Toward understanding the light propagation in clumpy universe

--- Perturbation theory of N point mass gravitational lens

Hideki Asada
(Hirosaki Univ.)
Today’s Menu

1. Intro:
   Obs in Clumpy Universe

2. Pertubation Theory:
   Arbitrary N
   point mass lens

3. Summary
§ 1. Introducción

Cosmic observations

Propagation of light ($\nu$, CR, GW\textellipsis)

Through our clumpy univ.
(Long-standing) Problem

Obs. in FLRW universe

Obs. in “averaged” universe

1) Both agree or not?

2) If not, what’s difference?
Distances play important roles in cosmological observations, especially in gravitational lens systems, but there is a problem in determining distances because they are defined in terms of light propagation, which is influenced gravitationally by the inhomogeneities in the universe. In this paper we first give the basic optical relations and the definitions of different distances in inhomogeneous universes. Next we show how the observational relations depend quantitatively on the distances. Finally, we give results for the frequency distribution of different distances and the shear effect on distances obtained using various methods of numerical simulation.

§1. Introduction

In optical relations among observed quantities, distances such as the luminosity distance and the angular diameter distances play an important role. They are clearly defined in the homogeneous Friedmann-Lemaitre-Robertson-Walker model (Weinberg, 1)

Schneider et al. 2) owing to the simple nature of light propagation in this case. In inhomogeneous universes, however, their behavior is complicated, due to gravitational lens effect which implies that light rays are deflected gravitationally by an inhomogeneous matter distribution. On the other hand, we also use distances to interpret the structure of gravitationally lensed systems.

To correctly treat distances in inhomogeneous universes, it is necessary first to have a reasonable formulation for the dynamics describing local matter motion and optics and clarify the validity condition of the formulation. A set of fluid dynamical equations and the Poisson equation in the cosmological Newtonian approximation was introduced and discussed by Nariai 3) and Irvine 4) under the conditions

\(|\Phi| \ll 1, \ (v/c)^2 \ll 1, \ L/L_H \ll 1\)

Lifshitz (1946)

Nariai and Ueno (1960), Irvine (1965)
On a New Approach to Cosmology. II
—The Problem of Local Gravitation—

Hidekazu NARIAI and Yoshio UENO

Research Institute for Theoretical Physics, Hiroshima University
Takehara-shi, Hiroshima-ken

(Received October 8, 1959)

As a sequel to the previous paper, an attempt is made to develop a general method for attacking at the problem of local gravitational field due to such a large scale aggregation of matter that the effect of the cosmic expansion cannot be ignored. The formalism of this paper will provide us with a basis for treating the dynamical motion of galaxies within the Supergalaxy, together with the reexamination of the velocity-distance relation of galaxies.

Topic is modern still now!
Light propagation through inhomogeneity

Zeldovich (1964)
Dashevskii, Slysh (1964)
Kantowski (1969)
Dyer, Roeder (1972,73)
OBSERVATIONS IN A UNIVERSE HOMOGENEOUS IN THE MEAN

Ya. B. Zel'dovich

Original article submitted June 12, 1963

A local nonuniformity of density due to the concentration of matter of the universe into separate galaxies produces a significant change in the angular dimensions and luminosity of distant objects as compared to the formulas for the Friedman model.

Fig. 1.

Fig. 2.
“Dyer-Roeder” distance

Dyer, Roeder (1972,73)

Raychaudhuri Eq

\[ \frac{d\theta}{dv} = \frac{1}{2} k^{\mu} k_{\mu} = -\frac{1}{2} \mathcal{R} - (\theta^2 + \sigma^2) \]

Assumption:

1) \( R = \alpha \rho_{\text{FLRW}} \) (clumpiness)

2) \( \sigma^2 = \text{Negligible} \)
\[ \frac{d^2}{dw^2} D + \frac{3}{2} (1 + z)^5 \alpha \Omega D = 0 \]

\[ \frac{dz}{dw} = (1 + z)^2 \sqrt{\Omega z (1 + z)^2 - \lambda z (2 + z) + (1 + z)^2} \]
FLRW
Homogeneous & Isotropic

$\alpha = 1$

feel average density
Empty

$\alpha < 1$

Clumps
Inequalities in Observables


Monotonicity: Dyer, Roeder

\[ D_{OL}(\alpha_1) > D_{OL}(\alpha_2) \]

for \( \alpha_1 < \alpha_2 \).
R = Source size fixed

\[ \phi \] weaker Ricci focus

smaller \( \alpha \)

larger \( \phi \)

\[ D_A = R / \phi \text{ decrease with } \alpha \]
OBSERVATION OF GRAVITATIONAL LENSING IN THE CLUMPY UNIVERSE

