho(\theta) = 1$$

$$\int \prod_i \left[\frac{d\alpha_i}{2\pi}\right] \to \int [N\mathcal{D}\rho(\theta)] = \prod_{n=1}^{\infty} \left[N^2 d\rho_n d\rho_{-n}\right]$$

The large N index & supergravity

- With zero fluxes, one finds from letter indices

$$\rho(\theta) = \frac{1}{N}\sum_{i=1}^{N}\delta(\theta-\alpha_i) = \frac{1}{2\pi} + \frac{1}{2\pi}\sum_{n\neq 0}\rho_n e^{in\theta} \qquad \rho_n \equiv \frac{1}{N}\sum_{i=1}^{N} e^{in\alpha_i}$$

- From definition,

$$\sum_{n=1}^{\infty}\frac{1}{n}f(x^n, x^{in\alpha}) \stackrel{\text{bifundamental}}{\longrightarrow} \sum_{i,j} e^{in(\alpha_i-\tilde{\alpha}_j)} = N^2\rho_n\tilde{\rho}_{-n}\ , \quad \text{etc.}$$

$$\sum_{n=1}^{\infty}\frac{1}{n}f_{\rm vec}(x^n, e^{in\alpha}) = -\sum_{n}\frac{1}{n}\sum_{i\neq j}e^{in(\alpha_i-\alpha_j)} \approx -\sum_{n}\frac{1}{n}\sum_{i}e^{in\alpha_i}\sum_{j}e^{-in\alpha_j} = -N^2\sum_{n}\frac{1}{n}\rho_n\rho_{-n}$$

- Gaussian integration at large N…

- gauge theory index also factorizes: $I_{\text{free CS}} I_{\text{flux}>0} I_{\text{flux}<0}$
- $I_{\text{IIA grav.}} = I_{\text{free CS}}$ was proven. [Bhattacharya-Minwalla]
- Nonperturbative in 1/k : compare D0 brane part & flux>0 part.

Single D0 brane

- one saddle point: unit flux on both gauge groups $H = \tilde{H} = (1, 0, 0, \cdots, 0)$

- Gauge theory result: $z \equiv e^{i(\tilde{\alpha}-\alpha)}$

$$I_{\text{CS}}(x, y_1, y_2) = \frac{1}{2\pi i}\oint \frac{dz}{z} z^{-k} \times$$

$$\frac{(1 - z\sqrt{x^3 y_1 y_2})(1 - z\sqrt{x^3 y_1^{-1} y_2^{-1}})(1 - z^{-1}\sqrt{x^3 y_1 y_2^{-1}})(1 - z^{-1}\sqrt{x^3 y_2 y_1^{-1}})}{(1 - z\sqrt{x y_1 y_2^{-1}})(1 - z\sqrt{x y_2 y_1^{-1}})(1 - z^{-1}\sqrt{x y_1 y_2})(1 - z^{-1}\sqrt{x y_1^{-1} y_2^{-1}})(1 - x^2)^2}$$

- Gravity: single graviton index in AdS_4 x S^7 projected to p_{11} = k

$$I_{\text{grav.}}(x, y_1, y_2) = \frac{1}{2\pi i}\oint \frac{dz}{z} z^{-k}\, I^{\text{single}}_{AdS_4 \times S^7}(x, y_1, y_2, z)$$

- One can show :

$$= I^{\text{single}}_{AdS_4 \times S^7}(x, y_1, y_2, z) + \frac{1 - x^2 + x^4}{(1 - x^2)^2}$$

Numerical tests: 2 & 3 KK momenta

- Example: KK momenta at CS level k=1

$$
\begin{array}{rl|lcccccccccll}
\hline
I_{(3)(3)} & = & x^{\frac{3}{2}}[& 4 & +4x & +0x^2 & +8x^3 & -4x^4 & +8x^5 & +2x^6 & +4x^7 & +0x^8 & +\mathcal{O}(x^9) &] \\
I_{(2,1)(2,1)} & = & x^{\frac{3}{2}}[& 6 & +20x & +24x^2 & +28x^3 & +64x^4 & +34x^5 & +34x^6 & +166x^7 & -32x^8 & +\mathcal{O}(x^9) &] \\
I_{(1,1,1)(1,1,1)} & = & x^{\frac{3}{2}}[& 4 & +12x & +30x^2 & +52x^3 & +52x^4 & +98x^5 & +170x^6 & +130x^7 & +106x^8 & +\mathcal{O}(x^9) &] \\
I_{(2,1)(1,1,1)} + I_{(1,1,1)(2,1)} & = & x^{\frac{3}{2}}[& & & & & 0x^4 & +12x^5 & -20x^6 & -44x^7 & +176x^8 & +\mathcal{O}(x^9) &] \\
I_{(3)(2,1)} + I_{(2,1)(3)} & = & x^{\frac{3}{2}}[& & & & & & & +4x^6 & -16x^7 & +32x^8 & +\mathcal{O}(x^9) &] \\
I_{(3)(1,1,1)} + I_{(1,1,1)(3)} & = & x^{\frac{3}{2}}[& & & & & & & & & & +\mathcal{O}(x^{12}) &] \\
\hline
I_3(x) & = & x^{\frac{3}{2}}[& 4 & +4x & +2x^2 & +4x^3 & +2x^4 & +4x^5 & +2x^6 & +4x^7 & +2x^8 & +\mathcal{O}(x^9) &] \\
I_1(x)I_2(x) & = & x^{\frac{3}{2}}[& 6 & +20x & +26x^2 & +36x^3 & +46x^4 & +52x^5 & +66x^6 & +68x^7 & +86x^8 & +\mathcal{O}(x^9) &] \\
\frac{1}{3}I_1(x^3)+\frac{1}{2}I_1(x)I_1(x^2)+\frac{1}{6}I_1(x)^3 & = & x^{\frac{3}{2}}[& 4 & +12x & +26x^2 & +48x^3 & +64x^4 & +96x^5 & +122x^6 & +168x^7 & +194x^8 & +\mathcal{O}(x^9) &] \\
\hline
\end{array}
$$

