Techniques for exact calculations in 4D SUSY gauge theories

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First lecture

- Motivations for studying four dimensional N=2 theories
- N=2 supersymmetric gauge theories
- Older examples of exact results in four-dimensional N=2 gauge theories
- Wilson loops in N=4 super Yang-Mills and AdS/CFT
- Localization for N=2 gauge theories on S^4
  - Strategy for localization
  - Localization equations and their solutions

Second lecture

- Localization for N=2 gauge theories on $\mathbf{S}^4$ (continued)
  - Gauge-fixing
  - Fluctuation determinants and the equivariant index
  - One-loop contributions
  - Instanton partition function on $\mathbf{R}^4$
  - Instanton contributions on $\mathbf{S}^4$
- 4D/2D correspondence
- S-duality

Third lecture

- Localization for 't Hooft loops on $\mathbf{S}^4$
  - Motivations, set-up and localization
  - Monopole/instanton correspondence
  - Results and comparison with 2D theories
- Line operators on $\mathbf{S}^1\times\mathbf{R}^3$
- Instanton counting with surface operators
- Instanton counting on ALE spaces
- Superconformal index
- Concluding remarks

First lecture

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- N=2 supersymmetric gauge theories
- Older examples of exact results in four-dimensional N=2 gauge theories
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  - Localization equations and their solutions

Motivations

N=2 theories are not chiral, so not suitable as a model for the physics beyond the Standard Model. Why are we interested?

Motivations

- Good theoretical laboratories for studying non-perturbative effects and dualities.
- Rich mathematical structure: Donaldson theory, geometric Langlands program, etc.
- May help understand M5-brane theory.
- $\exists$Holographic duals. cond-mat applications?

Motivations

N=2 gauge theories are related to

- 2D CFTs
- Hitchin systems
- Other integrable models

N=2 theories admit interesting operators

- Wilson-'t Hooft loop operators
- Surface operators
- Domain walls

Basics of N=2 SUSY gauge theories

- Vector multiplet for gauge group G

$$\begin{array}{ccc} & A_\mu & \\ \lambda & & \psi \\ & \phi & \end{array}$$

- Hypermultiplet in representation R

$$\begin{array}{ccc} & \psi_q & \\ q & & \widetilde{q}^\dagger \\ & \psi_{\widetilde{q}}^\dagger & \end{array}$$

Basics of N=2 SUSY gauge theories

- Vector multiplet for gauge group G

$$A_\mu \quad \lambda \quad \psi \quad \phi$$

N=1 multiplets

- Hypermultiplet in representation R

$$\psi_q \quad q \quad \widetilde{q}^\dagger \quad \psi^\dagger_{\widetilde{q}}$$

Basics of N=2 SUSY gauge theories

- Vector multiplet for gauge group G

$$A_\mu$$

$$\lambda \qquad \psi$$

$$\phi$$

SU(2)
R-symmetry

- Hypermultiplet in representation R

$$\psi_q$$

$$q \qquad \widetilde{q}^\dagger$$

$$\psi_{\widetilde{q}}^\dagger$$

- N=2 SUSY is restrictive. After the gauge group and the matter representations are specified, variable parameters are gauge couplings, theta angles, and masses. Given them, the renormalizable Lagrangian on $\mathbf{R}^4$ is unique.

(Older) exact results in N=2 theories

- Witten '88: topological twist of N=2 gauge theory
- Seiberg and Witten '94: exact low-energy effective action in terms of the prepotential
- Nekrasov '02: instanton partition functions

Topological twist of N=2 gauge theory

- Fields in N=2 theory transform under the Lorentz and R-symmetry groups

$$SU(2)_{\text{left}} \times SU(2)_{\text{right}} \times SU(2)_R$$

- Interpret $SU(2)_{\text{left}} \times [SU(2)_{\text{right}} \times SU(2)_R]_{\text{diag}}$

  as the new Lorentz group. Twisted theory.

- There exists a scalar supercharge Q.

- Can put the twisted theory on an arbitrary curved 4-manifold X_4.

