Nonrelativistic conformal invariance

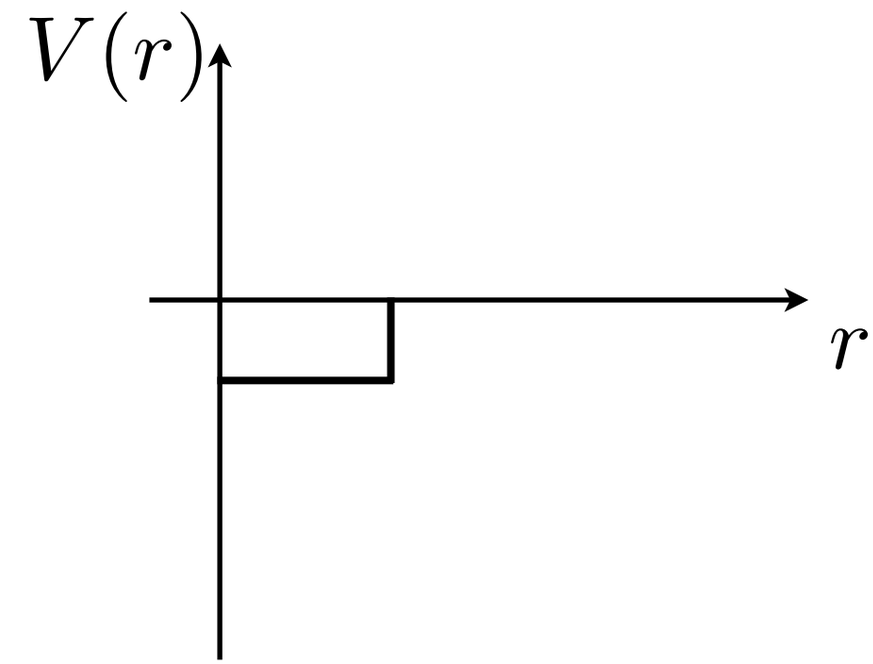
Dam T. Son (INT)

Plan of the talk

- Fermions at unitarity
- Conformal invariance
- Classification of operators: primary, descendants
- Operator-state correspondence
- Attempt for a holographic description

Fermions at unitarity

Consider 2 particles interacting though a potentials



Fermions at unitarity

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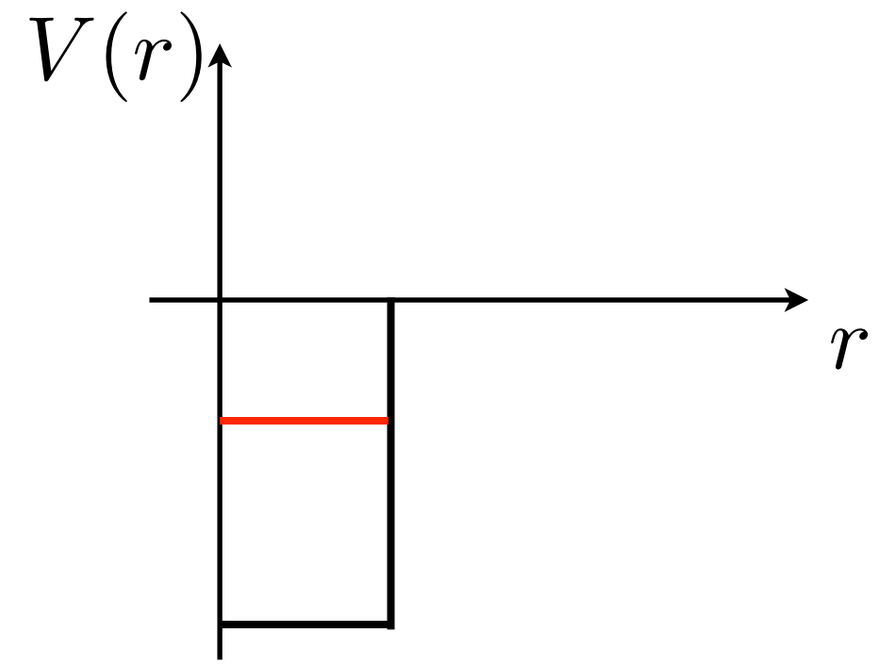

Fermions at unitarity

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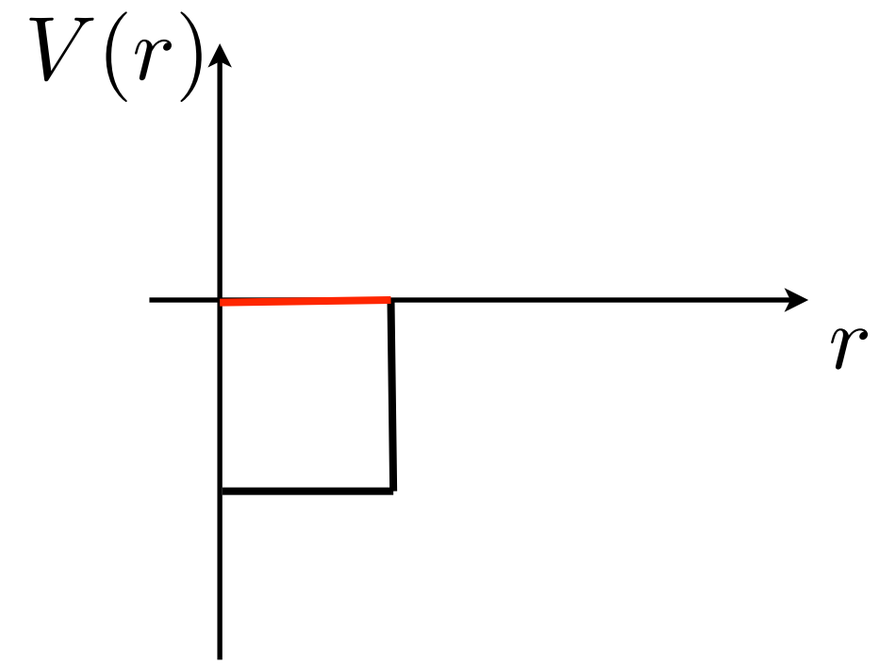

