The QCD static potential: perturbative calculations

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(based on work done with Nora Brambilla, Joan Soto and Antonio Vairo)
Outline of the talk

- The static potential
Outline of the talk

- The static potential
- Renormalon effects in the static potential
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- The static potential
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- Comparison with lattice
Outline of the talk

■ The static potential

■ Renormalon effects in the static potential

■ Comparison with lattice

■ Conclusions
The static potential

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Here we are interested in the short distance region ($r \ll 1/\Lambda_{QCD}$), where perturbative (weak coupling) calculations are reliable.

$$V_s = -C_F \frac{\alpha_s(1/r)}{r} \left( 1 + \tilde{a}_1 \frac{\alpha_s(1/r)}{4\pi} + \tilde{a}_2 \left( \frac{\alpha_s(1/r)}{4\pi} \right)^2 + \cdots \right)$$
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Here we are interested in the short distance region ($r \ll 1/\Lambda_{QCD}$), where perturbative (weak coupling) calculations are reliable.

(in this talk I will always refer to the weak coupling regime)
When calculated in perturbation theory infrared divergences are found, starting at three loops

Appelquist, Dine, Muzinich '78
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After selective resummation of certain type of diagrams, logarithmic contributions (starting at three loops) are generated

The use of Effective Field Theories allows us to calculate those contributions
Currents status of perturbative calculations
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\[ V_s(r, \mu) = -\frac{C_F}{r} \alpha_s(1/r) \left\{ 1 + (a_1 + 2\gamma_E/\beta_0) \frac{\alpha_s(1/r)}{4\pi} \right. \]

\[ + \left[ a_2 + \left( \frac{\pi^2}{3} + 4\gamma_E^2 \right) \beta_0^2 + \gamma_E \left( 4a_1 \beta_0 + 2\beta_1 \right) \right] \left( \frac{\alpha_s(1/r)}{4\pi} \right)^2 \]

\[ + \left[ \frac{16 \pi^2}{3} C_A^3 \ln r \mu + \tilde{a}_3 \right] \left( \frac{\alpha_s(1/r)}{4\pi} \right)^3 \]

\[ + \left[ a_4^L \ln^2 r \mu + \left( a_4^L + \frac{16}{9} \pi^2 C_A^3 \beta_0 (-5 + 6 \ln 2) \right) \ln r \mu \right. \]

\[ + \tilde{a}_4 \left. \right] \left( \frac{\alpha_s(1/r)}{4\pi} \right)^4 \} \]
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\[ + \left[ \frac{16\pi^2}{3} C_A^3 \ln r \mu + \tilde{a}_3^{[n_f]} + \tilde{a}_3^0 \right] \left( \frac{\alpha_s(1/r)}{4\pi} \right)^3 \]

\[ + \left[ a_4 L^2 \ln^2 r \mu + \left( a_4^L + \frac{16}{9} \pi^2 C_A^3 \beta_0 (-5 + 6 \ln 2) \right) \ln r \mu \right. \]

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Known \hspace{2cm} Not known
Currents status of perturbative calculations

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\[ + \tilde{a}_4 \left( \frac{\alpha_s(1/r)}{4\pi} \right)^4 \]

RG improved expressions also available for sub-leading ultrasoft logs (given later in the talk)
One and two loop coefficients have been known since ten years ago

\[ a_1 = \frac{31}{9} C_A - \frac{20}{9} T_F n_f \]  

\[ a_2 = \left( \frac{4343}{162} + 4\pi^2 - \frac{\pi^4}{4} + \frac{22}{3} \zeta(3) \right) C_A^2 - \left( \frac{1798}{81} + \frac{56}{3} \zeta(3) \right) C_A T_F n_f - \left( \frac{55}{3} - 16 \zeta(3) \right) C_F T_F n_f + \left( \frac{20}{9} T_F n_f \right)^2 \]  

Billoire'80

Peter'97 Schröder'98
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\[ - \left( \frac{55}{3} - 16 \zeta(3) \right) C_F T_F n_f + \left( \frac{20}{9} T_F n_f \right)^2 \]

Very recently the fermionic parts of \( a_3 \) have been calculated

\[ a_3 = a_3^{(3)} n_f^3 + a_3^{(2)} n_f^2 + a_3^{(1)} n_f + a_3^{(0)} \]
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The computation of $a_3^{(0)}$ is reported to be in progress

Consider the non-relativistic bound state scales

\[ m (\gg \Lambda_{QCD}) \text{ hard scale} \]
\[ p \sim mv \text{ soft scale} \]
\[ E \sim mv^2 \text{ ultrasoft scale} \]
\[ v \ll 1 \text{ (} \alpha_s(mv) \sim v \text{) } \]
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The expansion is organized around the Coulombic state
Logarithmic contributions

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QCD
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\[ \text{QCD} \quad m \gg m_v , m_v^2 \quad \xrightarrow{\text{NRQCD}} \]
All those bound state scales will get entangled in a typical diagram.
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\[
\text{QCD} \xrightarrow{m \gg mv, mv^2} \text{NRQCD} \xrightarrow{m \gg mv \gg mv^2} \text{pNRQCD}
\]
All those bound state scales will get entangled in a typical diagram.