- The action is Q-exact up to a topological term

$$t$$

$$S_{\text{twisted}} = \frac{1}{g^2} Q(\ldots) + \frac{\tau}{4\pi} \int_{X_4} \mathrm{Tr} F \wedge F \,.$$

$$\tau = \frac{\theta}{2\pi} + i \frac{4\pi}{g^2}$$

- If there is no matter, the Q-exact term is the Mathai-Quillen representative of the Euler class on the space of connections.

- The infinite dimensional path integral reduces to the finite dimensional ones

$$\int \mathcal{D}A \ldots e^{-S} \ldots = \sum_k q^k \int_{\mathcal{M}_{\mathrm{inst},k}(X_4)} (\ldots) \qquad q = e^{2\pi i \tau}$$

- These are Donaldson's invariants for the 4-manifold X_4.

- No IR divergence for compact X_4.

Seiberg-Witten prepotential

- In '94 Seiberg and Witten determined the low-energy effective action for N=2 gauge theories in the Coulomb branch.

$$S_{\text{eff}} = \frac{1}{4\pi}\int d^4x \mathrm{Im}\left[d^4\theta \frac{\partial \mathcal{F}}{\partial A}\overline{A} + \frac{1}{2}\int d^2\theta \frac{\partial^2 \mathcal{F}}{\partial A^2} W_\alpha W^\alpha\right]$$

- Exact prepotential F determined by consistency conditions, mainly holomorphy.
- Direct calculation difficult. ∃IR divergence on **R**^{4}.

Nekrasov's instanton partition function

- In '02, Nekrasov circumvented this IR problem by introducing a useful regularization called Omega deformation.
- $\mathbf{R}^4$ fibered over $\mathbf{S}^1$ with rotations in two planes. Reduce on $\mathbf{S}^1$. Parameters ϵ_1 , ϵ_2 .
- Omega deformation + topological twist → "Mathai-Quillen representative of the equivariant Euler class" of certain bundles.

- Nekrasov used an RG flow to argue that Z_{inst} should be related the prepotential:

$$Z_{\text{inst}} \sim e^{\frac{1}{\epsilon_1 \epsilon_2} \mathcal{F}_{\text{inst}}(a)} \text{ as } \epsilon_1, \epsilon_2 \to 0$$

- Nekrasov and Okounkov confirmed this by computing the limit and comparing with known results.

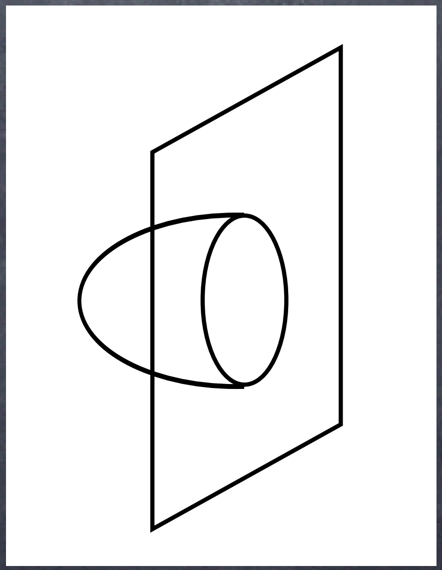
- Will compute Z_{inst} in the next lecture.

Wilson loops in N=4 super Yang-Mills

- For Pestun, the motivation for exact localization calculation came from a conjecture inspired by AdS/CFT.
- N=4 U(N) SYM $\Leftrightarrow$ Type IIB string on $AdS_5 x S^5$
- Maldacena-Wilson loop $\Leftrightarrow$ Minimal surface

$$W_R = \mathrm{Tr}_R e^{-\oint (iA + \Phi_0 ds)}$$

R: fundamental of U(N)



Gaussian matrix model conjecture

- Half-BPS loop: circular or straight line.
- Sum of planar rainbow diagrams for $\langle W_R \rangle$ (combinatorics captured by Gaussian matrix model) reproduce the regularized area of the minimal surface. [Erickson-Semenoff-Zarembo]
- Perturbatively, a conformal anomaly implies that $\langle W_R \rangle$ reduces to some matrix model. [Drukker-Gross]

- Conjecture:

$$\langle W_R \rangle = Z^{-1} \int_{\mathfrak{t}} dM e^{-\frac{2}{g^2}\mathrm{Tr} M^2} \mathrm{Tr}_R e^M$$

- Z: partition function of the matrix model
- $\mathfrak{t}$: Cartan of U(N) = hermitian matrices
- Many, many tests by AdS/CFT.
- Pestun wanted to prove the conjecture by localization.