Fermions at unitarity

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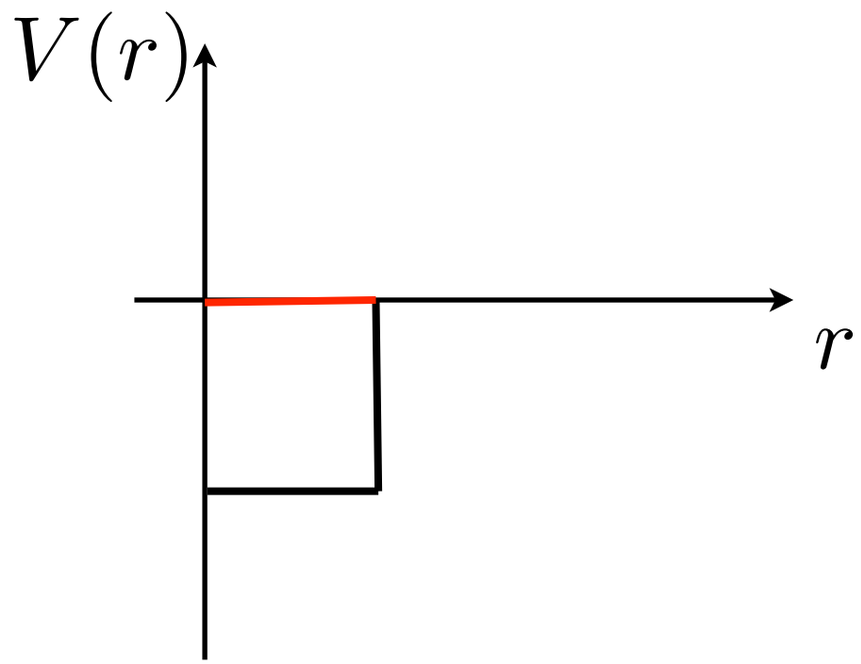
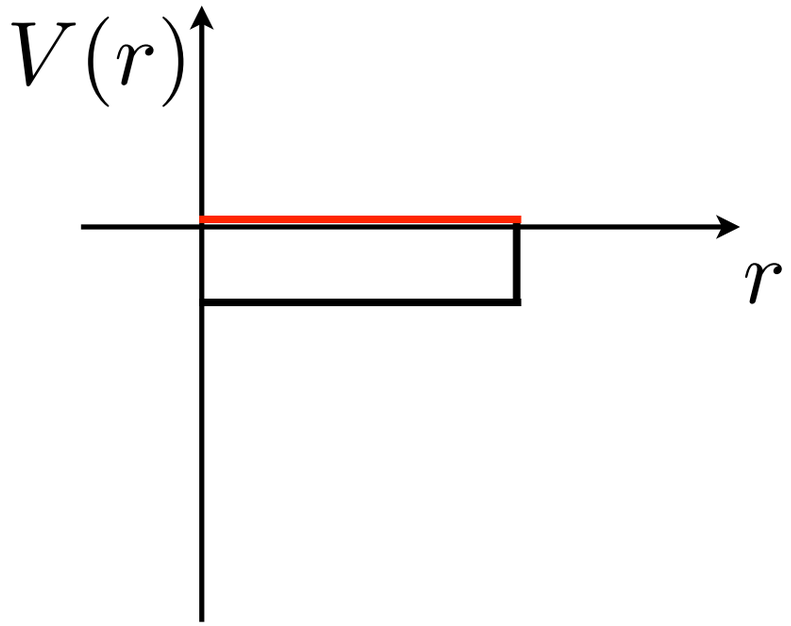


There is a critical depth when the bound state starts to appear

Zero-range limit



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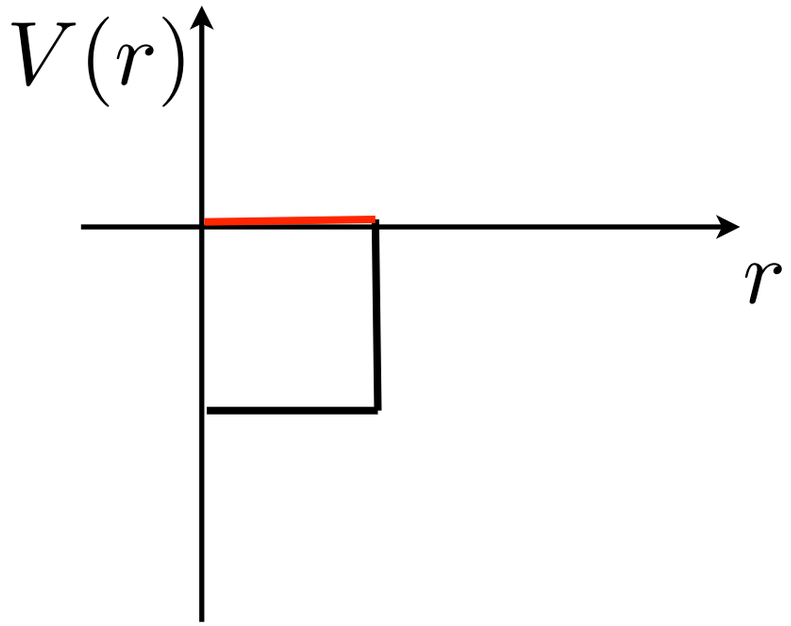

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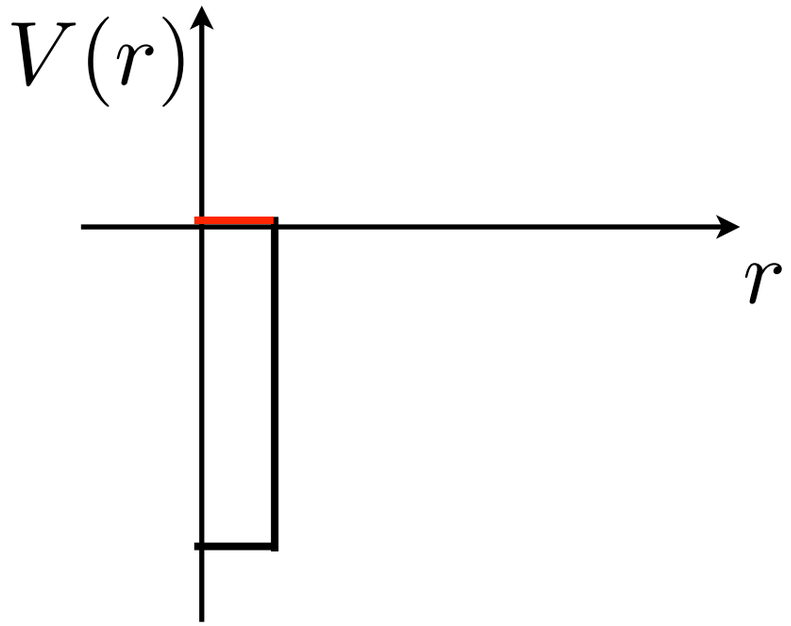

