Determination of low-energy constants at NNLO in the epsilon expansion

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Outline

1. Determination of low-energy constants
2. ε-expansion at NNLO
3. Role of the lattice geometry
4. Conclusions

- at $T = 0$ the low-energy phenomenology of QCD can be described by an effective chiral Lagrangian which in leading order is

$$\mathscr{L} = \frac{F^2}{4} \operatorname{tr} \partial_\nu U \partial_\nu U^{-1} - \frac{\Sigma}{2} \operatorname{tr}(MU + M^\dagger U^{-1})$$

  - everything in Euclidean space
  - $U(x) = \exp\left[i \frac{\sqrt{2}}{F} \xi(x) \right]$ parametrizes the Nambu-Goldstone manifold
  - M is the quark mass matrix (GOR: $2m\Sigma = m_\pi^2 F^2$ for $M = m\mathbb{1}_{N_f}$)
- Σ and F are low-energy constants (LEC)
  - important for phenomenology
  - computable in lattice QCD

- consider QCD in a finite volume V and for small quark masses m
- when $m_\pi^{-1} > V^{1/4} \longrightarrow$ reorder chPT $\longrightarrow$ ε-regime power counting

Gasser-Leutwyler 1987

$$V \sim \varepsilon^{-4}, \quad m \sim \varepsilon^4, \quad \partial_\nu \sim \varepsilon, \quad \xi(x) \sim \varepsilon$$

- separate constant (zero-momentum) pion mode U_0:

$$U(x) = U_0 \exp\left[i\frac{\sqrt{2}}{F}\xi(x)\right]$$

with $U_0 \in \mathrm{SU}(N_f)$ and $\int d^4x\, \xi(x) = 0$

- integrate out space-time dependence to each order in ε^2
  $\longrightarrow$ finite-volume effective theory in terms of U_0
- systematic expansion in powers of ε^2
  $\longrightarrow$ finite-volume corrections in powers of $1/(F^2\sqrt{V})$

- leading order = group integral (zero-momentum-mode)
  - described by random matrix theory (RMT)
  - RMT results for Dirac eigenvalue distributions known
  - fit of lattice data to RMT → LECs
    (to obtain F a suitable chemical potential needs to be included)
- NLO: RMT results still apply, but LECs receive finite-volume corrections

$$\Sigma_{\text{eff}}^{\text{NLO}} = \Sigma \left[1 + \frac{\beta_1 (N_f^2 - 1)}{F^2 \sqrt{V} N_f} \right]$$

Gasser-Leutwyler 1987

$$F_{\text{eff}}^{\text{NLO}} = F \left[1 + \frac{N_f}{F^2 \sqrt{V}} \left(\frac{\beta_1}{2} + \frac{k_{00} L_0^2}{2\sqrt{V}} \right) \right]$$

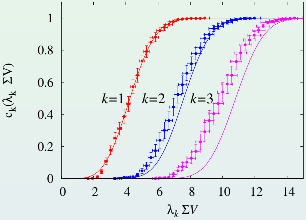
Damgaard et al. 2007
Akemann et al. 2008
Lehner-TW 2009

β_1 and k_{00} are shape coefficients (depend on the lattice geometry)

- NNLO: non-universal deviations from RMT (this work)

- lattice setup:
  - $a = 0.107(3)$ fm, $16^3 \times 32$ lattice points, $V^{1/4} \approx 1.7$ fm
  - $N_f = 2$ overlap fermions with $am_u = am_d = 0.002 \to m_\pi^2 \sqrt{V} \approx 1$
  - sea quarks at zero chemical potential
  - valence quarks at zero and nonzero imaginary chemical potential
- fit to RMT Dirac eigenvalue distributions to extract LECs:
  - Σ: Fukaya et al., PRL 98 (2007) 172001
  - F: this work (and to be published)

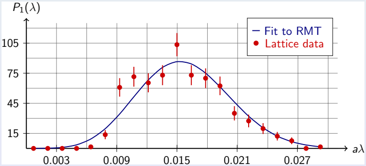
- JLQCD fit to cumulant Dirac eigenvalue distributions



$$a^3\Sigma_{\text{eff}} = 0.00212(6)$$

(before finite-volume corrections are applied)

