Strong coupling dynamics of 3d SCFT:

studies of the partition function on 3-sphere

Seok Kim

(Seoul National University)

6^{th} Asian Winter School on Strings, Particles and Cosmology

Kusatsu, Japan (10 – 20 January 2012)

References:

- Kapustin, Willett, Yaakov (2009)
- Jafferis (2010); Hama, Hosomichi, Lee (2010)

Related references:

- Drukker, Marino, Putrov (2010)
- Herzog, Klebanov, Pufu, Tesileanu (2010)
- Jafferis, Klebanov, Pufu, Safdi (2011); Martelli, Sparks (2011); Cheon, Kim, Kim (2011)
- Klebanov, Pufu, Safdi (2011)
- Amariti, Klare, Siani (2011); Gang, Hwang, S.K., Park (2011),

- Kapustin, Willett, Yaakov (2010), (2011);

- and many many more refs…

- Festuccia, Seiberg (2011)

Motivation

- Euclidean CFT on flat space can be put on any conformally flat manifold
- Sphere is such a manifold: partition function of Eulicean QFT: e.g.

$$ds^2(S^3) = \left(\frac{1}{1+\frac{|\vec{x}|^2}{4}}\right)^2 d\vec{x}\cdot d\vec{x}$$

- One may even put non-conformal QFT's on such manifolds: exists ambiguity, action fixed up to parameters if one demands certain SUSY.
- If the QFT (on flat space) flows to CFT in IR, one can study the RG fixed point by studying sphere partition functions for UV theory (r independent)

- Meaning? partition function does not depend on r (via r $g_{YM}{}^2$): consider partition function on large 3-sphere, so IR CFT is reached by RG flow.

Outline

- 3-sphere partition functions of CFT [Kapustin-Willett-Yaakov]
- Comments on extensions
- Applications:

- $N^{3/2}$ for M2 [Drukker-Marino-Putrov] [Herzog-Klebanov-Pufu-Tesileanu]
- other applications for ABJM
- determination of U(1) R-symmetry in 3d N=2 SCFT
- comments on other applications

SCFT on a 3-sphere

- Action: in principle can be written down via conformal map (impractical)
- More useful to start from a few simple facts about conformal map (Killing spinors, etc.), then just start from flat space action & SUSY complete
- (More systematic method by Festuccia-Seiberg)

- Full action (r=1) : (different convention… $\lambda \leftrightarrow \lambda^\dagger$)

$$S_m = \int \sqrt{g} \left(D^\mu \phi^\dagger D_\mu \phi + \frac{3}{4}\phi^\dagger \phi + i\psi^\dagger \not{D} \psi + F^\dagger F + \phi^\dagger \sigma^2 \phi + \phi^\dagger \sigma^2 \phi + i\phi^\dagger D \phi - i\psi^\dagger \sigma \psi + i\phi^\dagger \lambda^\dagger \psi - i\psi^\dagger \lambda \phi \right)$$

$$S_{CS} = \frac{k}{4\pi} \int \mathrm{tr} \left(A \wedge A + \frac{2i}{3} A^3 - \lambda^\dagger \lambda + 2D\sigma \right)$$

SCFT on a 3-sphere

- N=2 SUSY: 4 complex Killing spinors on S^3 (derivable from SUSY on R^3):

$$\nabla_\mu \epsilon = \pm \frac{i}{2}\gamma_\mu \epsilon \quad \text{(similar for } \eta\text{)}$$