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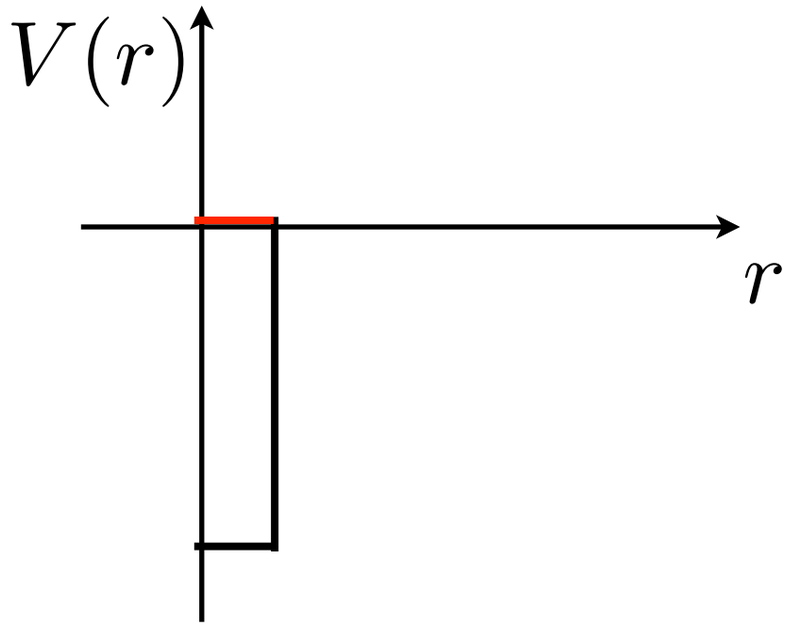


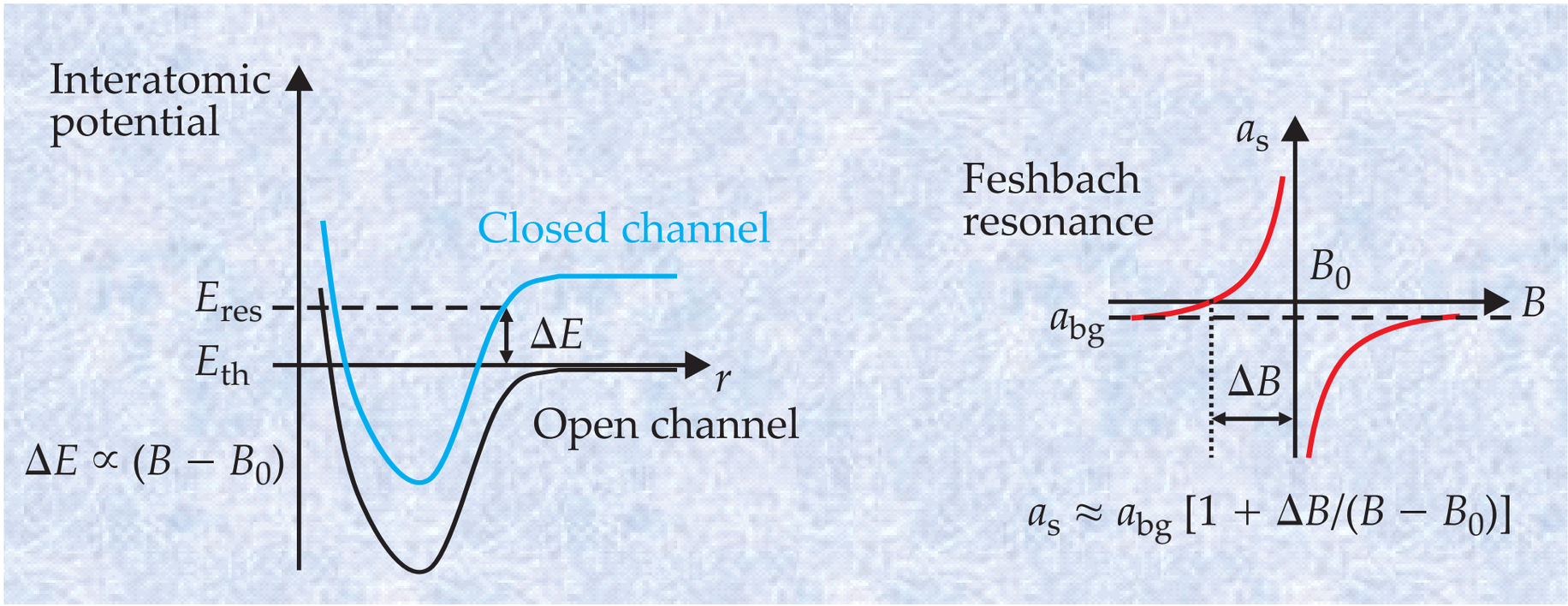
We can shrink the potential to zero size, keeping the zero-energy bound state

Will have potential with no intrinsic length scale (infinite scattering length, zero range)

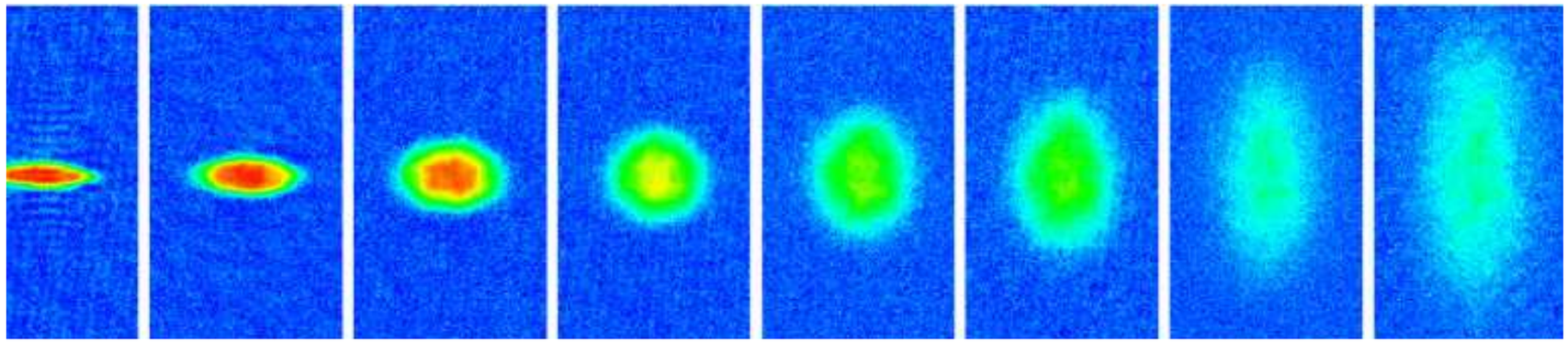
This regime is called the unitarity regime (s-wave scattering saturates unitarity)