$$
\begin{aligned}
&I_{(3)(3)} + I_{(2,1)(2,1)} + I_{(1,1,1)(1,1,1)} + 2I_{(2,1)(1,1,1)} + 2I_{(3)(2,1)} + 2I_{(3)(1,1,1)} \\
&\qquad = I_3(x) + I_1(x)I_2(x) + \frac{1}{3}I_1(x^3) + \frac{1}{2}I_1(x)I_1(x^2) + \frac{1}{6}I_1(x)^3 + \mathcal{O}(x^{\frac{3}{2}+9})
\end{aligned}
$$

Other examples (list)

- AdS_4/CFT_3
- ABJ: N=6, N=5 SUSY [Cheon, Gang, S.K. (unpublished)]
- N=4 SUSY, AdS_4 x (7 manifold with singularity): gravity modes localized on the singularity (twisted sectors) [Imamura, Yokoyama (2009)]
- N=3: tri-Sasakian N^{010} or N^{010}/Z_k [Chern, Gang, S.K., Park (2011)]

- Can also test various non-perturbative 3d dualities
- N=4 mirror dualities [Gang, Koh, Lee, Park (2011)]
- Seiberg-like dualities [Hwang, Kim, Park, Park (2011)]

More generalizations: N=2 SCFT

- In N=2 superconformal theories, R-charge is SO(2) ~ U(1)
- If there are many U(1) symmetries (one combination: R, others: flavors), determining the correct combination which appears in the SC algebra is a nontrivial problem.
- R-charges, and thus scale dimensions, of various fields are in general complicated.
- Writing down SUSY QFT (not necessarily conformal) on S^2 x R, preserving U(1) R-charge, one can compute an index which simply counts BPS states in this case. [Imamura-Yokoyama (2011)]

More generalizations: N=2 SCFT

- Result:

$$\text{matter}: \quad f(x,\alpha) = x^{|\rho(n)|}\frac{e^{i\rho(\alpha)}x^{\Delta} - e^{-i\rho(\alpha)}x^{2-\Delta}}{1-x^2} \quad ; \quad \text{vector det}: \quad \text{same}$$

$$\epsilon_0 = \frac{1}{2}(1-\Delta)\sum_{\text{chiral}}\sum_{\rho}|\rho(n)| - \sum_{\alpha>0}|\alpha(n)| \quad ; \quad \Delta S_{CS} = -\frac{1}{2}\sum_{\text{chiral}}\sum_{\rho}|\rho(n)|\rho(\alpha)$$

... and zero point shifts of flavor charges

- If the theory flows to CFT in IR (on flat space): superconformal index (compactifying on large S^2), supposing that correct R-charges assigned.

- Sometimes, R-charges determined by symmetry considerations, etc.
- Sometimes, can test properties like dualities without determining it.
- Can also determine it using 3-sphere partition function (tommorrow)

Applications of the N=2 index

- Spectrum on nontrivial $AdS_4 \times SE_7$
- Gravity spectrum known for some simple 7-manifolds… M^{32}, Q^{111}, V^{52}
- M^{32}: agreement [Cheon, Gang, S.K., Park (2011)] [Hwang, Gang, Kim, Park (2011)]
- Q^{111}: some tension, could be a problem on SUGRA KK spectrum ref.
- V^{52}: not completely studied…

- N=2 mirror duality: [Imamura-Yokoyama] [Krattenthaler-Spiridonov-Vartanov]
- N=2 Seiberg-like dualities: [Bashkirov (2011)] [Hwang, Kim, Park, Park (2011)]
- New 3d dualities [Jafferis-Yin] [Kapustin-Kim-Park] [Dimofte-Gaiotto-Gukov] , etc.

- Index can be used to **rule out** delicate duality proposals. [S.K., Park (2010)]

Summary of this lecture

- Calculation & studies of the index for strongly interacting theories
- Applications: monopole spectrum, AdS/CFT, symmetry enhancement, nonperturbative dualities, …

- Challenges:
- More techniques need for exact treatments (too much info…!)
- Complete analytic control is also lacking in 4d index: more math needed [Gadde, Pomoni, Rastelli, Razamat, Yan (2009-2011)] [Dolan-Osborn]
- With monopoles, added complications (See, however, [Krattenthaler-Spiridonov-Vartanov (2011)] for recent studies on Abelian monopole indices)

Summary of this lecture

- Partition functions for CFT with AdS gravity duals should undergo a large N deconfinement phase transition.
- Dual to Hawking-Page phase transition: thermal AdS vs. black holes.
- How about the index…?

- 2d index sees this: microscopic black hole countings with AdS_3 factors
- 4d index: does not see this, perhaps due to severe boson-fermion cancelations [Kinney, Maldacena, Minwalla, Raju]
- 3d: Unclear. Maybe we need large N techniques to sum over monopole sectors. Could the index contain $N^{3/2}$ & see SUSY black holes?