10D notation

- 4D N=4 super Yang-Mills is a dimensional reduction of the 10D super Yang-Mills.
- A_M: 10D gauge field, Ψ : 16-component chiral spinor.
- 10D Euclidean metric

$$ds^2 = \sum_{M=1,\ldots,9,0} dx^M dx^M$$

- 10D gamma matrices

$$\{\mathbf{\Gamma}^M, \mathbf{\Gamma}^N\} = 2\delta^{MN}$$

$$\mathbf{\Gamma}^M = \begin{pmatrix} & \widetilde{\Gamma}^M \\ \Gamma^M & \end{pmatrix}$$

- $\Gamma^M, \widetilde{\Gamma}^M$ are symmetric 16x16 matrices.
- Compactify on small T^6 to get a 4D theory.

- The 4D action for N=4 super Yang-Mills on $\mathbf{R}^4$ is simply

$$S = \frac{1}{g^2}\int d^4x \mathrm{Tr}\left[\frac{1}{2}F_{MN}F^{MN} - \Psi^T\Gamma^M D_M\Psi\right]$$

- Covariant derivative and field strength

$$D_M = \partial_M + iA_M \qquad F_{MN} \equiv -i[D_M, D_N]$$

- Pointcare SUSY $\quad \delta_\epsilon A_M = \epsilon^T\Gamma_M\Psi\,,$

$$\delta_\epsilon\Psi = \frac{1}{2}F_{MN}\Gamma^{[M}\Gamma^{N]}\epsilon$$

- 4D field content

gauge field	A_μ	$\mu = 1, \ldots, 4\,,$
real scalars	$\Phi_A \equiv A_A$	$A = 5, \ldots, 9, 0\,,$
fermions	Ψ	$(\mathbf{2}\,,\mathbf{1}\,,\mathbf{4}) \oplus (\mathbf{1}\,,\mathbf{2}\,,\bar{\mathbf{4}})$ of
		$SU(2)_{\text{left}} \times SU(2)_{\text{right}} \times SU(4)_R$

Superconformal symmetry of N=4 SYM

- N=4 superconformal transformations on $\mathbf{R}^4$ are generated by position dependent SUSY parameter $\epsilon(x) = \epsilon_s + x^\mu \widetilde{\Gamma}_\mu \epsilon_c$
- Fermionic transformations in N=4 SYM are

$$\delta_\epsilon A_M = \epsilon^T(x)\Gamma_M\Psi\,,$$

$$\delta_\epsilon\Psi = \frac{1}{2}F_{MN}\Gamma^{[M}\Gamma^{N]}\epsilon(x) + \frac{1}{2}\Gamma^{\mu A}\Phi_A\partial_\mu\epsilon(x)$$

- Generate N=4 superconformal algebra

$$PSU(2,2|4)$$

Conformal Killing spinor equations

- The spinor $\epsilon(x)$ satisfies the **conformal Killing spinor equations**

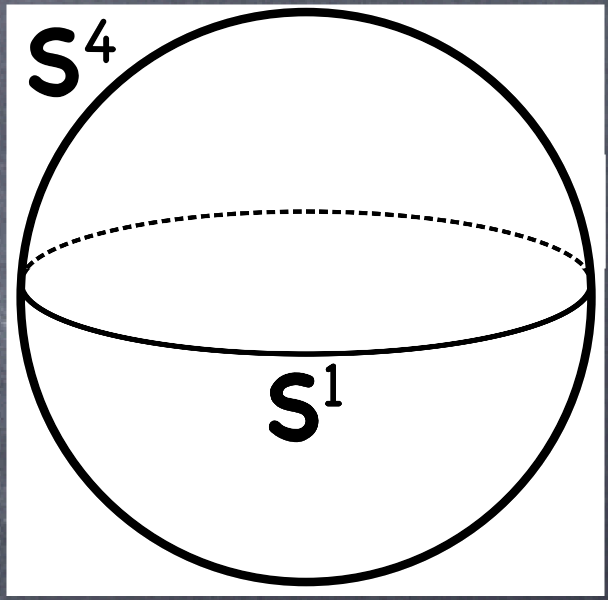
$$\nabla_\mu \epsilon = \widetilde{\Gamma}_\mu \widetilde{\epsilon}, \qquad \widetilde{\epsilon} = \frac{1}{4} \Gamma^\mu \nabla_\mu \epsilon$$