Realizations

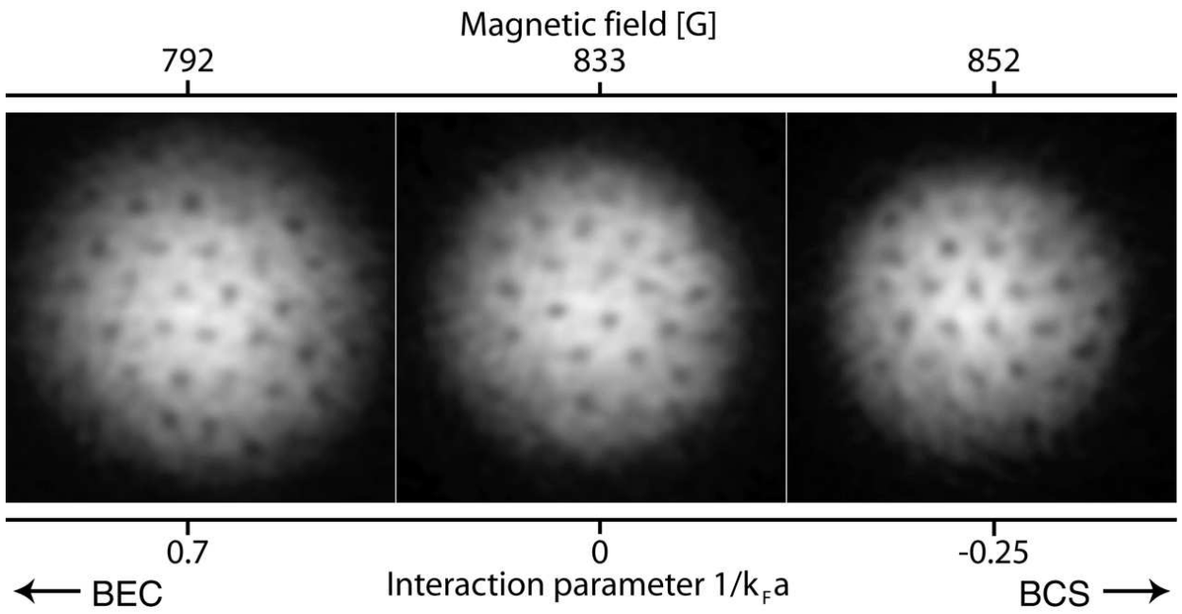
- Neutrons: almost bound state (a=-20 fm, r_0~fm)
- Fermions at unitarity are realized with atoms



A system at strong coupling



collective behavior



superfluidity

Bertsch problem

- What are the ground state properties of a gas of spin-1/2 particles at unitarity?
- In particular, what is the energy density?

dimensional analysis

$$\epsilon \equiv \frac{E}{V} = \# \frac{n^{5/3}}{m}$$

the same parametric dependence as a free gas

$$\epsilon(n) = \xi \epsilon_{\rm free}(n)$$

Bertsch parameter, ~ 0.4

Few-body physics

- In this talk we will concentrate mostly on few-body physics
- can be useful for many-body physics (through OPE)

Field theory interpretation

Consider the following model Sachdev, Nikolic; Nishida, DTS

$$S = \int dt\, d^d x \left(i\psi^\dagger \partial_t \psi - \frac{1}{2m} |\nabla \psi|^2 - c_0 \psi_\uparrow^\dagger \psi_\downarrow^\dagger \psi_\downarrow \psi_\uparrow \right)$$

Dimensional analysis:

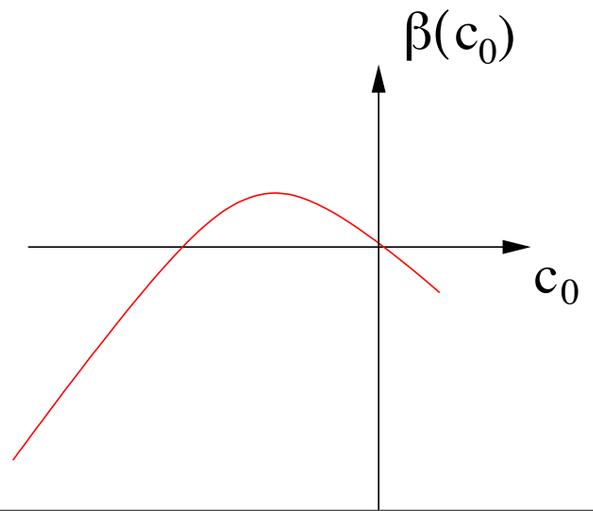
$$[t] = -2, \quad [x] = -1, \quad [\psi] = \frac{d}{2}, \quad [c_0] = 2 - d$$

Contact interaction is irrelevant at $d > 2$
RG equation in $d = 2 + \epsilon$:

$$\frac{\partial c_0}{\partial s} = -\epsilon c_0 - \frac{c_0^2}{2\pi}$$

Two fixed points:

- $c_0 = 0$: trivial, noninteracting
- $c_0 = -2\pi\epsilon$: unitarity regime



Field theory in $d = 4 - \epsilon$ dimensions

Sachdev, Nikolic; Nishida, DTS; Nussinov and Nussinov

$$S = \int dt\, d^d x \left(i\psi^\dagger \partial_t \psi - \frac{1}{2m} |\nabla \psi|^2 - g\phi \psi_\uparrow^\dagger \psi_\downarrow^\dagger - g\phi^* \psi_\downarrow \psi_\uparrow \right.$$
$$\left. + i\phi^* \partial_t \phi - \frac{1}{4m} |\nabla \phi|^2 + C\phi^* \phi \right)$$

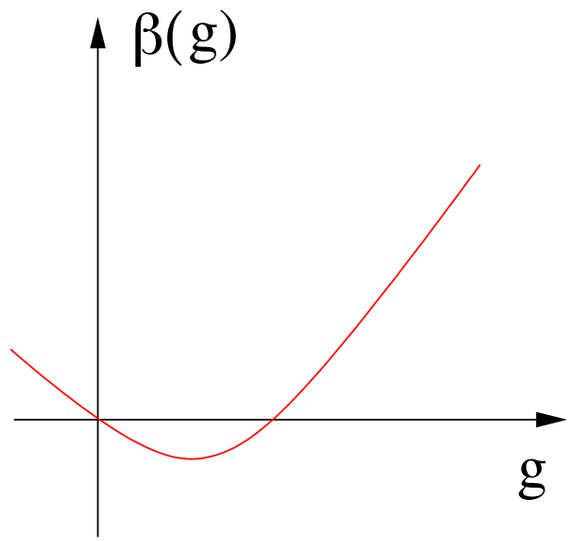
C fined tuned to criticality

Dimensions: $[g] = \frac{1}{2}(4-d) = \frac{1}{2}\epsilon$

RG equation for g:

$$\frac{\partial g}{\partial \ln \mu} = -\frac{\epsilon}{2} g + \frac{g^3}{16\pi^2}$$

