Single flavor staggered fermions

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1. Introduction

2. Staggered Wilson

3. Symmetries

4. New operator

5. Tests in 2D

6. Outlook
Lattice fermions

Naive fermions
\[ D_N = \gamma_\mu D_\mu \]
16 species

Staggered fermions
\[ D_{st} = \eta_\mu D_\mu \]
4 species

Wilson fermions
\[ D_W = \gamma_\mu D_\mu + rW \]
1+4+6+4+1 species

remove 4-fold degeneracy

add Wilson term

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Chiral symmetry

Staggered fermions

\[ D_{\text{st}} = \eta_\mu D_\mu \]

4 species, no anomaly

\[ U(1)_\epsilon : \{ D_{\text{st}}, \epsilon \} = 0 \]

\[ \epsilon = (-1)^{x_1+x_2+x_3+x_4} \]

\[ \sim (\gamma_5 \otimes \xi_5) \]

Wilson fermions \( D_W \)

1+4+6+4+1 species

Overlap fermions

\[ D_{\text{ov}} = \rho \left( 1 + \gamma_5 \text{sign} (\gamma_5 (D_W - \rho)) \right) \]

1 species, correct anomaly

Full \( SU(N_f) \) chiral Symmetry

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Wilson term

Add a Wilson term to naive operator:

Momentum dependent mass (remove doublers)

Positivity of det($D_W$):

- Naive operator: antihermitean, $\{D_N, \gamma_5\} = 0$
- Wilson term $W$: hermitean, $[W, \gamma_5] = 0$

$D_W \gamma_5 = \gamma_5 D_W^\dagger$

$\Rightarrow$ eigenvalues real or in complex conjugate pairs $\lambda_i = \lambda_i^*$

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Single flavor staggered fermions
Add a Wilson term to staggered operator:

Momentum (taste) dependent mass (remove doublers)

Positivity of $\text{det}(D_A)$:

- Staggered operator: antihermitean, $\{D_N, \epsilon\} = 0$
- Wilson term $A$: hermitean, $[A, \epsilon] = 0$

$D_A \epsilon = \epsilon D_A^\dagger$

$\Rightarrow$ eigenvalues real or in complex conjugate pairs $\lambda_i = \lambda_i^*$
Usual Wilson term:

\[ W = \sum_{\mu} (C_\mu + 1) \]

\[ C_\mu := \frac{1}{2} \left( V_\mu + V_\mu^\dagger \right) \quad (V_\mu)_{xy} := U_\mu(x) \delta_{x+\mu,y} \]

\[ W^\dagger = W \checkmark \quad [W, \gamma_5] = 0 \checkmark \]
Staggered Wilson term construction

Staggered Wilson term: (Adams, 2010)

\[ A = \epsilon \eta_5 (C_1 C_2 C_3 C_4)_{sym} \]

\[ C_\mu := \frac{1}{2} \left( V_\mu + V_\mu^\dagger \right) \]

\[ (V_\mu)_{xy} := U_\mu(x) \delta_{x + \mu, y} \]

\[ A^\dagger = A \checkmark \]

\[ [A, \epsilon] = 0 \checkmark \]

- \( \eta_5 = \eta_1 \eta_2 \eta_3 \eta_4 = (-1)^{x_1 + x_3} \)
- \( \eta_\mu = (-1)^{\sum_{\nu < \mu} x_\nu} \sim (\gamma_\mu \otimes 1) \)
- \( \epsilon = (-1)^{x_1 + x_2 + x_3 + x_4} \sim (\gamma_5 \otimes \xi_5) \)
- \( \{ C_\mu, \epsilon \} = 0 \)
- \( A \sim (1 \otimes \xi_5) + O(a) \)
Remnant flavor degeneracy

- Staggered flavor basis: \( A \sim \xi_5 = \text{diag}(1, 1, -1, -1) \)
- Twofold degeneracy left!
- Let us take

\[
M_{\mu\nu} = i \epsilon_{\mu\nu} \eta_\mu \eta_\nu (C_\mu C_\nu)_{\text{sym}}
\]

\[
M_{\mu\nu}^\dagger = M_{\mu\nu} \quad [M_{\mu\nu}, \epsilon] = 0
\]

\[
\epsilon_{\mu\nu} = -\epsilon_{\nu\mu} = (-1)^{x_\mu + x_\nu}, \mu < \nu
\]

\[
\eta_\mu = (-1)^\sum_{\nu < \mu} x_\nu
\]

\[
M_{\mu\nu} \sim (1 \otimes \sigma_{\nu\mu}) + O(a)
\]

\[
\sigma_{\nu\mu} = \text{diag}(1, -1, -1, 1)
\]

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Single flavor staggered fermions
Discrete staggered symmetries

Remnants of Poincare symmetry:

<table>
<thead>
<tr>
<th></th>
<th>(D_{st})</th>
<th>(A)</th>
<th>(M_{\mu\nu})</th>
</tr>
</thead>
<tbody>
<tr>
<td>shift (translation)</td>
<td>+</td>
<td>-</td>
<td>(\pm)</td>
</tr>
<tr>
<td>axis reversal</td>
<td>+</td>
<td>-</td>
<td>(\pm)</td>
</tr>
<tr>
<td>rotation</td>
<td>+</td>
<td>+</td>
<td>(M_{\alpha\beta})</td>
</tr>
</tbody>
</table>