- SUSY transformations

$$\begin{aligned}
\delta\phi &= \eta^\dagger \psi \\
\delta\phi^\dagger &= \psi^\dagger \epsilon \\
\delta\psi &= (-i\gamma^\mu D_\mu \phi - i\sigma\phi)\,\epsilon - \tfrac{i}{3}\gamma^\mu(\nabla_\mu \epsilon)\phi + \eta^* F \\
\delta\psi^\dagger &= \eta^\dagger\left(i\gamma^\mu D_\mu \phi^\dagger + i\sigma\phi^\dagger\right) + \tfrac{i}{3}\phi^\dagger(\nabla_\mu \eta^\dagger)\gamma^\mu + \epsilon^T F^\dagger \\
\delta F &= \epsilon^T\left(-i\gamma^\mu D_\mu \psi + i\lambda\phi + i\sigma\psi\right) \\
\delta F^\dagger &= \left(iD_\mu \psi^\dagger \gamma^\mu - i\lambda^\dagger \phi^\dagger + i\sigma\psi^\dagger\right)\eta^*
\end{aligned}$$

$$\delta A_\mu = \tfrac{i}{2}\left(\eta^\dagger \gamma_\mu \lambda - \lambda^\dagger \gamma_\mu \epsilon\right)$$

$$\delta\sigma = -\tfrac{1}{2}\left(\eta^\dagger \lambda + \lambda^\dagger \epsilon\right)$$

$$\delta D = \tfrac{i}{2}\left(\eta^\dagger \gamma^\mu (D_\mu \lambda) - (D_\mu \lambda^\dagger)\gamma^\mu \epsilon\right) - \tfrac{i}{2}\left(\eta^\dagger[\lambda,\sigma] - [\lambda^\dagger,\sigma]\epsilon\right) + \tfrac{i}{6}(\nabla_\mu \eta^\dagger \gamma^\mu \lambda - \lambda^\dagger \gamma^\mu \nabla_\mu \epsilon)$$

$$\delta\lambda = \left(-\tfrac{1}{2}\gamma^{\mu\nu}F_{\mu\nu} - D + i\gamma^\mu D_\mu \sigma\right)\epsilon \; + \tfrac{2i}{3}\sigma\gamma^\mu \nabla_\mu \epsilon$$

$$\delta\lambda^\dagger = \eta^\dagger\left(\tfrac{1}{2}\gamma^{\mu\nu}F_{\mu\nu} - D - i\gamma^\mu D_\mu \sigma\right) \; - \tfrac{2i}{3}\sigma\nabla_\mu \eta^\dagger \gamma^\mu$$

Partition function & localization

- Add Q-exact terms to the action [Kapustin-Willett-Yaakov]: pick a Killing spinor & use corresponding Q to localize: use $\nabla_\mu \epsilon_+ = +\frac{i}{2}\gamma_\mu \epsilon_+$

$$S \to S + tQV \ , \quad QV = Q\left[(Q\lambda)^\dagger \lambda + (Q\psi)^\dagger \psi + \psi^\dagger (Q\psi^\dagger)^\dagger\right]$$

- (ref. other localization methods by Hama-Hosomichi-Lee)

- Upon expanding SUSY variations,

$$Q((Q\lambda)^\dagger \lambda) = \mathrm{tr}\left(\frac{1}{2}F_{\mu\nu}F^{\mu\nu} + D_\mu \sigma D^\mu \sigma + (D+\sigma)^2 + i\lambda^\dagger \gamma^\mu D_\mu \lambda + i[\lambda^\dagger, \sigma]\lambda - \frac{1}{2}\lambda^\dagger \lambda\right)$$

$$Q((Q\psi)^\dagger \psi + \psi^\dagger (Q\psi^\dagger)^\dagger) = D_\mu \phi^\dagger D^\mu \phi + i v^\mu \phi^\dagger D_\mu \phi + \phi^\dagger \sigma^2 \phi + \frac{1}{4}\phi^\dagger \phi + F^\dagger F + \psi^\dagger \left(i\not{D} - i\sigma + \frac{1+\not{v}}{2}\right)\psi$$

$$v^\mu \equiv \epsilon^\dagger \gamma^\mu \epsilon \quad (\text{commuting spinors } \epsilon_+ \text{ satisfying } \epsilon^\dagger \epsilon = 1)$$

Some calculations: saddle points

- Saddle point equations:

$$\delta\lambda = \left(-\frac{1}{2}\gamma^{\mu\nu}F_{\mu\nu} + i\gamma^{\mu}\sigma - D - \sigma\right)\epsilon = 0 \;\rightarrow\; \star F_{\mu} = D_{\mu}\sigma\,,\;\; D = -\sigma \;\; (\text{valid for all } \epsilon)$$