We can construct Effective Field Theories to disentangle the effects from those scales.

pNRQCD can be organized as an expansion in $r$ (multipole expansion) and $1/m$

$$\mathcal{L} = \int d^3r \, \text{Tr} \left\{ S^\dagger \left[ i\partial_0 - V_s(r; \mu) \right] S + O^\dagger \left[ iD_0 - V_o(r; \mu) \right] O \right\} +$$

$$+ V_A(r; \mu) \text{Tr} \left\{ O^\dagger \cdot g\text{ES} + S^\dagger \cdot g\text{EO} \right\} +$$

$$+ \frac{V_B(r; \mu)}{2} \text{Tr} \left\{ O^\dagger \cdot g\text{EO} + O^\dagger \cdot O \cdot g\text{E} \right\} - \frac{1}{4} F_{\mu\nu}^a F^{\mu\nu} a$$

\[\text{Diagram:} \quad \begin{array}{c}
\text{S} \quad \text{O} \quad \text{O}^\dagger \cdot g\text{ES}
\end{array}\]
All those bound state scales will get entangled in a typical diagram. We can construct Effective Field Theories to disentangle the effects from those scales. pNRQCD can be organized as an expansion in $r$ (multipole expansion) and $1/m$. Potentials appear as Wilson coefficients in the EFT.
We obtain the potential by matching NRQCD to pNRQCD. Schematically (at order $r^2$)
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E_0(r) = \lim_{T \to \infty} \frac{i}{T} \ln \left\langle P \exp \left\{ -ig \int_{r \times T} dz^\mu A_\mu(z) \right\} \right\rangle = V_s(r; \mu) - i \frac{g^2}{N_c} \int_0^\infty dt e^{-it(V_0-V_s)} \langle r \cdot E \ r \cdot E \rangle(\mu)
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Expectation value of Wilson loop operator
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- Expectation value of Wilson loop operator
- Matching coefficient
- Ultrasoft contribution (retardation effects)
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- Expectation value of Wilson loop operator
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- Left hand side must be $\mu$ independent (static energy)
- The logarithmic contribution at three loops can be deduced from the leading ultrasoft contribution (Brambilla, Pineda, Soto, Vairo ’99), the logarithmic terms at four loops from the sub-leading contribution (Brambilla, X.G.T., Soto, Vairo ’06)
The ultrasoft logarithms can be resummed by solving the renormalization group equations.
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\[
\begin{align*}
\mu \frac{d}{d\mu} V_s &= -\frac{2}{3} \frac{\alpha_s C_F}{\pi} \left(1 + 6 \frac{\alpha_s}{\pi} B\right) V_A^2 (V_o - V_s)^3 r^2 \\
\mu \frac{d}{d\mu} V_o &= \frac{1}{N_c^2 - 1} \frac{2}{3} \frac{\alpha_s C_F}{\pi} \left(1 + 6 \frac{\alpha_s}{\pi} B\right) V_A^2 (V_o - V_s)^3 r^2 \\
\mu \frac{d}{d\mu} \alpha_s &= \alpha_s \beta(\alpha_s) \\
\mu \frac{d}{d\mu} V_A &= 0 \\
\mu \frac{d}{d\mu} V_B &= 0
\end{align*}
\]

\[(B = \frac{-5 n_f + C_A (6 \pi^2 + 47)}{108})\]
The solution of the renormalization group equation for the singlet and octet potential is
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\[
V_s(\mu) = V_s(1/r) + 2 \frac{N_c^2 - 1}{N_c^2} [(V_o - V_s)(1/r)]^3 r^2 \frac{\gamma_{os}^{(0)}}{\beta_0} \left\{ \ln \frac{\alpha_s(\mu)}{\alpha_s(1/r)} \right. \\
+ \left. \left( -\frac{\beta_1}{4\beta_0} + \frac{\gamma_{os}^{(1)}}{\gamma_{os}^{(0)}} \right) \left[ \frac{\alpha_s(\mu)}{\pi} - \frac{\alpha_s(1/r)}{\pi} \right] \right\}
\]

\[
V_o(\mu) = V_o(1/r) - \frac{2}{N_c^2} [(V_o - V_s)(1/r)]^3 r^2 \frac{\gamma_{os}^{(0)}}{\beta_0} \left\{ \ln \frac{\alpha_s(\mu)}{\alpha_s(1/r)} \right. \\
+ \left. \left( -\frac{\beta_1}{4\beta_0} + \frac{\gamma_{os}^{(1)}}{\gamma_{os}^{(0)}} \right) \left[ \frac{\alpha_s(\mu)}{\pi} - \frac{\alpha_s(1/r)}{\pi} \right] \right\}
\]

\[\gamma_{os}^{(0)} = \frac{N_c}{3}, \quad \gamma_{os}^{(1)} = 2BN_c\]
Renormalon effects in the static potential