HIDEKI ASADA¹

Yukawa Institute for Theoretical Physics, Kyoto University, Kyoto 606-01, Japan; asada@yukawa.kyoto-u.ac.jp

Received 1997 August 7; accepted 1998 February 17

ABSTRACT

We discuss how inhomogeneities of the universe affect observations of the gravitational lensing: (1) the bending angle, (2) the lensing statistics, and (3) the time delay. In order to take account of the inhomogeneities, the so-called Dyer-Roeder distance is used, which includes a parameter representing the clumpiness of the matter along the line of sight. It is shown analytically that all three combinations of distances appearing in the above observations, (1)–(3), are monotonic with respect to the clumpiness in general for any given set of the density parameter, cosmological constant, and redshifts of the lens and the source. Some implications of this result for the observations are presented; the clumpiness decreases both the bending angle and the lensing event rate, while it increases the time delay. We also discuss cosmological tests using the gravitational lensing in the clumpy universe.
1) Bending angle \( \frac{D_{LS}}{D_{OS}}(\alpha_1) < \frac{D_{LS}}{D_{OS}}(\alpha_2) \) for \( \alpha_1 < \alpha_2 \)

2) Lens statistics \( \frac{D_{OL}D_{LS}}{D_{OS}}(\alpha_1) < \frac{D_{OL}D_{LS}}{D_{OS}}(\alpha_2) \)

3) Time delay \( \frac{D_{OL}D_{OS}}{D_{LS}}(\alpha_1) > \frac{D_{OL}D_{OS}}{D_{LS}}(\alpha_2) \)
Monotonic in Lambda-term

--- Competing with clumpliness

\[
\frac{D_{LS}}{D_{OS}}(\lambda_1) < \frac{D_{LS}}{D_{OS}}(\lambda_2)
\]

\[
\frac{D_{OL}D_{LS}}{D_{OS}}(\lambda_1) < \frac{D_{OL}D_{LS}}{D_{OS}}(\lambda_2)
\]
Effects by Clumpiness
(Inhomogeneity)
and Lambda term
(Dark energy)
FLRW distance is valid?

“Average”

“Yes”

e.g., Tomita, HA, Hamana (1999)

Numerical Simulation approaches

$\alpha=1$

if $z > 1$
In reality ---

Need of 3D distribution

\[ \alpha = \alpha(z, \theta, \phi) \]
Distances in Inhomogeneous Cosmological Models

Kenji TOMITA,1,*) Hideki ASADA2,**) and Takashi HAMANA3,***)

1 Yukawa Institute for Theoretical Physics, Kyoto University
Kyoto 606-8502, Japan

2 Faculty of Science and Technology, Hirosaki University, Hirosaki 036-8561, Japan

3 Astronomical Institute, Tohoku University, Sendai 980-8578, Japan

(Received February 10, 1999)

Distances play important roles in cosmological observations, especially in gravitational lens systems, but there is a problem in determining distances because they are defined in terms of light propagation, which is influenced gravitationally by the inhomogeneities in the universe. In this paper we first give the basic optical relations and the definitions of different distances in inhomogeneous universes. Next we show how the observational relations depend quantitatively on the distances. Finally, we give results for the frequency distribution of different distances and the shear effect on distances obtained using various methods of numerical simulation.
Fig. 1. The percentage \((100N(\alpha)/N)\) of the distribution of \(\alpha\) in bins with the interval \(\Delta \alpha = 0.4\), for \(D_{1A}\) in the lens model 1 and model S with \((\Omega_0, \lambda_0) = (1.0, 0)\). Results for \(z = 0.5, 1, 2, 3, 4\) and \(5\) are denoted by dot-long dashed, dot-short dashed, long dashed, short dashed, dotted and solid lines, respectively.
“Variance”

“No”

Monte-Carlo Simulations

e.g., Rauch (1991)
Holz, Wald (1998)
Metcalf, Silk (1999)
Barber (2000)
Porciani, Madau (2000)
Valageas (2000)
Fig. 2.—Amplification probability distribution of a compact object-dominated $\Omega = 1$ universe for $\sigma = 0.01$ (corresponding to a redshift of $z_{\text{src}} = 0.22$). A line proportional to $A^{-3}$ has been drawn for comparison. The curve for the $\Lambda \neq 0, \sigma = 0.01$ case is nearly identical (see Fig. 5).
less than 5% of the photon beams.