- No monopole-like configurations on 3-sphere, solutions:

$$S^3:\;\; F_{\mu\nu} = 0\;,\;\; \sigma = \text{const}\;,\quad D = -\sigma$$

- A simple proof…

$$0 = D^{\mu}\star F_{\mu} = D^{\mu}D_{\mu}\sigma \;\rightarrow\; 0 = -\int\sqrt{g}\,\text{tr}\,\sigma D^{\mu}D_{\mu}\sigma = \int\sqrt{g}\,\text{tr}\,(D_{\mu}\sigma)^2 \;\rightarrow\; D_{\mu}\sigma = 0$$

- All matter fields are set to zero.

- Measure obtained by plugging in saddle points to (Chern-Simons) action

$$e^{i\frac{k}{4\pi}\int 2D\sigma} \;\longrightarrow\; e^{-ik\pi\,\text{tr}\,\sigma^2}$$

Some calculations: vector multiplet determinants

- Could have used index theorems: brutal calculations also do.
- Quadratic fluctuation action $\det\cancel{(\nabla^\mu D_\mu)}\delta[\nabla^\mu A_\mu]e^{-S}$

$$S = \int \sqrt{g}\,\mathrm{tr}\left(-A^\mu \Delta A_\mu - [A_\mu,\sigma]^2 - \cancel{\delta\sigma\partial^2\delta\sigma} + \lambda^\dagger\left(i\not{\nabla} + i[\sigma,\ ] - \frac{1}{2}\right)\lambda\right)$$

$$\delta^\nu_\mu\nabla^2 - \nabla^\nu\nabla_\mu \overset{eff}{=} \delta^\nu_\mu\nabla^2 - R_\mu{}^\nu = \delta^\nu_\mu(\nabla^2 - 2) \equiv \delta^\nu_\mu\Delta$$

- To be careful, should have reformulated localization after gauge fixing: just gauge fix the Q-deformed path integral…
- Decompose: $A_\mu = \partial_\mu\Phi + B_\mu \quad (\nabla^\mu B_\mu = 0):\ \Delta[\nabla^\mu A_\mu] = \cancel{\det^{-1/2}}(\nabla^2)\delta[\Phi]$

- Take σ in the Cartan using global gauge symmetry:

$$U(N):\ \sigma = \mathrm{diag}(\sigma_1, \sigma_2, \cdots, \sigma_N)$$

- Measure from FP determinant, etc.

$$\frac{1}{|\mathrm{Weyl}|}\int d\sigma \prod_\alpha \alpha(\sigma)\cdots$$

Vector multiplet determinants

- Various spherical harmonics on 3-sphere $\ell = 1, 2, 3, \cdots$

$$\text{vector}: \ (j_1, j_2) = \left(\frac{\ell+1}{2}, \frac{\ell-1}{2}\right) \oplus \left(\frac{\ell-1}{2}, \frac{\ell+1}{2}\right)$$

$$-\nabla^2+2 \to (j_1+j_2+1)^2 = (\ell+1)^2\,, \ \text{degeneracy} = 2(2j_1+1)(2j_2+1) = 2\ell(\ell+2)$$

$$\text{spinor}: (j_1, j_2) = \left(\frac{\ell}{2}, \frac{\ell-1}{2}\right) \oplus \left(\frac{\ell-1}{2}, \frac{\ell}{2}\right)$$

$$i\not\nabla \to \pm(j_1+j_2+1) = \pm\left(\ell + \frac{1}{2}\right)\ , \quad \text{degeneray} = \ell(\ell+1) \text{ for each}$$