- Preserved by staggered operator
- Preserved up to a sign flip by \(A\)
- Rotation introduces new terms for \(M_{\mu\nu}\)
  - Bad: new counterterms
  - Search for more symmetric construction
Symmetrized staggered Wilson

\[ M_s = \frac{1}{\sqrt{3}} \left( s_{12} (s_1 s_2 M_{12} + s_3 s_4 M_{34}) + s_{13} (s_1 s_3 M_{13} + s_4 s_2 M_{42}) + s_{14} (s_1 s_4 M_{14} + s_2 s_3 M_{23}) \right) \]

- Shift or axis reversal in \( \rho \): \( s_\rho \rightarrow -s_\rho \)

<table>
<thead>
<tr>
<th>rotation ((\rho, \sigma))</th>
<th>sign flip</th>
</tr>
</thead>
<tbody>
<tr>
<td>((1,4)) ((2,3)) ((3,1)) ((2,4))</td>
<td>(s_{12} \rightarrow -s_{12})</td>
</tr>
<tr>
<td>((1,2)) ((3,4)) ((4,1)) ((3,2))</td>
<td>(s_{13} \rightarrow -s_{13})</td>
</tr>
<tr>
<td>((1,3)) ((4,2)) ((2,1)) ((4,3))</td>
<td>(s_{14} \rightarrow -s_{14})</td>
</tr>
</tbody>
</table>

- \( M_s \sim \left(1 \otimes \xi^{(s)}\right) + O(a) \)

\[ \xi^{(s)} = \text{diag}(0, 0, 2, -2) \]
Single flavor staggered operator

\[ D_{s} = D_{st} + (2 + M_{s}) \]

\[ D_{1} = 1 + \epsilon \text{sign}(\epsilon(D_{s} - 1)) \]
Single flavor staggered operator

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Single flavor staggered operator

\[ D_{st} \]
\[ D_{st} + A \]
\[ D_{st} + A + M_{\mu\nu} \]
\[ D_s = D_{st} + (2 + M_s) \]
\[ D_1 = 1 + \varepsilon \text{sign}(\varepsilon (D_s - 1)) \]
Single flavor staggered operator

\[ D_{st} \]
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Single flavor staggered operator
Some tests in 2D

- Only 2-fold degeneracy in 2D
- $M_{12}$ uniquely lifts this degeneracy

$$D_{st}$$

$$D_{st} + 1 + M_{12} \quad Q = 0$$

$$1 + \epsilon \text{sign}(\epsilon(D_{st} + M_{12}))$$
Some tests in 2D

- Only 2-fold degeneracy in 2D
- $M_{12}$ uniquely lifts this degeneracy

$D_{\text{st}}$
$D_{\text{st}} + 1 + M_{12}$, $Q = 0$
$1 + \epsilon \text{sign}(\epsilon (D_{\text{st}} + M_{12}))$
Some tests in 2D

- Only 2-fold degeneracy in 2D
- \( M_{12} \) uniquely lifts this degeneracy

- \( D_{st} \)
- \( D_{st} + 1 + M_{12} \quad Q = 0 \)
- \( 1 + \epsilon \text{sign}(\epsilon(D_{st} + M_{12})) \)
Some tests in 2D

\[ D_{st} \]

\[ D_{st} + 1 + M_{12} \quad Q = 1 \]

\[ 1 + \epsilon \text{sign}(\epsilon(D_{st} + M_{12})) \]
Some tests in 2D

\begin{align*}
D_{\text{st}} \\
D_{\text{st}} + 1 + M_{12} & \quad Q = 1 \\
1 + \epsilon \text{sign}(\epsilon(D_{\text{st}} + M_{12}))
\end{align*}

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Some tests in 2D

\[ D_{st} \]
\[ D_{st} + 1 + M_{12}, \quad Q = 1 \]
\[ 1 + \epsilon \text{sign}(\epsilon(D_{st} + M_{12})) \]
Some tests in 2D

\[ \text{Re}(\lambda) \]

\[ \text{Im}(\lambda) \]

-1 0 1

-1 0 1

-2

-2

\[ D_{\text{st}} \]

\[ D_{\text{st}} + 0.05M_{12} \]

\[ D_{\text{st}} + M_{12} \quad Q = 2 \]
Some tests in 2D

- $D_{st}$
- $D_{st} + 0.05M_{12}$
- $D_{st} + M_{12}$, $Q = 2$
Some tests in 2D

- $D_{st}$
- $D_{st} + 0.05M_{12}$
- $D_{st} + M_{12}$  \( Q = 2 \)
Conclusion

Single flavor staggered operator is possible

Wilson fermions without remnants of spurious naive degeneracy exist

But is it useful?

✔ Better condition number  ❌ Breaks flavor symmetry
✔ Smaller matrix        ❌ 2-hop Wilson term
Staggered Wilson

To do list:

- Find counterterm structure
- Construct mesons, baryons
- $O(a)$ improvement
  - Clover “for free” due to 2-hop Wilson term? (some CPU, but no additional bandwidth)
- Optimize algorithms for the structure
- Check flavor breaking
- Study scaling
- Apply to real problem
  - Insensitive to flavor breaking
    - Ground states, bulk properties, spectral quantities
  - Hadron/quark masses? Thermodynamics?
Staggered overlap

To do list:
- Find counterterm structure
- Check locality
- Check flavor breaking
- Study scaling
- Apply to real problem
  - Insensitive to flavor breaking
  - Chiral symmetry essential
  - Spectral quantities?