Fixed point at $g^2 = 8\pi^2 \epsilon$



Boundary conditions

Unitarity fermions = system described by free Hamiltonian

$$H = \sum_i \frac{\mathbf{p}_i^2}{2m}$$

with nontrivial boundary condition on the wave function:

$$\Psi(\underbrace{\mathbf{x}_1, \mathbf{x}_2, \ldots}_{\mathrm{spin-up}}, \underbrace{\mathbf{y}_1, \mathbf{y}_2, \ldots}_{\mathrm{spin-down}})$$

$$\Psi \to \frac{C}{|\mathbf{x}_i - \mathbf{y}_j|} + 0 \times |\mathbf{x}_i - \mathbf{y}_j|^0 + O(|\mathbf{x}_i - \mathbf{y}_j|) \qquad |\mathbf{x}_i - \mathbf{y}_j| \to 0$$

Free fermions corresponds to another boundary condition:

$$\Psi \to \frac{0}{|\mathbf{x}_i - \mathbf{y}_j|} + C + O(|\mathbf{x}_i - y_j|)$$

Galilean algebra

Conserved quantities:

Hamiltonian: H $t \to t + \delta t$

Momentum: P_i $\mathbf{x} \to \mathbf{x} + \mathbf{a}$

Particle number (mass): M $\psi \to e^{i\alpha}\psi$

} commute

$N = M$

and Galilean boosts: K_i $\mathbf{x} \to \mathbf{x} + \mathbf{v}t$

M, P, K can be expressed in terms of local density and current:

$$M = \int d\mathbf{x}\, n(\mathbf{x}) \qquad \mathbf{P} = \int d\mathbf{x}\, \mathbf{j}(\mathbf{x}) \qquad \mathbf{K} = \int d\mathbf{x}\, \mathbf{x}\, n(\mathbf{x})$$

$$n = \psi^\dagger\psi, \quad \mathbf{j} = -\frac{i}{2}(\psi^\dagger\nabla\psi - \nabla\psi^\dagger\psi)$$

also angular momentum

Galilean algebra (II)

Using commutation relations between n and j:

$$[n(\boldsymbol{x}),\, n(\boldsymbol{y})] = 0, \qquad [n(\boldsymbol{x}),\, j_i(\boldsymbol{y})] = -in(\boldsymbol{y})\partial_i\delta(\boldsymbol{x}-\boldsymbol{y}),$$
$$[j_i(\boldsymbol{x}),\, j_j(\boldsymbol{y})] = -i\left(j_j(\boldsymbol{x})\partial_i + j_i(\boldsymbol{y})\partial_j\right)\delta(\boldsymbol{x}-\boldsymbol{y}).$$

Landau 1941

and $\quad [H,\, n] = -i\partial_t n = i\nabla\cdot\mathbf{j}$

$$[K_i,\, P_j] = i\delta_{ij}M$$
$$[K_i,\, H] = iP_i$$

Other commutators are zero

Note: K is not conserved, but Galilean invariance has physical consequences: generating family of solutions

Scale invariance

$$\mathbf{x} \to \lambda \mathbf{x}, \quad t \to \lambda^2 t$$

Should be an invariance of fermions at unitarity: no length scale

$$D = \int d\mathbf{x}\, \mathbf{x} \cdot \mathbf{j}$$

$$[D,\, O] = i\Delta_O O$$

dim of O

$$[D,\, \mathbf{P}] = i\mathbf{P}$$

$$[D,\, \mathbf{K}] = -i\mathbf{K}$$

$[D,\, H] = 2iH$ for scale-invariant Hamiltonian

Conformal invariance

If ψ satisfies the time-dependent Schrödinger equation

$$i\frac{\partial}{\partial t}\psi(t,\mathbf{x}_i) = -\sum_i \frac{\nabla_i^2}{2m}\psi(t,\mathbf{x}_i)$$

then

$$\psi_\lambda(t,\mathbf{x}_i) = \frac{1}{(1-\lambda t)^{d/2}} \exp\left[-\frac{im\lambda}{2(1-\lambda t)}\sum_i \mathbf{x}_i^2\right] \psi\left(\frac{t}{1-\lambda t}, \frac{\mathbf{x}}{1-\lambda t}\right)$$

is also a solution to the time-dependent Schr. eq. for any λ

Short-distance boundary condition is preserved.

Was known a long time ago, first applied unitarity fermions by Mehen, Stewart and Wise

Conformal algebra

Contain Galilean operators, dilatation D, and

$$C = \frac{1}{2}\int d\mathbf{x}\,\mathbf{x}^2 n(\mathbf{x})$$

Nonzero commutators involving C:

$$[C,\, P_i] = iK_i \qquad [D,\, C] = -2iC \qquad [C,\, H] = iD$$

$$[D,\, H] = 2iH$$

SO(2,1) subalgebra

Particle number N: center of the algebra

Local operators

Include ψ, $\psi^\dagger$, $\partial_i\psi$, composites like $\psi_\uparrow(x)\psi_\downarrow(x)$ which in general needs renormalization

Commutators with H and P: $[H,\, O(t,\mathbf{x})] = -i\partial_t O(t,\mathbf{x})$

$[P_i,\, O(t,\mathbf{x})] = -i\partial_i O(t,\mathbf{x})$

Classification:

Particle number: $[N,\, O(x)] = iN_O O(x)$

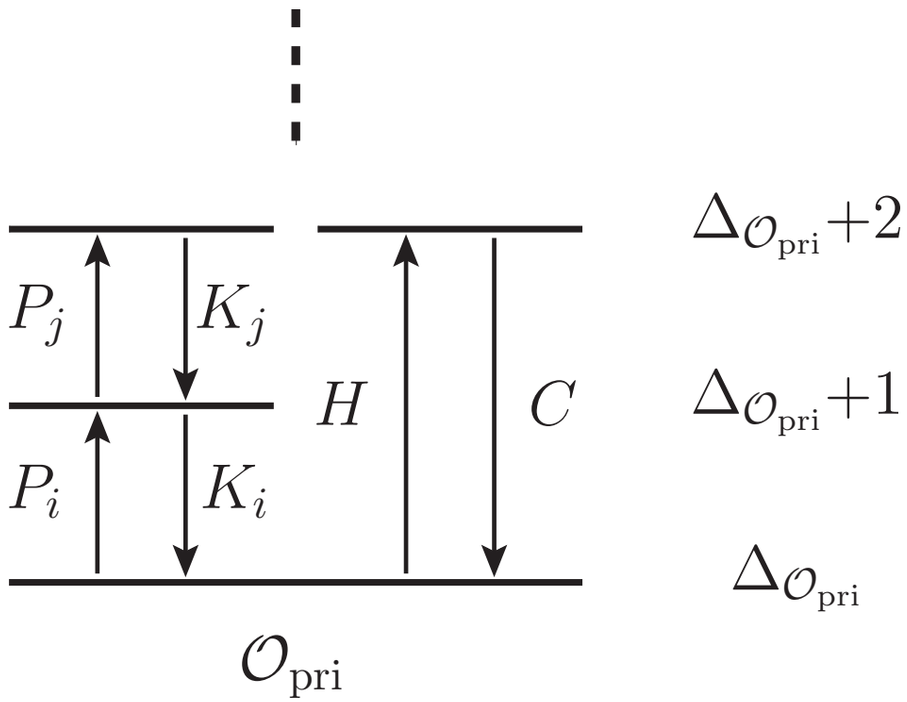
Dimension: $[D,\, O(0)] = i\Delta_O O(0)$