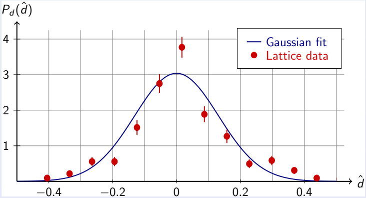
- check: fit to distribution of smallest eigenvalue



$$a^3\Sigma_{\text{eff}} = 0.00208(2) \quad \checkmark$$

$$\chi^2/\text{dof} = 2.9$$

- experimental value: $F \approx 93$ MeV
- JLQCD fit to meson correlators in ε-regime: $F_{\text{meson}} = 87(6)$ MeV
  Fukaya et al. PRD 77 (2008) 074503
- other approach:
  (first proposed in Damgaard et al. 2005 and Akemann et al. 2006)
  - add small imaginary chemical potential (couples to F), here $a\mu = 0.01$
    $\longrightarrow$ results in a shift d of the Dirac eigenvalues
  - shift can be computed in RMT ($\hat{d} = d\Sigma V$)
    $\longrightarrow$ Gaussian distribution with $\sigma^2 = \mu^2 F^2 V$
    $\longrightarrow$ fit to lattice data yields F



$F_{\text{eff}} = 67(5)$ MeV

$\chi^2/\text{dof} = 4.2$ bad!

- experimental value:

$$F_{\text{exp}} \approx 93 \text{ MeV}$$

- from meson correlators:

$$F_{\text{meson}} = 87(6) \text{ MeV}$$

- from Dirac eigenvalue shift due to imaginary chemical potential:

$$F_{\text{eff}} = 67(5) \text{ MeV}$$

- including NLO corrections:

$$F = 51(4) \text{ MeV}$$

agreement gets worse!

- the problem:
  - χ^2/dof of fit to RMT is large
  - there are non-universal deviations from RMT starting in NNLO
    $\rightarrow$ we are not fitting to the right function
- this work:
  - go to NNLO
  - understand the systematic errors
  - minimize them to find a reasonable value for F

- LO Lagrangian with imaginary chemical potential:

$$\mathcal{L}_1 = \frac{F^2}{4}\operatorname{tr}[\nabla_\rho U(x)^{-1}\nabla_\rho U(x)] - \frac{\Sigma}{2}\operatorname{tr}[MU(x) + M^\dagger U(x)^{-1}]$$

with

$$\nabla_\rho U(x) = \partial_\rho U(x) - i\delta_{\rho 0}[C, U(x)]$$

$$U(x) = \underset{\uparrow}{U_0} \exp\left[i\frac{\sqrt{2}}{F}\xi(x)\right]$$

zero-momentum mode of NG manifold

the imaginary chemical potentials $i\mu_f$ are in $C = \operatorname{diag}(\mu_1, \ldots, \mu_{N_f})$

- power counting: $V \sim \varepsilon^{-4}$, $M \sim \varepsilon^4$, $\partial_\nu \sim \varepsilon$, $\xi(x) \sim \varepsilon$, $C \sim \varepsilon^2$
- zero-dimensional limit: $U(x) \to U_0$

$$S_{\text{eff}}^{\text{LO/RMT}} = -\frac{V\Sigma}{2}\operatorname{tr}(M^\dagger U_0 + MU_0^{-1}) - \frac{VF^2}{2}\operatorname{tr}(CU_0CU_0^{-1})$$

- the NLO Lagrangian $\mathcal{L}_2$ in the p-expansion contains many more terms, with LECs $L_1, \dots, L_8$ and HEC H_2 (L_9, L_{10}, H_1 do not appear)
- now expand partition function to NNLO in the $\xi(x)$ fields and average over them (using Christoph's C++ library for tensor algebra)
  $\longrightarrow$ effective action at NNLO in ε and to second order in C:

$$
\begin{aligned}
S_{\text{eff}}^{\text{NNLO}} = & -\frac{V\Sigma_{\text{eff}}^{\text{NNLO}}}{2}\operatorname{tr}(M^\dagger U_0 + MU_0^{-1}) - \frac{V(F_{\text{eff}}^{\text{NNLO}})^2}{2}\operatorname{tr}(CU_0CU_0^{-1}) \\
& + \Upsilon_1 \Sigma(VF)^2 \operatorname{tr}(C)[\operatorname{tr}(U_0\{M^\dagger, C\}) + \operatorname{tr}(U_0^{-1}\{C, M\})] \\
& \quad \vdots \\
& + \Upsilon_8 (V\Sigma)^2 [\operatorname{tr}(MU_0^{-1}MU_0^{-1}) + \operatorname{tr}(M^\dagger U_0 M^\dagger U_0)] \\
& + \mathcal{H}_2 (V\Sigma)^2 \operatorname{tr}(M^\dagger M) + \dots
\end{aligned}
$$

$\longrightarrow$ NNLO finite-volume corrections to Σ and F
non-RMT terms proportional to Υ_i and $\mathcal{H}_2$ (they are $\sim \varepsilon^4$)

- the results for $\Sigma_{\text{eff}}^{\text{NNLO}}$, $F_{\text{eff}}^{\text{NNLO}}$, Υ_i and $\mathscr{H}_2$ are rather lengthy, e.g.

