Factorization in Exclusive Quarkonium Production

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Statement of Factorization

- Factorization in QCD is the separation of short-distance, perturbatively calculable processes from long-distance, inherently nonperturbative processes.
- Factorization is the basis for all perturbative calculations in QCD.
- I will discuss the proofs of factorization for two specific exclusive quarkonium production processes:
  - $e^+e^-$ annihilation to two quarkonia (focus on this in examples),
  - $B$-meson decay to a light meson plus a quarkonium.
Evolution of a $Q\bar{Q}$ pair into a quarkonium

- A part of any exclusive quarkonium production process is the nonperturbative amplitude for a heavy $Q\bar{Q}$ pair to evolve into a heavy quarkonium $H$ and nothing else.

- It can be calculated as a matrix element in nonrelativistic QCD (NRQCD):

$$\langle H | \psi^{\dagger} \kappa_i \chi | 0 \rangle = \langle H | \mathcal{O}_i | 0 \rangle.$$ 

- $\psi$ is the Pauli spinor field that annihilates $Q$.
- $\chi$ is the Pauli spinor field that creates $\bar{Q}$.
- The $\kappa_i$ are combinations of Pauli and color matrices and covariant derivatives, usually chosen to put the $Q\bar{Q}$ pair in a definite color, spin, and orbital-angular-momentum state.
Factorization for Exclusive Production of a Single Quarkonium ($B$-meson decay)

- $Q$ is the momentum of the quarkonium in the CM frame (of the $B$ meson).
- In the limit $Q \to \infty$, the production amplitude can be written as

$$A = \sum_{n} A_{i} \langle H | O_{i} | 0 \rangle.$$

- The coefficients $A_{i}$ contain the Lorentz-invariant amplitude to create a $Q\bar{Q}$ pair in the color-angular-momentum state of $O_{i}$.
  - Includes the Born amplitude plus corrections of higher-order in $\alpha_{s}$ involving momenta with high virtualities $\sim Q$.
  - The high virtuality allows one to calculate the higher-order corrections in a perturbation series in $\alpha_{s}(Q^{2}) << 1$.
- In the case of $B$-meson decays to a charmonium plus a light hadron, the coefficients $A_{i}$ also contain a nonperturbative $B$-meson-to-light-meson form factor.
- The sum over operator matrix elements is an expansion in powers of $v$, where $v$ is the typical velocity of the $Q$ or $\bar{Q}$ in the quarkonium rest frame.
Factorization for Exclusive Production of Two Quarkonia \((e^+e^- \text{ annihilation})\)

- In the limit \(Q \to \infty\), the production amplitude can be written as
  \[
  A = \sum_{i,j} B_{ij} \langle H_1 | O_i | 0 \rangle \langle H_2 | O_j | 0 \rangle.
  \]

- The coefficients \(B_{ij}\) contain the Lorentz-invariant amplitude to create a \(Q\bar{Q}\) pair in the color-angular-momentum states of \(O_i\) and a \(Q\bar{Q}\) pair in the color-angular-momentum state of \(O_j\).

Ingredients in a Proof of Factorization

- A proof of factorization is complicated because gluons can connect the quarkonium to the other parts of the production process.
- The gluons can be soft (low-energy) or nearly collinear to an emitting particle.
  - In either case, propagators with low virtuality appear.
  - These contributions do not fit into the factorized form and must be re-organized in order to prove factorization.
- Goal:
  - Re-organize soft contributions so that they cancel or can be absorbed into the \(B\)-meson-to-light-meson form factor.
  - Re-organize collinear contributions so that they cancel or can be absorbed into NRQCD matrix elements or the \(B\)-meson-to-light-meson form factor.
Leading Regions

- Work in the $B$-meson or $e^+e^-$ CM frame with the momenta of the produced mesons along the plus or minus 3 axis.

- Detailed power-counting arguments and/or Landau analyses of pinch singularities (Sterman) show that low-virtuality propagators arise from logarithmic integrals that are associated with soft or collinear gluons.