See note added in proof in Sec. III C. This is not a very
good way of determining which
image—and we shall do so below. However, as discussed in
argued in Sec. III C, each source should have exactly one
luminosity for the primary image of a randomly placed
luminosity, except for the brightest sources.

ity of the primary images should be accurate for the total
images should be merely to further brighten a few of the
of a source cannot be resolved
strongly lensed. Thus, if the primary and secondary images
are filled with randomly distributed point masses. A plot of
Figure 2 to determine
P
A
0 filled with randomly distributed point masses. A plot of
A
0.5 will be carried in secondary images. Undoubtedly,
probability distribution, to give an indication of what one might roughly view as
magnification
Frieman's estimates are consistent with those of
and/or bias, and so it is particularly instructive to examine
magnification
0.5 was pre-

corresponding to redshift
The probability distribution shown is for primary images only; in-
clusion of the flux from secondary images presumably would
parent luminosity of 1 in the underlying Robertson-Walker model. The

Note that, according to Fig. 2, approxi-
mately 5% of the total luminosity is carried by secondary images.

It should be noted that, since at any
given redshift, for any given cosmological pa-

Since the apparent luminosity,

The vertical dashed line represents the empty beam apparent lumi-

Our approach can be used to obtain the spread in image
Consider, first, the case of a universe with
and
0.5, so that
0.5, as we
rennomena in our universe if most of the matter in the universe
with point mass galaxies, at a redshift of 1/2. The absolute lumi-
probability distribution,

The values corresponding to these lines

Thus, as mentioned in Sec. IV B,

radius

As we
rennomena in our universe if most of the matter in the universe

Figure 2. We wish to convert this figure into a

luminosity, which is the minimum possible apparent luminosity for pri-

The dotted lines show the lower and upper 16% of this probability dis-

magnification

tho
tough more are demagnified...
Recent, more sophisticated work

Fig. 9. The scale parameter \( \theta \) is depicted as a function of the source redshift.

Fig. 10. MPDF for the lens model (b) at \( z_S = 1.2 \) is shown. Values of \( R \sqrt{H_0/M_L} \) of clumps are distributed uniformly within \( 6 \leq R \sqrt{H_0/M_L} \leq 10 \). The smooth lines are the gamma distributions that fit the MPDFs.

Random \( RH_0^{1/2}/M_L^{1/2} \) in [6,10]

Yoo, Ishihara, Nakao, Tagoshi (2008)
In Reality, Strong Lensing

Multiple paths
§2-1. Intro to N pt.

Gravitational Lens (GL)

Direct Probe of Gravity (Mass)

Dark Energy

Dark Matter

Dark Object (Exoplanet etc)
Main Result

HA, arXiv:0809.4122

First systematic attempt to determine lensed image positions for arbitrary N using Perturbation Theory
Analogy of Sun-Jupiter-Saturn

Gaudi et al. Science (08)
Approaches (Modelling)

Fluid Approx. (Continuum)

Cosmological GL

Lens = galaxies, LSS

Particle Approx. (Discrete)

Microlens

Lens = stars, planets, etc
Question (#1)

Lens = $N$ particles

$N \Rightarrow \infty$

Lens = Continuum

have to agree (proof?)

$N$-finite Effects?
N-finite Effects have been observed.

QSO microlens

Lens=galaxy (+star)

Source=quasar
Q2237+0305 = Einstein Cross

Wambsganss, LRR (91)
Time Variability

Wambsganss, LRR (01)
Q2237+0305
= Einstein Cross
Time Variability

Eigenbrod et al.
ArXiv:0709.2828
Question (#2)

We want to get

Roots for lens eq.

Positions of images

--- unknown
Problem

$N=1 \Rightarrow$ Quadratic Eq.

$N=2 \Rightarrow$

Complex Quintic Eq.

(Witt 90)

Real Quintic Eq.

(Asada 02, Asada et al 04)
Theorem (Galois)

Algebraic Eq. cannot be solved algebraically, if 5th or higher order.

Algebraic method

= +, −, ×, ÷, \(n\sqrt{\text{ }}\)

Thus, Formula is unknown.
Our goal

First attempt to get perturbative roots

Approximate ones can be sufficient for observation
§ 2-2. Complex Formalism

Bourassa and Kantowski (73, 75)

GL

= 2D mapping (thin lens)

Source Pl.  $\overset{\rightarrow}{\beta}$  Lens Pl.  $\overset{\rightarrow}{\theta}$

$w = w_x + iw_y$  $z = x + iy$
Assumption

1) thin lens approx.