- Vector multiplet determinant:

$$\det{}_{\text{vec}} = \prod_{\alpha}\prod_{\ell=1}^{\infty} \frac{[(\ell+i\alpha(\sigma))(\ell+1-i\alpha(\sigma))]^{\ell(\ell+1)}}{[(\ell+1)^2+\alpha(\sigma)^2]^{2\ell(\ell+2)}} = \prod_{\alpha}\prod_{\ell=1}^{\infty} \frac{(\ell+i\alpha(\sigma))^{\ell+1}}{(\ell-i\alpha(\sigma))^{\ell-1}}$$

$$\stackrel{\sigma\to-\sigma}{\longrightarrow} \prod_{\alpha}\prod_{\ell=1}^{\infty} \frac{(\ell^2+\alpha(\sigma)^2)^{\frac{\ell+1}{2}}}{(\ell^2+\alpha(\sigma)^2)^{\frac{\ell-1}{2}}} = \prod_{\alpha}\prod_{\ell=1}^{\infty}(\ell^2+\alpha(\sigma)^2) \sim \prod_{\alpha} \frac{2\sinh(\pi\alpha(\sigma))}{\pi\alpha(\sigma)}$$

Some calculations: matter determinants

- Bosonic determinant:

$$\det{}_{\mathrm{b}} = \det\left(-\nabla^2 + iv^\mu\nabla_\mu + \frac{1}{4} + \rho(\sigma)^2\right) = \prod_{\ell=0}^{\infty}\prod_{m=-\ell}^{\ell}\left(\ell(\ell+2) - m + \frac{1}{4} + \rho(\sigma)^2\right)^{\ell+1}$$

- Fermionic determinant: $i\nabla\!\!\!/ + \frac{1+v\!\!\!/}{2} - i\rho(\sigma) \to -4\vec{S}\cdot\vec{L} + S_3 - 1 - i\rho(\sigma)$

$$\text{use } \nabla_i = \partial_i + \frac{i}{2}\gamma_i \quad \text{(using left invariant 1−forms as vielbeins)}$$

$$\partial_i = 2iL_i\ , \quad \gamma_i = 2S_i \quad \text{(two } SU(2) \text{ generators)}$$

- Matter:

$$\frac{\det_{\mathrm{f}}}{\det_{\mathrm{b}}} = \prod_{\rho}\prod_{n=1}^{\infty}\left(\frac{n+\frac{1}{2}+i\rho(\sigma)}{n-\frac{1}{2}-i\rho(\sigma)}\right)^n$$

- For self-conjugate representation, keep pieces which are even in
- Then, [KWY] shows (after suitable renormalizations, etc.):

$$\frac{\det_{\mathrm{f}}}{\det_{\mathrm{b}}} \sim \prod_{\rho}\frac{1}{[2\cosh(\pi\rho(\sigma))]^{1/2}}$$

Final result

- Final result: integrate over saddle points, Cartans & Vandermonde

$$Z = \boxed{\frac{1}{|\mathrm{Weyl}|}} \int d\sigma\, e^{-ik\pi \mathrm{tr}(\sigma^2)} \frac{\prod_\alpha 2\sinh(\pi\alpha(\sigma))}{\prod_\rho 2\cosh(\pi\rho(\sigma))}$$

Correct normalization for ABJM ~ $(N!)^2$
[Drukker-Marino-Putrov]

- One can also introduce mass terms & FI terms: as this is not an index, Z may depend on continuous parameters (unless the deformation is Q-exact) (some comments on it later…)

Application for ABJM

- ABJM at large N: AdS_4 dual? $N^{3/2}$? E.g., entropy of thermal black M2-branes [Klebanov-Tseytlin] $S/V \sim N^{3/2}T^2$
- Z identified as (renormalized) Euclidean action of AdS_4
- Gravity result:

$$I(AdS_4) = \frac{\pi \ell^2}{2G_N} \quad (\ell : \ AdS_4 \ \text{radius}) \ , \quad AdS_4 \times S^7/\mathbb{Z}_k : \ \frac{\pi\sqrt{2}}{3} k^{1/2} N^{3/2}$$