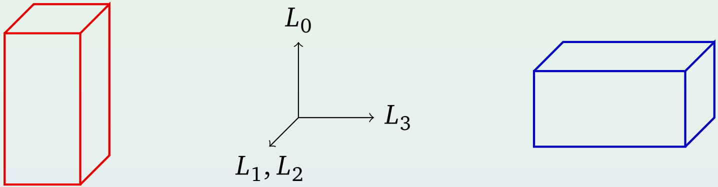
$$\frac{(F_{\text{eff}}^{\text{NNLO}})^2}{F^2} = 1 - \frac{2N_f(P_2+P_3)}{F^2\sqrt{V}} + \frac{1}{F^4V}\Big\{N_f^2(P_2^2+P_3^2+2P_2P_3+2P_4+4P_5+P_6) + 16\left[(N_f^2-1)L_1+L_2+(N_f-N_f^{-1})L_3\right] + 16P_1\left[2L_1+N_f^2L_2+(N_f-2N_f^{-1})L_3\right]\Big\}$$

- they depend on
  - NLO LECs of chPT
  - shape coefficients P_i resulting from finite-volume one- and two-loop diagrams (renormalization necessary)

$\longrightarrow$ finite-volume corrections and non-universal terms depend on geometry of space-time box

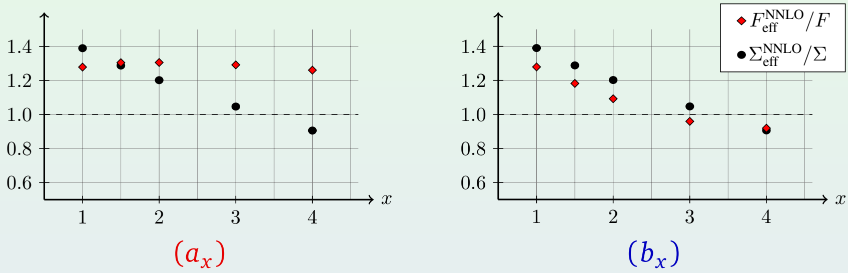
Role of the lattice geometry

define the following geometries:
(from now on the L_i are no longer LECs but lattice extensions)



$$
\begin{array}{lll}
(a_x) & L_0 = xL\,, & L_1 = L_2 = L_3 = L \\
(b_x) & L_3 = xL_0\,, & L_0 = L_1 = L_2
\end{array}
$$

- JLQCD uses (a_2)
- L_0 is the direction in which μ is included, but for valence quarks only
  $\rightarrow$ can rotate the lattice and still use JLQCD's dynamical configurations



Parameters:

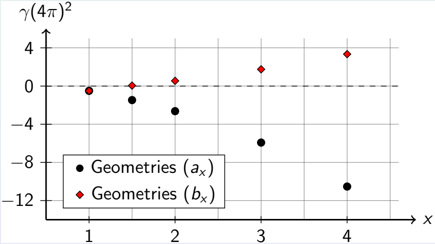
- $m_\pi^2 \sqrt{V} = 1$
- $F = 90$ MeV
- $L = 1.71$ fm

$\longrightarrow$ corrections to Σ same for (a_x) and (b_x)
corrections to F smaller for geometry (b_x)

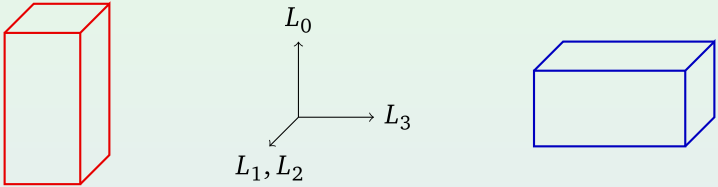
- $\Upsilon_4, \ldots, \Upsilon_8, \mathscr{H}_2$ do not depend on geometry (a_x) or (b_x) for the same x
- $\Upsilon_1, \Upsilon_2, \Upsilon_3$ depend on geometry (a_x) or (b_x) for the same x:

$$\Upsilon_1, \Upsilon_2, \Upsilon_3 \propto \underset{\uparrow}{\gamma}$$

contains the dependence on the geometry



→ nonuniversal terms smaller for geometry (b_x)



(a_x)	$L_0 = xL\,,$	$L_1 = L_2 = L_3 = L$
(b_x)	$L_3 = xL_0\,,$	$L_0 = L_1 = L_2$