- The equations are invariant under Weyl rescaling $ds^2 \to \Omega^2 ds^2$ if the spinor transforms as $\epsilon \to \Omega^{1/2} \epsilon$

From $\mathbf{R}^4$ to $\mathbf{S}^4$

- Now stereographically map $\mathbf{R}^4$ to $\mathbf{S}^4$ of radius r by taking a conformal factor

$$\Omega = \left(1 + \frac{x^\mu x^\mu}{4r^2}\right)^{-1}$$



- Still have $PSU(2,2|4)$ as symmetry.

- Place a Wilson loop along the equatorial $\mathbf{S}^1$

$$x_1^2 + x_2^2 = 4r^2\,, \quad x_3 = x_4 = 0$$

Breaking to N=2 superconformal algebra

- To restrict to N=2 subalgebra, impose the projection condition

$$\widetilde{\Gamma}^5\Gamma^6\widetilde{\Gamma}^7\Gamma^8\epsilon = -\epsilon$$

- Generate N=2 superconformal algebra

$$SU(2,2|2)\times SU(2)_{\mathrm{F}} \subset PSU(2,2|4)$$

$$\overbrace{A_1,\dots A_4}^{SU(2)_{\mathrm{left}}\times SU(2)_{\mathrm{right}}} \qquad \overbrace{\Phi_5,\dots,\Phi_8}^{SU(2)_R\times SU(2)_{\mathrm{F}}} \qquad \overbrace{\Phi_9,\Phi_0}^{U(1)_R}$$

Breaking to N=2 SUSY algebra on S^4

- It is possible to further restrict SUSY parameters so that the anti-commutators of fermionic charges do not generate dilatation or $U(1)_R$ symmetry. This will allow mass.

- After choosing a U(1) subgroup of $SU(2)_R$ generated by self-dual 4x4 anti-symmetric matrix R_{kl} normalized as $R_{kl}R_{kl}=4$, these SUSY parameters are characterized by

$$D_\mu \epsilon = -\frac{1}{8r}\widetilde{\Gamma}_\mu \widetilde{\Gamma}^k \Gamma^l R_{kl} \widetilde{\Gamma}^0 \epsilon$$

- The restricted SUSY parameters geenerate OSp(2|4) : N=2 SUSY algebra on $\mathbf{S}^4$. [Pestun]

Adding mass

- OSp(2|4) does not include dilatation that would be broken by mass. Can we add a mass term?
- Choose M: a generator of the flavor symmetry $SU(2)_F$. M=diag(m,-m).
- Weakly gauge the flavor symmetry $SU(2)_F$ and set a real scalar in the new vector multiplet to M.

- Redefine $D_0 \equiv i[\Phi_0\,, \bullet\,] + iM$
- Can be regarded as a background Wilson line in the compactified 0-direction. Green-Schwarz mechanism.
- "10D field strength" $F_{MN} \equiv -i[D_M, D_N]$ includes mass.
- SUSY transformations get deformed by mass.

- M acts as an anti-self-dual 4x4 anti-symmetric matrix on hypermultiplet scalars:

$$(M \cdot \Phi)_j = M_{jk} \Phi_k$$

- On spinors, $M \cdot \Psi = \frac{1}{4} M_{jk} \widetilde{\Gamma}_j \Gamma_k \Psi$

- Real work: find a mass-deformed action preserving N=2 SUSY.