- CFT dual: $F = -\log(Z)$
- Originally, obtained in the 't Hooft limit [Drukker-Marino-Putov]

$$N, k \to \infty, \ \lambda = \frac{N}{k} = \text{finite} \gg 1$$

- Perhaps more illustrative to discuss the simple large N limit, keeping k finite (Many AdS_4 duals are singular at general k with orbifolds) [Herzog-Klebanov-Pufu-Tesileanu]

The large N partition function

- Integration over large N number of variables: saddle point approx.

$$Z = \frac{1}{(N!)^2} \int \left(\prod_{i=1}^{N} \frac{d\lambda_i \, d\tilde{\lambda}_i}{(2\pi)^2} \right) \frac{\prod_{i<j} \left(2\sinh \frac{\lambda_i - \lambda_j}{2}\right)^2 \left(2\sinh \frac{\tilde{\lambda}_i - \tilde{\lambda}_j}{2}\right)^2}{\prod_{i,j} \left(2\cosh \frac{\lambda_i - \tilde{\lambda}_j}{2}\right)^2} \exp\left(\frac{ik}{4\pi} \sum_i (\lambda_i^2 - \tilde{\lambda}_i^2) \right)$$

$$\begin{aligned} F(\lambda_i, \tilde{\lambda}_i) = -i\frac{k}{4\pi} \sum_j (\lambda_j^2 - \tilde{\lambda}_j^2) - \sum_{i<j} \log \left[\left(2\sinh \frac{\lambda_i - \lambda_j}{2}\right)^2 \left(2\sinh \frac{\tilde{\lambda}_i - \tilde{\lambda}_j}{2}\right)^2 \right] \\ + 2\sum_{i,j} \log \left(2\cosh \frac{\lambda_i - \tilde{\lambda}_j}{2}\right) + 2\log N! + 2N\log(2\pi) \end{aligned}$$

- Saddle point equations (complex due to CS term):

$$-\frac{\partial F}{\partial \lambda_i} = \frac{ik}{2\pi}\lambda_i - \sum_{j\neq i} \coth \frac{\lambda_j - \lambda_i}{2} + \sum_j \tanh \frac{\tilde{\lambda}_j - \lambda_i}{2} = 0$$

$$-\frac{\partial F}{\partial \tilde{\lambda}_i} = -\frac{ik}{2\pi}\tilde{\lambda}_i - \sum_{j\neq i} \coth \frac{\tilde{\lambda}_j - \tilde{\lambda}_i}{2} + \sum_j \tanh \frac{\lambda_j - \tilde{\lambda}_i}{2} = 0$$

The large N partition function

- Distribution of eigenvalues on the complex plane?
- From some intuition from numerics [HKPT], one takes the following large N scalings of eigenvalue distributions on complex plane:

will be 1/2

O(1) & real

$$\lambda_j = N^\alpha x_j + i y_j \,, \qquad \tilde{\lambda}_j = N^\alpha x_j - i y_j$$

- Continuum limit: $x_j = x(j/N) \,, \qquad y_j = y(j/N)$

  $y(x)$ is an increasing function

- Eigenvalue density: $\int dx\, \rho(x) = 1; \rho(x) \geq 0$

The large N partition function

- "free energy" from CS term:

$$F_{\rm ext} = -i\frac{k}{4\pi}\sum_j\left[(N^\alpha x_j + iy_j)^2 - (N^\alpha x_j - iy_j)^2\right] \approx \frac{kN^\alpha}{\pi}\sum_j x_j y_j \to \frac{kN^{1+\alpha}}{\pi}\int_{-x_*}^{x_*} xy(x)\rho(x)dx$$