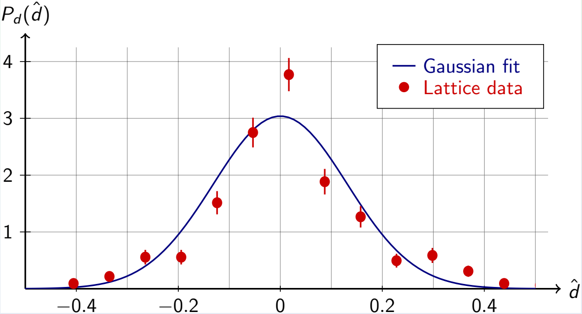
→ rotate lattice and use (b_2) instead of (a_2)

(i.e., include μ in a spatial direction) numbers for JLQCD data:

	(a_1)	$(a_{3/2})$	(a_2)	(a_3)	(a_4)
$\Sigma_{\text{eff}}^{\text{NLO}}/\Sigma$	1.3455	1.2477	1.1454	0.9404	0.7355
$\Sigma_{\text{eff}}^{\text{NNLO}}/\Sigma$	1.39(1)	1.288(7)	1.202(5)	1.047(3)	0.906(3)
$F_{\text{eff}}^{\text{NLO}}/F$	1.3004	1.3182	1.3192	1.3193	1.3193
$F_{\text{eff}}^{\text{NNLO}}/F$	1.279(9)	1.305(4)	1.306(2)	1.292(1)	1.261(2)
		$(b_{3/2})$	(b_2)	(b_3)	(b_4)
$F_{\text{eff}}^{\text{NLO}}/F$		1.1894	1.06816	0.7710	0.2186
$F_{\text{eff}}^{\text{NNLO}}/F$		1.182(8)	1.092(7)	0.959(6)	0.919(5)

$\longrightarrow$ ε-expansion converges for JLQCD data

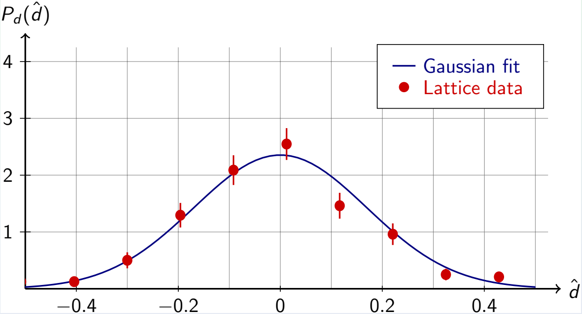
- fit of lattice data to the RMT result for the Dirac eigenvalue shift due to an imaginary chemical potential $a\mu = 0.01$
- RMT prediction: Gaussian distribution with $\sigma^2 = \mu^2 F^2 V$
- geometry (a_2):



$$\chi^2/\text{dof} = 4.2\,, \qquad F_{\text{eff}}^{(a_2)} = 67(5)\ \text{MeV} \quad \longrightarrow \quad F_{\text{NLO}}^{(a_2)} = 51(4)\ \text{MeV}$$

$$F_{\text{NNLO}}^{(a_2)} = 51(6)\ \text{MeV}$$

- fit of lattice data to the RMT result for the Dirac eigenvalue shift due to an imaginary chemical potential $a\mu = 0.01$
- RMT prediction: Gaussian distribution with $\sigma^2 = \mu^2 F^2 V$
- geometry (b_2):



$$\chi^2/\text{dof} = 0.9, \qquad F_{\text{eff}}^{(b_2)} = 86(5)\ \text{MeV} \quad \rightarrow \quad F_{\text{NLO}}^{(b_2)} = 81(5)\ \text{MeV}$$

$$F_{\text{NNLO}}^{(b_2)} = 79(5)\ \text{MeV}$$

Conclusions

- starting in NNLO, there are non-universal deviations from RMT
  - responsible for the bad fit to RMT
  - can be mimized by a suitable choice of the lattice geometry
- LECs from JLQCD ε-regime configurations in geometry (b_2), using RMT and finite-volume corrections:

$$\Sigma_{\text{NLO}}^{\overline{\text{MS}}} = [235(6)\ \text{MeV}]^3 \qquad F_{\text{NLO}} = 81(5)\ \text{MeV}$$
$$\Sigma_{\text{NNLO}}^{\overline{\text{MS}}} = [231(6)\ \text{MeV}]^3 \qquad F_{\text{NNLO}} = 79(5)\ \text{MeV}$$

  (F compatible within errors with $F_{\text{meson}} = 87(6)$ MeV)

- quoting an NNLO result is not really consistent since we are not fitting to the right function → additional $1/V$ corrections
  however, the non-RMT terms are small in geometry (b_2)
- Outlook: calculation of spectral density in ε-expansion beyond RMT