- Use light-cone coordinates:

$$
\begin{align*}
  k^+ &= (k^0 + k^3)/\sqrt{2}, \\
  k^- &= (k^0 - k^3)/\sqrt{2}, \\
  k_\perp &= (k_1, k_2), \\
  k_1 \cdot k_2 &= k_1^+ k_2^- + k_1^- k_2^+ - k_1 \cdot k_2 \perp.
\end{align*}
$$

- The soft and collinear gluons have momenta whose orders of magnitude are as follows:
  
  - Soft gluons: $(k^+, k^-, k_\perp) \sim Q(\lambda, \lambda, \lambda)$
  
  - Collinear-to-plus gluons: $(k^+, k^-, k_\perp) \sim Q(1, \lambda^2, \lambda)$
  
  - Collinear-to-minus gluons: $(k^+, k^-, k_\perp) \sim Q(\lambda^2, 1, \lambda)$

$$
\lambda \sim \Lambda_{\text{QCD}}/Q.
$$
• Detailed power-counting arguments and/or a Landau analysis show that the contributions of leading order in $\lambda$ are arranged in specific topologies.

• For example, for $e^+e^-$ annihilation to two quarkonia, we have

\[ H J_2 J_1 S \]

\[ H \]

The hard subdiagram $H$ contains the contributions of high virtuality $\sim Q$.

• The “jet” subdiagrams $J_1$ and $J_2$ contain the plus-moving and minus-moving quarkonia, respectively, and the corresponding collinear gluons (plus ghost and quark loops).

• $J_1$ and $J_2$ connect to the hard subdiagram through the $Q$ and $\bar{Q}$ lines and any number of collinear gluons.

• The soft subdiagram $S$ contains the soft gluons (plus ghost and quark loops).

• $S$ connects to the jet subdiagrams through any number of soft gluons.
Collinear Approximations

- For a collinear-to-plus gluon with momentum $k$, we can approximate the $g_{\mu\nu}$ in its propagator:

$$g_{\mu\nu} \rightarrow \frac{k_{\mu} \bar{n}_{\nu}}{k \cdot \bar{n}}.$$ 

- $\bar{n}$ is a light-like vector in the minus direction.
- $\nu$ is the index on the collinear-to-plus line to which the gluon attaches.
- This approximation holds if the plus component dominates in both $k$ and in the current at the vertex to which the $\nu$ end of the soft gluon attaches.
- It holds up to corrections of order $\lambda$ if the $\mu$ index attaches to a hard or collinear-to-minus line.
- It does not hold if the $\mu$ index attaches to a collinear-to-plus line.
• Similarly, for a collinear-to-minus gluon with momentum $k$, we can approximate the $g_{\mu\nu}$ in its propagator:

$$g_{\mu\nu} \rightarrow \frac{k_\mu n_\nu}{k \cdot n},$$

- $n$ is a light-like vector in the plus direction.
- The approximation holds up to corrections of order $\lambda$ if the $\mu$ index attaches to a hard or collinear-to-plus line, but not to a collinear-to-minus line.

• With these approximations, the collinear gluons have polarizations proportional to $k$, i.e., they are longitudinally polarized.
Factorization of Collinear Gluons

• For a longitudinally polarized gluon attached to a fermion line carrying momentum $p$, one can apply the Feynman identity

\[ k \cdot \gamma = [(p+k) \cdot \gamma + m] - [p \cdot \gamma + m] \]

to cancel propagators to the left and right of the attachment.

• Repeated application of the Feynman identity and its non-Abelian generalizations leads to a Ward identity (GTB; Collins, Soper, Sterman):

\[
P_{n,a_n}P_{1,a_1} = \sum_{n=1}^{\infty} a_n a_1 P_{n,a_n}P_{1,a_1}
\]

• Holds for any number of collinear-to-plus (minus) gluons that attach in all possible ways to a Green’s function with $n$ truncated external legs.