2) arbitrary N co-planar mass

#) any configuration without symmetry
We consider a lens system with $N$ point mass. The mass and two-dimensional location of each body is denoted as $M_i$ and the vector $E_i$, respectively. For a later convenience, let us define the Einstein ring radius angle as

$$\theta_E = \sqrt{\frac{4GM_{\text{tot}}D_{\text{LS}}}{c^2D_LD_S}},$$

where $G$ is the gravitational constant, $c$ is the light speed, $M_{\text{tot}} = \sum_{i=1}^{N} M_i$ and $D_L$, $D_S$ and $D_{\text{LS}}$ denote distances between the observer and the lens, between the observer and the source, and between the lens and the source, respectively.

In the unit normalised by the Einstein ring radius angle, the lens equation becomes

$$\beta = \theta - \sum_{i} \nu_i \left( \frac{\theta - e_i}{|\theta - e_i|^2} \right),$$

where $\beta = (\beta_x, \beta_y)$ and $\theta = (\theta_x, \theta_y)$ denote the vectors for the position of the source and image, respectively and we defined the mass ratio and the angular separation vector as $\nu_i = \frac{M_i}{M_{\text{tot}}}$ and $e_i = \frac{E_i}{\theta_E} = (e_x, e_y)$.

In a formalism based on complex variables, two-dimensional vectors for the source, lens and image positions are denoted as $w = \beta_x + i\beta_y$, $z = \theta_x + i\theta_y$, and $\epsilon_i = e_x + ie_y$, respectively.

By employing this formalism, the lens equation is rewritten as

$$w = z - \sum_{i} \nu_i z^* - \epsilon_i^*,$$

where the asterisk $^*$ means the complex conjugate. The lens equation is non-analytic because it contains both $z$ and $z^*$.

3 EMBEDDING THE LENS EQUATION INTO AN ANALYTIC POLYNOMIAL

The complex conjugate of Eq. (3) is expressed as

$$w^* = z^* - \sum_{i} \nu_i z - \epsilon_i.$$

This expression can be substituted into $z^*$ in Eq. (3) to eliminate the complex variable $z^*$. As a result, we obtain a $(N^2 + 1)$-th order analytic polynomial equation as (Witt 1990)

$$\left(z - \frac{\theta - e_i}{|\theta - e_i|^2} \prod_{k=1}^{N} |z - \epsilon_k| + \sum_{k=1}^{N} \nu_k \prod_{j \neq k} |z - \epsilon_j| \right)^2 = 0.$$
The equation is non-analytic because it contains both perturbation variables and the vector size of the Einstein ring, the lens equation becomes significant for the observer and the source, and between the lens and the source distances between the observer and the lens, between the observer and the lens. Let us define the angular size of the Einstein ring as \( E \).

In a formalism based on complex variables, two-dimensional location of each body is denoted as \( z, \theta, \beta \).

The lens equation is embedded into a single-complex description of gravitational lensing is briefly summarized. The lens equation is extended to a dual-complex-variables formalism and its perturbation scheme is used to eliminate the complex variable \( \nu \).
Complex Notation

\[ w = z - \sum_{i}^{N} \frac{\nu_i}{z^* - \epsilon_i^*} \]
Single-Complex-Variable Polynomial

Witt (90)

$\mathbb{Z}^*$ deleted

Polynomial in Only $\mathbb{Z}$

$N^{2+1}$th Order
\[ w = z - \sum_{i}^{N} \frac{\nu_i}{z^* - \epsilon_i^*} \]

C.C.

\[ w^* = z^* - \sum_{i}^{N} \frac{\nu_i}{z - \epsilon_i} \]
Only z but no z*

\[(z - w) \prod_{l=1}^{N} \left( (w^* - \epsilon_l^*) \prod_{k=1}^{N} (z - \epsilon_k) + \sum_{k=1}^{N} \nu_k \prod_{j \neq k}^{N} (z - \epsilon_j) \right) \]

\[= \sum_{i=1}^{N} \nu_i \prod_{l=1}^{N} (z - \epsilon_l) \]

\[\times \prod_{m \neq i}^{N} \left( (w^* - \epsilon_m^*) \prod_{k=1}^{N} (z - \epsilon_k) + \sum_{k=1}^{N} \nu_k \prod_{j \neq k}^{N} (z - \epsilon_j) \right) \]
Perturbation

Mass Ratio

\[ \nu_i = \frac{M_i}{M_{tot}} < 1 \]