For example, $N_\psi = -1, \quad \Delta_\psi = 3/2$

Primary operators

$[D,\, P_i] = iP_i$ if dim[O]=Δ, then dim[P_i, O]=Δ+1

$$[D,\, H] = 2iH, \quad \dim[H,\, O] = \Delta + 2$$
$$[D,\, K_i] = -iK_i, \quad \dim[K_i,\, O] = \Delta - 1$$



O is primary operator if it cannot be lowered further

$$[K_i,\, O(0)] = [C,\, O(0)] = 0$$

Examples of primary operators

$\psi(x)$: $\quad [K_i\,\psi(0)] = \int d\mathbf{x}\, x_i [n(\mathbf{x}),\, \psi(0)] = -\int d\mathbf{x}\, \psi(\mathbf{x})\delta(\mathbf{x}) = 0$

$\psi_\uparrow \partial_i \psi_\downarrow - \partial_i \psi_\uparrow \psi_\downarrow$

but not $\quad \psi_\uparrow \partial_i \psi_\downarrow + \partial_i \psi_\uparrow \psi_\downarrow = \partial_i (\psi_\uparrow \psi_\downarrow)$

Operator-state correspondence

Consider operators made out of annihilation operators

Primary operator with dimension Δ ↔ Eigenstate in harmonic potential with energy $\hbar\Delta\omega$

Proof:

$$H_{\rm osc} = H + \omega^2 C \qquad C = \frac{1}{2}\int d\mathbf{x}\,\mathbf{x}^2 n$$

$$|\Psi_O\rangle \equiv e^{-H/\omega} O^\dagger(0)|0\rangle$$

$$\begin{aligned} H_\omega|\Psi_\mathcal{O}\rangle &= \left(H+\omega^2 C\right) e^{-H/\omega}\mathcal{O}^\dagger(0)|0\rangle \\ &= e^{-H/\omega}\left(\omega^2 C - i\omega D\right)\mathcal{O}^\dagger(0)|0\rangle = \omega\Delta_\mathcal{O}|\Psi_\mathcal{O}\rangle \end{aligned}$$

H,D,C form SO(2,1)

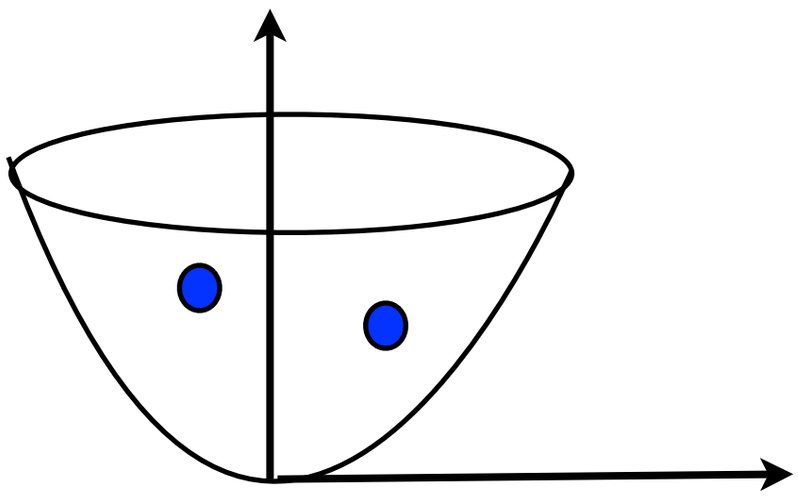
Operator-state correspondence: examples

$$\dim[\psi] = \frac{3}{2}$$

ground state of one particle in harmonic potential: E=3/2 ħω

Two particles

Ground state known exactly



$$\psi(\mathbf{x},\mathbf{y}) = \frac{e^{-(x^2+y^2)/2}}{|\mathbf{x}-\mathbf{y}|}$$

E=2ħω

So the operator $\psi_\uparrow\psi_\downarrow$ has dimension 2

Naive dimension=3, anomalous dimension= -1?

Dimer operator

Consider a 2-body state characterized by a wavefunction Ψ(x,y), call that state $|\Psi\rangle$

$$\langle 0|\psi_\uparrow(\mathbf{x})\psi_\downarrow(\mathbf{y})|\Psi\rangle = \Psi(\mathbf{x},\mathbf{y})$$

But recall the unitary boundary condition:

$$\Psi(\mathbf{x},\mathbf{y}) = \frac{C}{|\mathbf{x}-\mathbf{y}|} + \cdots$$

The operator $\Psi_\uparrow(x)\Psi_\downarrow(x)$ has infinite matrix elements

The properly defined two-body operator is

$$\phi(\mathbf{x}) = \lim_{\mathbf{x}\to\mathbf{y}} 4\pi|\mathbf{x}-\mathbf{y}|\psi_\uparrow(\mathbf{x})\psi_\downarrow(\mathbf{y})$$

Matrix elements of ϕ are finite

Operator product expansion

$$A(t,\mathbf{x})B(0,\mathbf{0}) = \sum_i |\mathbf{x}|^{\Delta_i-\Delta_A-\Delta_B} f_i\left(\frac{|\mathbf{x}|^2}{t}\right) O_i(0)$$

$$G_{AB}(\omega,\mathbf{q}) = \sum_i \frac{1}{\omega^{(5+\Delta_i-\Delta_A-\Delta_B)/2}} c_i\left(\frac{q^2}{\omega}\right)\langle O_i\rangle, \qquad q\equiv|\mathbf{q}|.$$

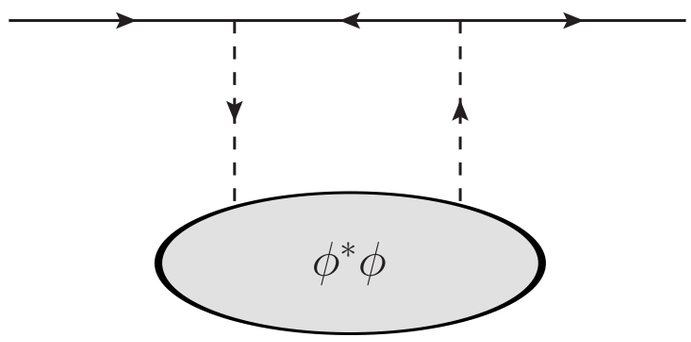
large ω and q regime dominated by operators with lowest dimensions