- Strategy: start with the familiar action with a conformal coupling to the curvature and compute

$$\delta_\epsilon \mathrm{Tr} \left[\frac{1}{2} F_{MM} F^{MN} - \Psi^T \Gamma^M D_M \Psi + \frac{2}{r^2} \Phi_A \Phi_A \right]$$

- This turns out to equal $\delta_\epsilon \left[\frac{1}{4r} R_{jl} M_{kl} \Phi^j \Phi^k\right]$

- Thus the action

$$S = \frac{1}{g^2}\int d^4x\sqrt{h}\mathrm{Tr}\left[\frac{1}{2}F_{MM}F^{MN} - \Psi^T\Gamma^M D_M\Psi + \frac{2}{r^2}\Phi_A\Phi_A - \frac{1}{4r}R_{jl}M_{kl}\Phi^j\Phi^k\right]$$

is supersymmetric.

- Action for mass-deformed N=4 SYM (known as N=2*) can be generalized to other N=2 theories. Will use N=2* to simplify notation, but will get results for general N=2 theories.

Off-shell SUSY

- Will need just one supercharge $Q = \delta_\epsilon$ for localization.
- In the usual formulation, its square is a bosonic symmetry generator up to the equation of motion (on-shell).
- For localization we need it to square to a bosonic symmetry off-shell.
- Need auxiliary fields.

Berkovits' method

- First we construct seven chiral spinors ν_j (j=1, . . . , 7) satisfying

$$\epsilon^T \Gamma^M \nu_j = 0\,,$$

$$\nu_j \nu_j^T + \epsilon\epsilon^T = \frac{1}{2}(\epsilon^T \Gamma_N \epsilon)\widetilde{\Gamma}^N\,,$$

$$\nu_i^T \Gamma^M \nu_j = \delta_{ij}\epsilon^T \Gamma^M \epsilon$$

- Add real bosonic auxiliary fields K_j.

- Off-shell SUSY transformations are

$$\delta_\epsilon A_M = \epsilon^T \Gamma_M \Psi \,,$$

$$\delta_\epsilon \Psi = \frac{1}{2} F_{MN} \Gamma^{[M} \Gamma^{N]} \epsilon + \frac{1}{2} \Gamma^{\mu A} \Phi_A D_\mu \epsilon + i K^j \nu_j \,,$$

$$\delta_\epsilon K_j = i \nu_j^T \Gamma^M D_M \Psi$$

- The supercharge squares to bosonic symmetries off-shell.

- Define $v^M = \epsilon^T \Gamma^M \epsilon$ and normalize ϵ so that
$$\epsilon^T \Gamma^0 \epsilon = ir$$

- Then $\quad Q^2 = iJ + iR + r\Phi_0 + rM$

- Here J is the isometry generated by the vector field v^μ, R is the R-symmetry generator, Φ_0 acts as a gauge transformation, and M is the flavor symmetry generator.

- The action with auxiliary fields for N=2* SYM on $\mathbf{S}^4$ is given by

$$S = \frac{1}{g^2}\int d^4x\sqrt{h}\mathrm{Tr}\left[\frac{1}{2}F_{MM}F^{MN} - \Psi^T\Gamma^M D_M\Psi \right.$$
$$\left. + \frac{2}{r^2}\Phi_A\Phi_A - \frac{1}{4r}R_{jl}M_{jk}\Phi^j\Phi^k - K_jK_j\right]$$

Strategy for localization

- Pestun's calculation in 2007 is the prototype of localization calculations discussed in S. Kim's lectures.

- Choose a supercharge Q that preserves the operator $\mathcal{O} = \mathrm{Tr}_R e^{-\oint (iA + i\Phi_0 ds)}$ and modify the action

$$\int \mathcal{D}A \ldots e^{-S}\mathcal{O} \to \int \mathcal{D}A \ldots e^{-S-tQ\cdot V}\mathcal{O}$$

- For Q^2-invariant V, the path integral is independent of t. To see this, compute

$$\frac{d}{dt}\int \mathcal{D}A\mathcal{D}\Psi\mathcal{D}K e^{-S-tQ\cdot V}\mathcal{O}$$

$$= -\int \mathcal{D}A\ldots(Q\cdot V)e^{-S-tQ\cdot V}\mathcal{O}$$

$$= -\int \mathcal{D}A\ldots Q\left[e^{-S-tQ\cdot V}\mathcal{O}\right]$$

- We used Q-invariance of S and O, and Q^2-invariance of V.