- 2-body interactions:

$$F_{\rm int} \approx -\sum_{i>j}\log\frac{4\sinh^2\frac{\lambda_i-\lambda_j}{2}\cdot 4\sinh^2\frac{\tilde\lambda_i-\tilde\lambda_j}{2}}{4\cosh^2\frac{\lambda_i-\tilde\lambda_j}{2}\cdot 4\cosh^2\frac{\tilde\lambda_i-\lambda_j}{2}} = -\sum_{i>j}\log\frac{(1-e^{-\lambda_i+\lambda_j})^2(1-e^{-\tilde\lambda_i+\tilde\lambda_j})^2}{(1+e^{-\lambda_i+\tilde\lambda_j})^2(1+e^{-\tilde\lambda_i+\lambda_j})^2}$$

$$= \sum_{i>j}\sum_{n=1}^{\infty}\frac{2}{n}\left[e^{n(-\lambda_i+\lambda_j)} + e^{n(-\tilde\lambda_i+\tilde\lambda_j)} - (-1)^n\left(e^{n(-\lambda_i+\tilde\lambda_j)} + e^{n(-\tilde\lambda_i+\lambda_j)}\right)\right]$$

$$\to \sum_{n=1}^{\infty}\frac{2N^2}{n}\int_{-x_*}^{x_*}dx\rho(x)\int_{-x_*}^{x}dx'\rho(x')\left[\boxed{e^{n(-\lambda(x)+\lambda(x'))}} + e^{n(-\tilde\lambda(x)+\tilde\lambda(x'))} - (-1)^n\left(\boxed{e^{n(-\lambda(x)+\tilde\lambda(x'))}} + e^{n(-\tilde\lambda(x)+\lambda(x'))}\right)\right]$$

- x' integrals dominated near x: e.g.

$$\int_{-x_*}^{x} dx'\rho(x')e^{nN^\alpha(x'-x)+in(y(x')-y(x))} \approx \frac{\rho(x)}{nN^\alpha}$$

$$\int_{-x_*}^{x} dx'\rho(x')e^{-nN^\alpha(x'-x)-in(y(x)+y(x'))} \approx \frac{\rho(x)}{nN^\alpha}e^{-2ny(x)}$$

- Result: $\approx \sum_{n=1}^{\infty}\frac{4N^{2-\alpha}}{n^2}\int_{x_*}^{x^*}dx\rho(x)^2\left[1-(-1)^n\cos(2ny(x))\right] = N^{2-\alpha}\int_{-x_*}^{x_*}dx\rho(x)^2(\pi^2-4y(x)^2)$

when $-\pi/2 \le y(x) \le \pi/2$

The large N partition function

- Long-range interactions canceled: take $\alpha = \frac{1}{2}$ (for saddle point to exist) $f(t) = \pi^2 - t^2$

$$F/N^{3/2} \approx \frac{k}{\pi}\int_{-x_*}^{x_*} xy(x)\rho(x)dx + \int_{-x_*}^{x_*} dx\rho(x)^2 f(2y(x)) - \frac{\mu}{2\pi}\left[\int dx\rho(x) - 1\right]$$

$$\delta\rho(x):\ 4\pi\rho f(2y) = \mu - 2kxy\ ,\quad \delta y(x):\ 2\pi\rho f'(2y) = -kx$$

$$\text{solution}:\ \rho(x) = \frac{\mu}{4\pi^3}\ ,\quad y(x) = \frac{\pi^2 k}{2\mu}x$$

$$\int_{-x_*}^{x_*} dx\rho(x) = 1 \ \rightarrow\ \mu = \frac{2\pi^3}{x_*}$$

- Free energy:

$$F = \frac{N^{3/2}(12\pi^4 + k^2 x_*^4)}{24\pi^2 x_*} \xrightarrow{x_* = \pi\sqrt{2/k}} \frac{\pi\sqrt{2}}{3}k^{1/2}N^{3/2}$$