$\psi^\dagger\psi$ Δ=3 $\phi^\dagger\phi$ Δ=4

c_i reduces to few-body physics

$\langle O_i\rangle$ nonperturbative parameters of many-body state

Tan's parameter



$$\int d\mathbf{x}\, e^{-i\mathbf{q}\cdot\mathbf{x}} \langle \psi_1^\dagger(0,\mathbf{0})\psi_1(t,\mathbf{x})\rangle = \frac{1}{q^4}\exp\left(i\frac{q^2}{2}t\right)\langle\phi^*\phi\rangle + \cdots$$

t=0: high-momentum tail of the distribution function

$$n_{\mathbf{q}} = \frac{\langle\phi^*\phi\rangle}{q^4},$$ ⟵ Shina Tan's "contact"

Dynamic structure factor

Dynamic structure factor S(q,ω) of unitarity fermions at high frequency, high momentum

$$S(\mathbf{q},\omega) = \sum_n \frac{1}{\omega^{\Delta_n - 1/2}} f_n\left(\frac{\mathbf{q}^2}{\omega}\right) \langle O_n^\dagger O_n \rangle$$

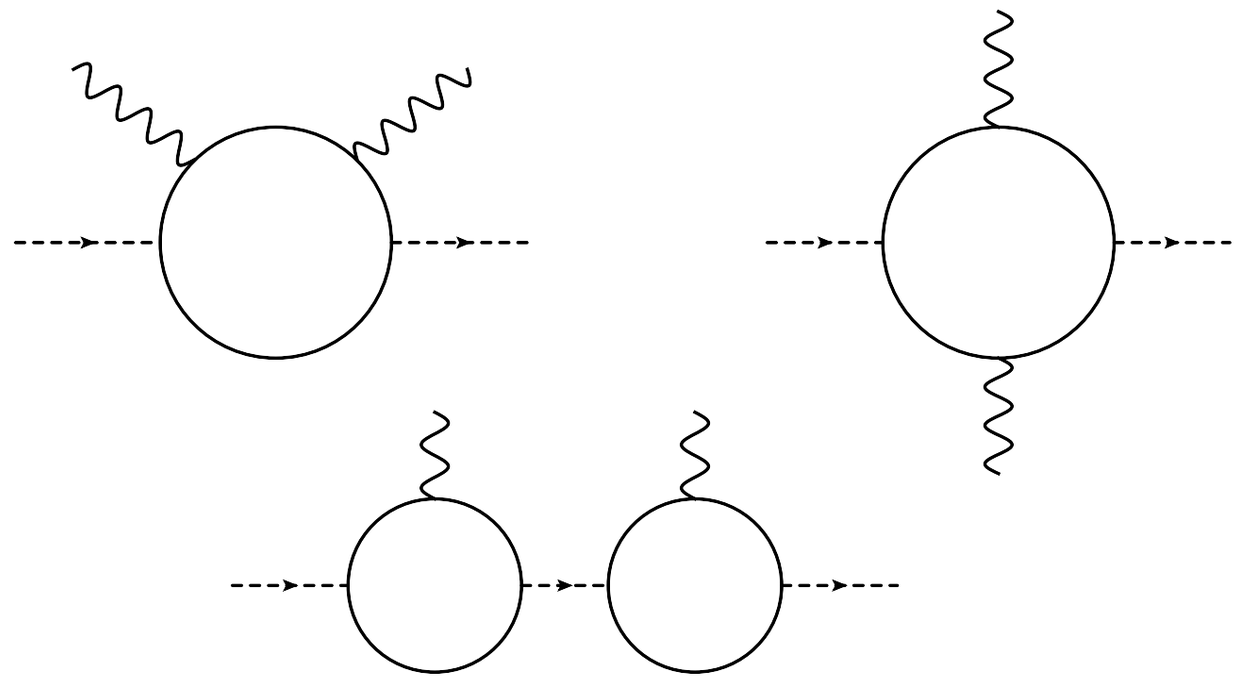
$$\mathrm{Im}\,\langle nn \rangle_{\omega,\mathbf{q}}$$

Dynamic structure factor

DTS, Thompson

$$S(\mathbf{q},\omega) = -\frac{1}{\pi}\,\mathrm{Im}\,G_{nn}(\omega,\mathbf{q})$$

$$G_{nn}(\omega,\mathbf{q}) = -i\int dt\,d\mathbf{q}\,e^{i\omega t - i\mathbf{k}\cdot\mathbf{x}}\,\langle Tn(t,\mathbf{x})n(0,\mathbf{0})\rangle = C_n\langle n\rangle + C_{\phi^*\phi}\langle\phi^*\phi\rangle + \cdots$$



Dynamic structure factor

$$\frac{S(\mathbf{q},\omega)}{\langle\phi^*\phi\rangle} = -\frac{1}{\pi}\,\mathrm{Im}\,C_{\phi^*\phi} = \frac{1}{2\pi^2}\frac{\sqrt{\omega-q^2/4}}{(\omega-q^2/2)^2} + \frac{1}{2\pi^2\omega q}\ln\frac{\omega+q\sqrt{\omega-q^2/4}}{|\omega-q^2/2|}$$
$$- \frac{1}{\pi^2 q^2\sqrt{\omega-q^2/4}}\left(\ln^2\frac{\omega+q\sqrt{\omega-q^2/4}}{|\omega-q^2/2|} - \pi^2\theta(q^2/2-\omega)\right) \longrightarrow \frac{4}{45\pi^2}\frac{q^4}{\omega^{7/2}}\langle\phi^*\phi\rangle$$

when $q^2/2m \ll \omega$

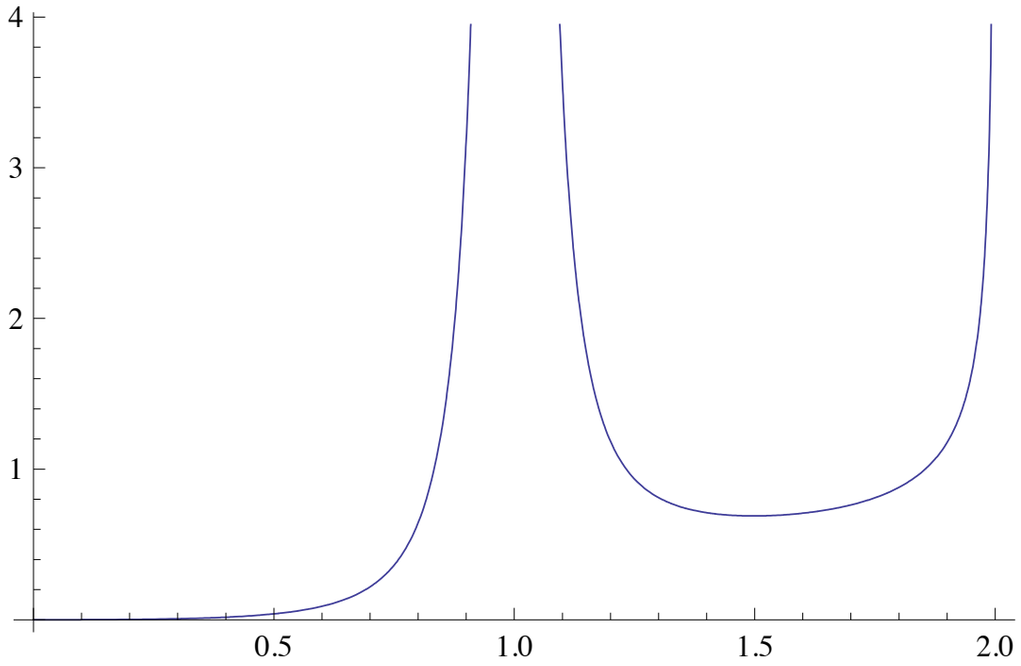


FIG. 7: A plot of $\omega^{3/2}S(\mathbf{q},\omega)/\langle\phi^*\phi\rangle$ as a function of $x = q^2/2\omega$.