- $V_s$ does not present a good convergent behavior

- $r_0 V_s(r)\, r_0 \sim 0.5fm$, $\mu = 2.5r_0^{-1}$, $\alpha_s$ determined according to Capitani et al. '99

Dotted blue: tree level; Dot-dashed magenta: 1 loop; dashed brown: 2 loops (plus leading us log resummation); Solid green: 3 loops (plus next-to-leading us log resummation)
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- This bad convergence can be interpreted as coming from a singularity close to the origin in the Borel plane, and signaling that non-perturbative contributions are important.
Renormalon effects in the static potential

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- The strategy is to find operators that account for the non-perturbative effects. Then we impose that the ambiguities in the Borel transform are accounted for those operators and reshuffle contributions from the perturbative series to the operators
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The strategy is to find operators that account for the non-perturbative effects. Then we impose that the ambiguities in the Borel transform are accounted for those operators and reshuffle contributions from the perturbative series to the operators.

We will implement the renormalon cancellation along the lines of the so-called RS scheme.
The lower dimensional operators, that account for the ambiguities, are those related to the residual mass term in HQET, which get inherited in pNRQCD

\[ \mathcal{L}_{HQET} = \bar{h}_v (iD_0 - \delta m_Q) h_v + \mathcal{O} \left( \frac{1}{m_Q} \right) \]
The lower dimensional operators, that account for the ambiguities, are those related to the residual mass term in HQET, which get inherited in pNRQCD.

In the weak coupling regime at the static limit, we account for them with the shift

\[ V_{s,o} \rightarrow V_{s,o} + \Lambda_{s,o} \quad \Lambda_{s,o} \sim \Lambda_{QCD} \]
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\[ V_{s,o} \rightarrow V_{s,o} + \Lambda_{s,o} \quad \Lambda_{s,o} \sim \Lambda_{QCD} \]

The renormalization group properties of \( \Lambda_{s,o} \) fix the renormalon singularity up to a normalization constant

Beneke '94
The renormalization group equations for $\Lambda_{s,o}$ are given by
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$$
\mu \frac{d}{d\mu} \Lambda_s = -2 \frac{\alpha_s C_F}{\pi} \left( 1 + 6 \frac{\alpha_s}{\pi} B \right) V_A^2 r^2 \left[ (V_o - V_s) (1/r) \right]^2 (\Lambda_o - \Lambda_s)
$$

$$
\mu \frac{d}{d\mu} \Lambda_o = \frac{2}{N_c^2 - 1} \frac{\alpha_s C_F}{\pi} \left( 1 + 6 \frac{\alpha_s}{\pi} B \right) V_A^2 r^2 \left[ (V_o - V_s) (1/r) \right]^2 (\Lambda_o - \Lambda_s)
$$

The ultrasoft effects introduce anomalous dimensions and mixing between singlet and octet
The renormalization group equations for $\Lambda_{s,o}$ are given by

Brambilla, X.G.T., Soto, Vairo '08 (in preparation)

\[
\begin{align*}
\mu \frac{d}{d\mu} \Lambda_s &= -2 \frac{\alpha_s C_F}{\pi} \left(1 + 6 \frac{\alpha_s}{\pi} B\right) V_A^2 r^2 \left[(V_o - V_s) \left(\frac{1}{r}\right)\right]^2 (\Lambda_o - \Lambda_s) \\
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\end{align*}
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And the solution is

\[
\begin{align*}
\Lambda_s(\mu) &= N_s \Lambda + 2 C_F (N_o - N_s) \Lambda r^2 \left[(V_o - V_s) \left(1/r\right)\right]^2 \\
&\quad \times \left(\frac{2}{\beta_0} \ln \alpha_s(\mu) + \eta_0 \alpha_s(\mu)\right)
\end{align*}
\]

\[
\begin{align*}
\Lambda_o(\mu) &= N_o \Lambda - \frac{1}{N_c} (N_o - N_s) \Lambda r^2 \left[(V_o - V_s) \left(1/r\right)\right]^2 \\
&\quad \times \left(\frac{2}{\beta_0} \ln \alpha_s(\mu) + \eta_0 \alpha_s(\mu)\right)
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We have to match those structures to the ambiguities in a proper definition of the Borel integral.
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Without ultrasoft effects (no anomalous dimension) we have

\[ I_{s,o} = \rho \frac{4\pi}{\beta_0} \int_0^\infty du \, e^{-\frac{4\pi}{\beta_0} \frac{u}{\alpha_s}} \times \left\{ \frac{R_{s,o}}{(1 - 2u)^{1+b}} \left[ 1 + c_1(1 - 2u) + c_2(1 - 2u)^2 + c_3(1 - 2u)^3 + \ldots \right] \right\} \]