- Agrees with AdS_4 Euclidean action dual to ABJM

More applications for ABJM

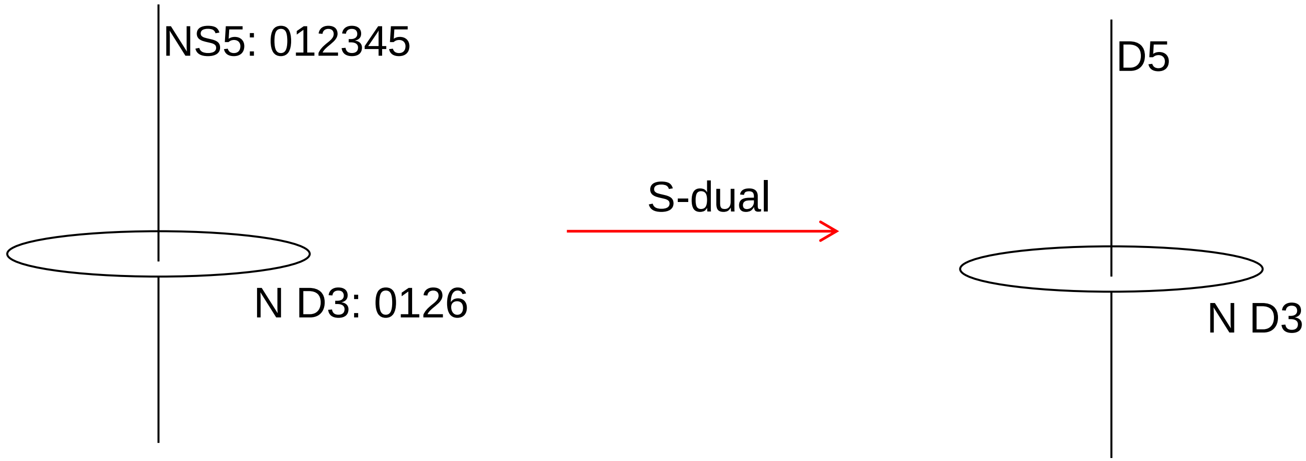
- ABJM: only N=6 manifest
- Can compare with other descriptions with manifest N=8 Poincare SUSY

- N=8 SYM: impossible to take U(1) subgroup of SO(7) in UV
- One way to see this: consider free theory on single D2
- choose any U(1) in SO(7) as the R-charge.
- Gauge field dualizes to a scalar: combines with one of the 7 scalars neutral under this U(1) to form a chiral field: $\Delta = \frac{1}{2} (\neq R = 0)$

- U(1) R-symmetry cannot be embedded in the UV SO(7): IR enhancement to SO(8) is crucial…

More applications for ABJM

- Mirror dual of N=8 SYM (with manifest N=4 SUSY in UV)



- Field contents: N=4 vector + adjoint hyper + fundamental hyper

- Results [Kapustin-Willett-Yaakov (2010)]:

$$Z_{ABJM} = Z_{\text{mirror SYM}}$$

- Similar agreement for ABJM & mirror-SYM superconformal indices

  [Bashkirov-Kapustin (2010)] [Gang-Koh-Lee-Park (2011)]

Generalizations: N=2 SCFT & non-conformal QFT

- In all examples encountered so far (essential with N=3 or higher SUSY), R-charges & scaling dimensions of matter fields were $\Delta = \frac{1}{2}$
- In N=2 SCFT, U(1) R-charge of fields can deviate from this "canonical" value by mixing with other U(1) flavor charges: $\text{assume general } \Delta$
- Lagrangian descriptions of SCFT with anomalous dimensions (w/ manifest symmetry) are more difficult: could play with (non-conformal) UV theories.
- Can write non-conformal QFT on S^3 [KWY] [Jafferis] [Hama-Hosomichi-Lee]
- Osp(2|2) part of Osp(2|4) symmetry preserved on S^3 : proposed to use this partition function to study "IR CFT"

Results

- UV Yang-Mills term is Q-exact: g_{YM} does not appear in Z
- Thus, S^3 radius r cannot appear either. (can appear only as r $g_{YM}{}^2$)

- Results:

- Vector multiplet determinant same as before:
- Matter determinant (chiral multiplet): $\prod_{n=1}^{\infty}\left(\frac{n+1+i\rho(\sigma)-\Delta}{n-1-i\rho(\sigma)+\Delta}\right)^n$