--- expansion parameter

Iterative calculations

\[ z = \sum_{p_2=0}^{\infty} \sum_{p_3=0}^{\infty} \cdots \sum_{p_N=0}^{\infty} \nu_2^{p_2} \nu_3^{p_3} \cdots \nu_N^{p_N} \tilde{z}(p_2)(p_3)\cdots(p_N) \]
0th order root

\[
\alpha_i \equiv -1/w_i^*
\]

\[
\alpha_\pm = \frac{w}{2} \left( 1 \pm \sqrt{1 + \frac{4}{ww^*}} \right)
\]

\[
\epsilon_i
\]

\[
\omega_i = \omega - \epsilon_i
\]
Problem

\[ \alpha_i \]

does not satisfy Lens Eq.

mixed with unphysical roots
Dual-Complex-Variables Formalism

Both $Z$ and $Z^*$

Merit

Equivalent to Lens Eq.

No unphysical root
\[
C(z, z^*) = \sum_{k=2}^{N} \nu_k D_k(z^*)
\]

**Linear in \( \nu \)**

\[
D_k(z^*) = \frac{1}{z^*} - \frac{1}{z^* - \epsilon_k^*}
\]

**Iteration**

\[
z = \sum_{p_2=0}^{\infty} \sum_{p_3=0}^{\infty} \cdots \sum_{p_N=0}^{\infty} (\nu_2)^{p_2} (\nu_3)^{p_3} \cdots (\nu_N)^{p_N} z(p_2)(p_3)\cdots(p_N)
\]
1st Order

\[ z(0)\ldots(1_k)\ldots(0) = \frac{b(0)\ldots(1_k)\ldots(0) - a(0)\ldots(1_k)\ldots(0)b^*(0)\ldots(1_k)\ldots(0)}{1 - a(0)\ldots(1_k)\ldots(0)a^*(0)\ldots(1_k)\ldots(0)} \]

\[ a(0)\ldots(1_k)\ldots(0) = \frac{1}{(z^*(0)\ldots(0))^2} \]

\[ b(0)\ldots(1_k)\ldots(0) = \frac{\epsilon_k^*}{z^*(0)\ldots(0)(z^*(0)\ldots(0) - \epsilon_k^*)} \]
2nd Order

\[ z^{(0)}(0) \cdots (2k) \cdots (0) = \frac{b(0) \cdots (2k) \cdots (0) - a(0) \cdots (2k) \cdots (0) b^*}{1 - a(0) \cdots (2k) \cdots (0) a^*} \]

\[ z^{(0)}(1_k) \cdots (1_l) \cdots (0) = \frac{b(0) \cdots (1_k) \cdots (1_l) \cdots (0) - a(0) \cdots (1_k) \cdots (1_l) \cdots (0) b^*}{1 - a(0) \cdots (1_k) \cdots (1_l) \cdots (0) a^*} \]
Figure 5. Graph representations of interactions among point masses for images at the second order level. The top and bottom graphs represent a mutually-interacting image and a self-interacting one, respectively.
Similarly,

3rd Order, 4th Order

--- Systematic!
### Convergence: On/Off-axis

<table>
<thead>
<tr>
<th>Case 1 (On-axis)</th>
<th>$\nu = 0.1$</th>
<th>$e = 1$</th>
<th>$w = 2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Root</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>1st.</td>
<td>2.43921</td>
<td>-0.389214</td>
<td>0.95</td>
</tr>
<tr>
<td>2nd.</td>
<td>2.43855</td>
<td>-0.388551</td>
<td>0.95</td>
</tr>
<tr>
<td>3rd.</td>
<td>2.43858</td>
<td>-0.388519</td>
<td>0.949938</td>
</tr>
<tr>
<td>Lens Eq.</td>
<td>2.43858</td>
<td>-0.388517</td>
<td>0.949937</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Case 2 (Off-axis)</th>
<th>$\nu = 0.1$</th>
<th>$e = 1$</th>
<th>$w = 1 + i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Root</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>1st.</td>
<td>1.33716+1.40546 i</td>
<td>-0.337158-0.355459 i</td>
<td>0.95-0.05 i</td>
</tr>
<tr>
<td>2nd.</td>
<td>1.33632+1.40363 i</td>
<td>-0.336316-0.354881 i</td>
<td>0.95-0.05 i</td>
</tr>
<tr>
<td>3rd.</td>
<td>1.33634+1.40371 i</td>
<td>-0.336275-0.354839 i</td>
<td>0.95-0.05025 i</td>
</tr>
<tr>
<td>Lens Eq.</td>
<td>1.33633+1.40371 i</td>
<td>-0.336272-0.354835 i</td>
<td>0.950015-0.0502659 i</td>
</tr>
</tbody>
</table>
Next, we investigate the vicinity of a root around $z = 7.5$. For both 0th and 1st order ($n = 1, 2, 3$), we may use Eq. (53) to obtain $z$ shown below. At $z = 7.4$, for 3rd order and other orders, we consider the lens equation at $z_0$.