DTS, Thompson
Zwerger et al

Embedding the Schrödinger algebra

- Sch(d): is the symmetry of the Schrödinger equation

$$i\frac{\partial\psi}{\partial t}+\frac{\nabla^2}{2m}\psi=0$$

- CFT$_{d+2}$: is the symmetry of the Klein-Gordon equation

$$\partial_\mu^2\phi=0, \qquad \mu=0,1,\cdots,d+1$$

In light-cone coordinates $x^\pm=x^0\pm x^{d+1}$ the Klein-Gordon equation becomes

$$-2\frac{\partial}{\partial x^+}\frac{\partial}{\partial x^-}\phi+\partial_i\partial_i\phi=0, \qquad i=1,\cdots,d$$

Restricting ϕ to $\phi=e^{-imx^-}\psi(x^+,\mathbf{x})$: Klein-Gordon eq. $\Rightarrow$ Schrödinger eq.:

$$2im\frac{\partial}{\partial x^+}\psi+\nabla^2\psi=0, \qquad \nabla^2=\sum_{i=1}^{d}\partial_i^2$$

This means Sch$(d)\subset$ CFT$_{d+2}$

Embedding (2)

Sch(d) is the subgroup of CFT$_{d+2}$ containing group elements which does not change the ansatz

$$\phi = e^{imx^-}\psi(x^+, x^i)$$

Algebra: sch(d) is the subalgebra of the conformal algebra, containing elements that commute with P^+

$$[P^+, O] = 0$$

One can identify the Schrödinger generators:

$$M = P^+, \quad H = P^-, \quad K^i = M^{i-},$$

$$D_{\rm nonrel} = D_{\rm rel} + M^{+-}, \quad C = \frac{1}{2}K^+$$

Nonrelativistic dilatation

$$
\begin{array}{ccccc}
 & & D_{\rm rel} & & M^{+-} \\
x^+ & \to & \lambda x^+ & \to & \lambda^2 x^+ \\
x^- & \to & \lambda x^- & \to & x^- \\
x^i & \to & \lambda x^i & \to & \lambda x^i
\end{array}
$$

Geometric realization of Schrödinger algebra

Start from AdS$_{d+3}$ space:

$$ds^2 = \frac{-2dx^+dx^- + dx^i dx^i + dz^2}{z^2}$$

Invariant under the whole conformal group, in particular with respect to relativistic scaling

$$x^\mu \to \lambda x^\mu, \qquad z \to \lambda z$$

and boost along the x^{d+1} direction:

$$x^+ \to \tilde{\lambda} x^+, \qquad x^- \to \tilde{\lambda}^{-1} x^-$$

Break the symmetry down to Sch(d):

$$ds^2 = \frac{-2dx^+dx^- + dx^i dx^i + dz^2}{z^2} - \frac{2(dx^+)^2}{z^4}$$

The additional term is invariant only under a combination of relativistic dilation and boost:

$$x^+ \to \lambda^2 x^+, \quad x^- \to x^-, \quad x^i \to \lambda x^i, \qquad z \to \lambda z$$

Model

$$ds^2 = \frac{-2dx^+dx^- + dx^i dx^i + dz^2}{z^2} - \frac{2(dx^+)^2}{z^4}$$

Is there a model where this is a solution to the Einstein equation?

The additional term gives rise to a change in $R_{++} \sim z^{-4}$: we need matter that provides $T_{++} \sim z^{-4}$.

Can be provided by A_μ with $A_+ \sim 1/z^2$: has to be a massive gauge field.

$$S = \int d^{d+3}x \sqrt{-g} \left(\frac{1}{2}R - \Lambda - \frac{1}{4}F_{\mu\nu}^2 - \frac{m^2}{2}A_\mu^2 \right)$$

Can be realized in string theory ($d = 2$)

Herzog, Rangamani, Ross;

Maldacena, Martelli, Tachikawa;
Adam, Balasubramanian, McGreevy (2008)

Black-hole solutions also constructed: allow studying hydrodynamics

Nonrelativistic hydro from relativistic hydro

Relativistic hydrodynamic equations

$$\partial_\mu T^{\mu\nu} = 0, \qquad T^{\mu\nu} = (\epsilon + P)u^\mu u^\nu + P g^{\mu\nu} + \text{viscous term}$$

Look at solutions that are independent of x^-:

$$\begin{aligned}
\partial_+ T^{++} + \partial_i T^{+i} &= 0 \\
\partial_+ T^{+i} + \partial_j T^{ij} &= 0 \\
\partial_+ T^{+-} + \partial_j T^{j-} &= 0
\end{aligned}
\qquad\qquad
\begin{aligned}
\partial_t \rho + \partial_i (\rho v^i) &= 0 \\
\partial_t (\rho v^i) + \partial_j \Pi^{ij} &= 0 \\
\partial_t \varepsilon + \partial_j j_\epsilon^i &= 0
\end{aligned}$$

Identify pressure on two sides: $P_{\rm rel} = P_{\rm nonrel}$

Identify u^+ with the density:

$$\begin{aligned}
u^+ &= \sqrt{\frac{\rho}{(d+2)P}} \\
u^+ u^i &= v^i + \text{viscous term}
\end{aligned}$$

Transport coefficients

Relativistic hydrodynamics: one transport coefficient η

Nonrelativistic hydrodynamics: viscosity η and thermal conductivty κ

$$\frac{\eta}{s} = \frac{1}{4\pi}, \qquad \kappa = 2\eta \frac{\varepsilon + P}{2\rho T}$$

"Prandtl number" equal to 1 (air: 0.7) Rangamani, Ross, DTS, Thompson

Conclusions

- Unitary fermions: a system realizing nonrelativistic conformal invariance
- Large q, large ω response can be calculated using operator product expansion
- OPE is a bridge between few- and many-body physics
- Very tentative holographic proposals