- Zeta function regularization of div. $e^{\ell(1-\Delta+i\sigma)}$

$$\ell'(z) = -\pi z \cot(\pi z)\ , \quad \ell(z) = -z\log(1-e^{2\pi i z}) + \frac{i}{2}\left(\pi z^2 + \frac{1}{\pi}\mathrm{Li}_2(e^{2\pi i z})\right) - \frac{i\pi}{12}$$

Flavor symmetry & real masses

- Anomalous dimensions Δ of all fields are determined by mixing flavor charges to R-charge $R = R_0 + \sum_j a_j F_j$

- Can also introduce real mass associated with each U(1) symmetry: introduce background vector supermultiplet & VEV

$$\text{flat}: \ \text{real mass} \int d^4\theta\, \Phi^\dagger e^{m\theta\bar{\theta}} \Phi$$

$$V_j = \sigma_j \theta\bar{\theta} + \cdots + \theta^2\bar{\theta}^2 D_j = m_j \theta\bar{\theta} - \frac{m_j}{r}\theta^2\bar{\theta}^2 \quad (\text{SUSY}: \ D_j = -\sigma_j/r)$$

- Recall the holomorphic dependence of matter det: $r\sigma + i\Delta$

- Holomorphic appearance of mass parameters:

$$rm_j + ia_j \ \rightarrow \ Z(a_j - irm_j)$$

Superconformal U(1) R-charge

- Z extremization [Jafferis]: differentiating with masses, obtains expectation values for real mass operators (which can possibly mix with other ops.)
- Holomorphy relates them to Δ derivatives
- nonzero expectation value only allowed for identity op. in CFT

$$\partial_{\Delta_i} Z = \frac{i}{r} \partial_{m_i} Z$$

$$0 = \mathrm{Im}\left(Z^{-1}\partial_{m_i} Z\right) \sim \mathrm{Re}\left(Z^{-1}\partial_{\Delta_i} Z\right) \;\; \rightarrow \;\; \partial_{\Delta_i} |Z|^2 = 0$$

- (Determined up to possible discrete degeneracy)
- (Works with the absence of accidental symmetry in IR)
- Also checked via examples: examples in which R-charge can be inferred from symmetry considerations, comparison with perturbative results…

Other applications to N=2 theories

- Comparison with gravity partition function: CFT dual for M-theory on AdS_4 times Q^{111}, V^{52}, etc. [Jafferis-Klebanov-Pufu-Safdi] [Martelli-Sparks] [Cheon, H. Kim, H. Kim]

- “non-chiral models” (matters in self-conjugate rep.): short-ranged force & $N^{3/2}$
- “chiral models” have been studied to yield N^2 with certain saddle points they found. [JKPS]

- Global minima of F: other saddle points, trickier to find
- M^{32}: trickier techniques needed [Amariti, Klare, Siani] [Hwang, Gang, Kim, Park]
- Could also work for infinite class of Ypq Sasaki-Einstein

More applications

- Test of other 3d dualities
- N=2 Mirror dualities [Kapustin-Willett-Yaakov] …
- N=2 Seiberg-like dualities [Kapustin] [Willett-Yaakov] [Benini et.al.] [Hwang, Kim, Park, Park] …
- And more…
- Squashed 3-sphere [Hama-Hosomichi-Lee], gravity dual [Martelli-Sparks]

Concluding remarks

- $N^{3/2}$ for S^3 free energy; many other dualities studied.

- Various relations between different observables…
- Relations between partition functions in different dimensions… [Dolan, Spiridonov, Vartanov] [Gadde et.al.] [Imamura] [Benini et.al.]

$$S^3 \times S^1 \to S^3 \,, \quad S^3/\mathbb{Z}_n \times S^1 \stackrel{n\to\infty}{\longrightarrow} S^2 \times S^1$$

- More partition functions? squashed S^3 partition function [Hama-Hosomichi-Lee], vortex partition function, generalized SC index [Kapustin-Willett], …
- $F(S^3)$ as a measure of d.o.f. ; large N scaling of "thermal" quantities (like index); baryonic M5-branes in AdS_4 x Y_7 ; …