And the coefficients \( c_i \) are determined just by the coefficients in the beta function.
Without ultrasoft effects (no anomalous dimension) we have

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I_{s,o} = \rho \frac{4\pi}{\beta_0} \int_0^\infty du \ e^{-\frac{4\pi}{\beta_0} u} \left\{ \frac{R_{s,o}}{(1 - 2u)^{1 + b}} \left[ 1 + c_1 (1 - 2u) + c_2 (1 - 2u)^2 + c_3 (1 - 2u)^3 + \ldots \right] \right\}
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With ultrasoft effects we have

\[ I_{s,o} = \rho \frac{4\pi}{\beta_0} \int_0^\infty du \, e^{-\frac{4\pi}{\beta_0} \frac{u}{\alpha_s}} \times \left\{ \frac{R_{s,o}}{(1-2u)^{1+b}} \left[ 1 + c_1(1-2u) + c_2; s,o(1-2u)^2 \\
+ c_3; s,o(1-2u)^3 + \ldots + d_1; s,o(1-2u)^2 \ln(1-2u) \\
+ d_2; s,o(1-2u)^3 \ln(1-2u) + \ldots \right] \right\} \]

\(c_2, c_3, \ldots\) are now different for singlet and octet and we have new non-analytic terms \((d_i)\)
The previous expression tells us which terms we have to subtract from $V_{s,o}$, to get rid of the bad behavior of the perturbative series.
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$R_{s,o}$ is determined (approximately) through

$$R_{s,o} = V_{s,o}^{BT} (u)(1 - 2u)^{1+b}|_{u=\frac{1}{2}}, \rho = \mu = 2.5r_0^{-1}$$

Dotted blue: tree level; Dot-dashed magenta: 1 loop; dashed brown: 2 loops (plus leading us log resummation); Solid green: 3 loops (plus next-to-leading us log resummation). RS scheme implemented by just subtracting the most singular term.
The previous expression tells us which terms we have to subtract from $V_{s,o}$, to get rid of the bad behavior of the perturbative series.

$$ R_{s,o} \text{ is determined (approximately) through } R_{s,o} = V_{s,o}^{BT} (u) (1 - 2u)^{1+b} \bigg|_{u=\frac{1}{2}}, \rho = \mu = 2.5 r_0^{-1} $$

Dotted blue: tree level; Dot-dashed magenta: 1 loop; dashed brown: 2 loops (plus leading us log resummation); Solid green: 3 loops (plus next-to-leading us log resummation). RS scheme implemented by just subtracting the most singular term.

The RS scheme provides us with a better perturbative behavior.
Comparison with lattice

We will compare the singlet static potential to the lattice data

(Necco, Sommer '01)
Comparison with lattice

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We have to plot $V_s + \Lambda_s$ as a function of $r$

$$V_s + \Lambda_s = V_s (r, \mu, \rho) + K_1 + K_2 f(r, \mu)$$

($K_1 = N_s \Lambda$ and $K_2 = (N_o - N_s) \Lambda$ are the two integration constants coming from the diff. eqs.)
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To have a definite way to organize the different terms we will use the counting

$$\frac{1}{r} \gg \frac{\alpha_s}{r} \gg \Lambda_{QCD} \quad \Lambda \sim N_s \Lambda \sim N_o \Lambda \sim \Lambda_{QCD} \sim \frac{\alpha_s^2}{r}$$
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Starting at three loop level we have the presence of two arbitrary constants. We fix the constants by forcing the curves to go through the first one or two lattice data points
\[ r_0(V_s(r) + \Lambda_s(r)) \]

(Dotted blue: tree level; Dot-dashed magenta: 1 loop; dashed brown: 2 loops (plus leading us log resummation); Solid green: 3 loops (plus next-to-leading us log resummation))
Uncertainties of the result
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Impact of varying the Pade estimate for $a_3^{(0)}$ by 30%
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Impact of the variation of $\alpha_s$ ($\Lambda_{\overline{MS}} = 0.602(48) r_0^{-1}$)
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Impact of varying the Pade estimate for $\alpha_3^{(0)}$ by 30%

Impact of the variation of $\alpha_s$ ($\Lambda_{\overline{MS}} = 0.602^{(48)} r_0^{-1}$)

Effect of higher order ($\alpha_s^5$) terms
Conclusions

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- Comparison with lattice data is very good