• On the RHS, the polarized gluons attach to eikonal lines that end on the truncated legs.

  – Eikonal vertex: $i g T_a \bar{n}_\mu \ (i g T_a n_\mu)$.
  – Eikonal propagator: $i/[k \cdot \bar{n} + i\epsilon] \ (i/[k \cdot n + i\epsilon])$. 
The eikonal lines correspond to path-ordered integrals of the gauge field:

\[ U = P \exp \left[ \int dx^- i g A(x) \cdot \vec{n} \right]. \]

Application of this result to the collinear-to-plus gluons in the leading topology leads to factorization of those gluons:

In the case of \( e^+ e^- \) annihilation to two quarkonia, a similar procedure can be used to factor the collinear-to-minus gluons.
For a soft gluon with momentum $k$, we can approximate the $g_{\mu \nu}$ in its propagator (Grammer, Yennie; Collins, Soper, Sterman):

$$g_{\mu \nu} \rightarrow \frac{k_{\mu} p_{\nu}}{k \cdot p},$$

- $p$ is the momentum of the line to which the $\mu$ end of the soft gluon attaches.
- The soft approximation holds provided that $k$ can be neglected compared to $p$ in the current at the $\mu$ vertex.
- It holds up to corrections of order $\lambda$ if the $\mu$ index attaches to a collinear line line or soft line and the $\nu$ index attaches to an anti-collinear line or soft line.
- Unlike the collinear approximations, the soft approximation depends on $p$.
- In the soft approximation, the soft gluon is longitudinally polarized.

We would like to use the soft approximation to factor soft gluons from the collinear-to-plus (or minus) jet.

But, in a given jet, the $Q$, $\bar{Q}$, and binding gluons all have different momenta.
• Key insight: Because of the large boost from the quarkonium rest frame to the $B$-meson or $e^+e^-$ CM frame, the $Q$, $\bar{Q}$ and binding gluons for the collinear-to-plus (minus) jet all have momenta in which the plus (minus) component is dominant.

  – All other components are suppressed by at least a factor $m_c/Q$.

• Allows us to use a uniform soft approximation for all of the constituents of a jet.

• For the collinear-to-plus (minus) jet, approximate the momenta of all of the constituents with $\tilde{n}$, where $\tilde{n} = n(\bar{n})$.

  – This leads to a modified soft approximation:

$$g_{\mu\nu} \rightarrow \frac{k_\mu \tilde{n}_\nu}{k \cdot \tilde{n}}.$$  

  – Holds up to corrections of order $m_c/Q$.  


Factorization of Soft Gluons

- As for collinear gluons, there is a Ward identity for soft gluons (Collins, Soper, Sterman):

\[ P_n; a_n = a_n a_1 P_1; a_1 \]

- Applying this Ward identity, we can factor the soft gluons from the collinear-to-plus jet:

- To factor the soft gluons completely from the jet, we need connections of soft gluons to collinear eikonal lines. These are missing, but they are power suppressed, and so can be added “for free.”

- Then the soft eikonal lines lie to the inside of the collinear eikonal lines (not shown).
Cancellation of Eikonal Lines

- The collinear-to-plus eikonal lines on the $Q$ and $\bar{Q}$ lines cancel.
  - They end on approximately the same space-time point because the hard subdiagram is local to within order $1/Q$.
  - Because quarkonium is a color singlet, the eikonal factors are $U_{ab}^\dagger U_{bc} = \delta_{ac}$.

- The soft eikonal lines also cancel in this way.

- In the case of $B$-meson decay, the soft subdiagram is decoupled from the jet subdiagram, but it remains as part of the $B$-meson-to-light-meson form factor.

- In the case of $e^+e^-$ annihilation to two quarkonia, we can carry out a similar factorization and cancellation of soft and collinear gluons for the $J_2$ subdiagram.
  The soft subdiagram completely decouples in this case:
Arriving at the Factorized Form

• Now the collinear gluons in each jet are completely decoupled from the hard subdiagram.