This shows a clear diagram.

Figure 2. Perturbative image positions for a binary lens case. This plot corresponds to Tables 1 and 2. The lenses ($e_1 = 0, e_2 = 1$) and sources ($w = 2$ and $w = 1 + i$) are denoted by filled squares. The image positions are denoted by filled disks. Perturbative images at the 1st, 2nd and 3rd orders are overlapped so that we cannot distinguish them in this figure.
There are numerous possible applications along the course of the perturbation theory of N point-mass gravitational lens systems. For instance, it will be interesting to extend the perturbation theory to a domain near (and possibly inside) the caustics. Then, positions of images with the same source point will be influenced only by the nearest lens object at the linear order as shown in Eqs. (109). Figures 6 and 7 show an example of a large-amplification case, where the two curves are overlapped, where the Einstein cross time, which is defined as the intersection of the source trajectory as the function multiplied by the number of the particles, namely the number of perturbed image positions. In the perturbation theory, the present perturbation method cannot be reproduced by the present method. However, double peaks due to caustic crossings cannot be reproduced by the present method.

Under a small mass-ratio approximation, this paper developed the dual-complex-variables formalism. The lens parameters are that the two curves are overlapped, where the source trajectory as the theorem on the maximum number of lensed images (Rhie and O. Asada) point-mass gravitational lens systems with gravitational fields but unphysical roots is consistent with the earlier work on the perturbative roots are listed in Table 3. Although, the validity of the present result may be limited in the weak field regions. It is important also to extend the perturbation theory to a domain near (and possibly inside) the caustics. Then, positions of images with the same source point will be influenced only by the nearest lens object at the linear order as shown in Eqs. (109). Figures 6 and 7 show an example of a large-amplification case, where the two curves are overlapped, where the Einstein cross time, which is defined as the intersection of the source trajectory as the function multiplied by the number of the particles, namely the number of perturbed image positions. In the perturbation theory, the present perturbation method cannot be reproduced by the present method. However, double peaks due to caustic crossings cannot be reproduced by the present method.
Truncated Isothermal

$N=10^3$
w. Ampl.
Gravitational Lensing by N Point Mass

Figure 6. Example of a large N case. Here, we assume a truncated isothermal sphere projected onto a single lens plane with $N = 1000$, where the truncation radius is the unity. For the simplicity, we assume equal masses. The source located at 0.25 is denoted by the circle. The top figure shows locations of the N point masses on the lens plane. The bottom shows a plot of image positions by using the perturbative solutions at the second order. In practice, the linear-order and second-order roots make no difference distinguishable by eyes in the figure. The present perturbation method is less than the maximum number for a N point-mass lens. This suggests that other images do not have the small mass limit. This is also in agreement with previous works. For instance, the appearance of the maximum number of images for a binary lens requires a finite mass ratio and the caustic crossing (Schneider and Weiss 1986).

Figure 7. Plot of image positions with lensing amplification for a case of $N = 1000$. The source and lenses are the same as those in Fig. 6. Here, we take account of amplifications by lensing. The area of a disk corresponding to each image is proportional to the magnification factor in arbitrary units. Large amplifications near $\pm 1$ are caused by the mutually-interacting images. On the other hand, a concentration of small but many images around the center are due to the self-interacting images, because lens objects have a large number density there. These three regions may correspond to three images for a singular isosphere lens in the limit of $N \to \infty$.

Figure 8. Einstein ring broken by the lens discreteness due to the finite-N effect. The lenses are the same as those in Figs. 6 and 7. The source is located at the origin of the coordinates. Amplifications are taken into account. The area denotes the magnification factor in arbitrary units.
§3. Summary

HA, arXiv:0809.4122

First attempt to get lensed image positions for arbitrary $N$
Future Works

1. Applications to N-Finite Effects
   Ex) Mean, Variance in Mag.

2. Extension to Multiple Lens Planes
   Ex) Cosmological GL
Thank you!

asada@phys.hirosaki-u.ac.jp