- We may write

$$Q = \int d^4x\sqrt{h}\left(\delta_\epsilon A_M(x)\frac{\delta}{\delta A_M(x)} + \delta_\epsilon \Psi(x)\cdot\frac{\delta}{\delta\Psi(x)} + \ldots\right)$$

- This is a functional analog of the exterior derivative d. Thus

$$\frac{d}{dt}\int \mathcal{D}A\mathcal{D}\Psi\mathcal{D}K e^{-S-tQ\cdot V}\mathcal{O} = 0$$

- If we take $V = \langle \Psi, \overline{Q \cdot \Psi} \rangle$ the bosonic terms are given by $||Q \cdot \Psi||^2 \subset Q \cdot V$

- This is positive-semidefinite. In the limit $t \to +\infty$

  the path integral localizes to the solutions of

$$Q \cdot \Psi = 0$$

General procedure for localization

1. Pick a supercharge Q that is preserved by the operator to compute. Add auxiliary fields so that Q squares to bosonic symmetries off-shell.

2. Choose a Q^2-invariant functional V such that the bosonic terms of $Q \cdot V$ are positive-semidefinite. Add $tQ \cdot V$ to the action.

3. Find the saddle points of $e^{-tQ \cdot V}$.

4. Compute the fluctuation determinants at the saddle points. This involves gauge-fixing and the inclusion of ghost fields. Either expand fields in the eigenmodes of kinetic operators, or use **the equivariant index theorem**.

Our choice of Q

- OSp(2|4) (N=2 on S^4) has 8 supercharges.
- Wilson loop preserves 4 out of 8

$$Q\left(iA_\mu \frac{dx^\mu}{ds} + \Phi_0\right) = 0$$

- Require that ϵ is right-handed at the north pole (Q$\sim$scalar supercharge for twisted theory) . 2 out of 4.
- Any linear combination of the two will do.

Completing squares

- To solve the localization equation, eliminate gamma matrices from $||Q \cdot \Psi||^2$ and complete squares.
- For the vector multiplet

$$||Q \cdot \Psi||^2 \supset \sin^2 \frac{\theta}{2}(F^-_{\mu\nu} + w^-_{\mu\nu}\Phi_9)^2 + \cos^2 \frac{\theta}{2}(F^+_{\mu\nu} + w^+_{\mu\nu}\Phi_9)^2$$
$$+ (D_\mu \Phi_a)^2 + \frac{1}{2}[\Phi_a, \Phi_b][\Phi^a, \Phi^b] + (K_i + w_i \Phi_0)^2$$

- $w^{\pm}_{\mu\nu}$ and w_i are expressions constructed from ϵ and gamma matrices. θ is the longitude such that the north and south poles are at θ =0 and π .

- Away from the north and south poles (i.e., $\theta \neq 0, \pi$),

$$F_{\mu\nu} = -(w^+ + w^-)_{\mu\nu}\Phi_9$$

- Bianchi identity implies that

$$D_{[\rho}(w^+ + w^-)_{\mu\nu]}\Phi_9 = 0$$

- $D_{[\rho}(w^+ + w^-)_{\mu\nu]}$ is non-zero, so Φ_9 and hence $F_{\mu\nu}$ have to vanish.

- Φ_0 can be a non-zero constant.

- A similar expression (complete squares) for the hypermultiplet contains positive-definite terms

$$\frac{3}{4r^2}\sum_{k=5}^{8}\Phi_k^2$$

- Hypermultiplet scalars must vanish.

Solutions of $Q \cdot \Psi = 0$

$$A_\mu = 0 \quad \mu = 1, \ldots, 4$$

$$\Phi_A = 0 \quad A = 5, \ldots, 8, 9$$

$$\underline{\Phi_0 = a = \text{constant}}$$

$$K_j = -w_j a \quad j = 5, 6, 7$$

$$K_j = 0 \quad j = 1, \ldots, 4$$

Non-perturbative saddle points

- The vanishing of $F_{\mu\nu}$ assumed that we were away from the north and south poles.
- Configurations that are localized at the poles may contribute.
- The localization action is approximately the action for the Omega-deformed theory of Nekrasov. Small instantons will contribute.