• The remaining gluon interactions in each jet simply produce the evolution of a $Q\bar{Q}$ pair into a quarkonium.

• Starting from this decoupled form, it is straightforward to expand the convolution of the $H$ sub-diagram with the $J_1$ and $J_2$ subdiagrams in terms of the NRQCD matrix elements to obtain the factorized results quoted earlier.

• This involves standard effective-field theory technology:
  – Taylor series in the $Q\bar{Q}$ relative momenta,
  – equations of motion,
  – projectors for color, spin, orbital angular momentum.
Corrections to Factorization

- The soft approximation for a given jet subdiagram accounts for soft contributions up to corrections of order $m_c/Q$.

- Therefore, there can be uncanceled soft logarithms of order $(m_c/Q) \log(m_c/\Lambda_{\text{QCD}})$.
  - Such power suppressed soft-logarithms appear as corrections in other hard-scattering factorization theorems, e.g., Drell-Yan lepton-pair production.
  - They show up as formal IR divergences in perturbative calculations, but are cut off physically by the confinement scale $\Lambda_{\text{QCD}}$.

- In $e^+e^-$ annihilation, there is a cancellation of soft-gluon connections to both jet subdiagrams: The corrections to factorization are suppressed as $m_c^2/Q^2 \sim m_c^2/s$ ($\sim 10\%$ at the $B$ factories).

- The corrections to factorization in $B$-meson decays are suppressed as only one power of $m_c/Q \sim m_c/m_b$.
  - Corrections to factorization of order $m_c^2/m_b^2$ were discovered earlier in a one-loop calculation (Song, Meng, Gao, Chao).
  - We find larger corrections, of order $m_c/m_b$, because we use more general forms for the meson wave functions.
• The corrections to factorization arise from nonperturbative soft physics, and so the coefficient of $m_c/m_b$ or $m_c^2/s$ is unknown.
  – One might guess that it is of order one.
  – One could determine its size by comparing theoretical predictions with experiment.

• The factor $m_c/m_b$ does not provide a large suppression.
  – In $B$-meson decays to $\chi_c$ states, only the $\chi_{c1}$ state is produced in the leading-order factorized formula.
  – Experimentally, in $B \to \chi_c + K$, $\text{Br}[\chi_{c0}]/\text{Br}[\chi_{c1}] = 0.37$.
  – This nonzero result could be from corrections of higher order in $\alpha_s$ or from power corrections to factorization.

• At order $v^0$ ($S$-wave charmonium states), the corrections to factorization vanish exactly in one-loop diagrams because the $Q$ and $\bar{Q}$ have the same momenta, and the same soft approximation applies to both.
  – Explicit calculation for $e^+e^-$ annihilation by Zhang, Ma, Chao.
  – Explicit calculation for $B$-meson decays by Chay and Kim.
  – We do not expect such an exact cancellation to hold at higher orders in $v$ or $\alpha_s$, even for $S$-wave charmonium states.
Discussion

Other Exclusive Processes

• The techniques used in these proofs (in particular the uniform soft approximation for quarkonium jets) can be applied to other exclusive production processes, e.g.,

  – $e^+e^-$ annihilation to a quarkonium and a light meson,
  – $\Upsilon$ decays to charmonium,
  – $\gamma\gamma$ production of two quarkonia or a quarkonium and a light meson.

Factorization for Inclusive Production

• The proofs of factorization for exclusive quarkonium production contain some of the technology that is needed to prove factorization theorems for inclusive quarkonium production.

• However, in the inclusive case, one needs to deal with new difficulties (Nayak, Qiu, Sterman):

  – additional charm quarks that are co-moving with the quarkonium,
  – additional light quanta that are co-moving with the quarkonium.

• Additional charm quarks might be controlled with an experimental isolation cut.

• Additional light quanta present a problem for factorization proofs that has been solved only through two-loop order (Nayak, Qiu, Sterman).