Automated Calculation Scheme for α^n Contributions of QED to Lepton $g-2$

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APPLICATION OF QUANTUM FIELD THEORY TO PHENOMENOLOGY

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Introduction

Primary concern of the talk is:

- ▷ $A_1^{(10)}$ term of α^5 correction of electron anomalous magnetic moment.
- ▷ Automated scheme for diagrams with no closed lepton loops.

- Anomalous magnetic moment is the best source of α, and the most stringent test of QED as well.
- From recent improvement of measurement (Harvard Univ.), we find

(*pure guess*)

$$\alpha^{-1}(a_e) = 137.035999708(12)(31)(68)$$

$(\alpha^4)(\alpha^5)$(expr)

Preliminary.
Do not quote until published.

- *cf.* Kinoshita's talk.
- Reliable estimates of α^5 term should be requested.

- A_1 term is QED correction purely due to electron contributions. It is evaluated by perturbation theory in terms of α:

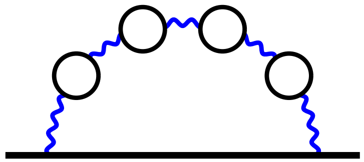
$$A_1 = A_1^{(2)} \left(\frac{\alpha}{\pi}\right) + A_1^{(4)} \left(\frac{\alpha}{\pi}\right)^2 + A_1^{(6)} \left(\frac{\alpha}{\pi}\right)^3 + \cdots$$

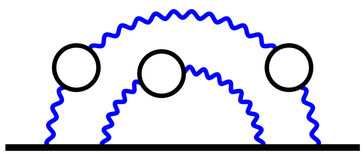
  α^5 contribution is denoted by $A_1^{(10)}$.

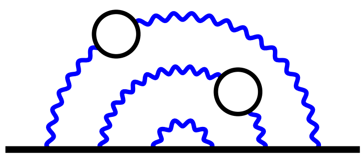
- The number of diagrams contributing to $A_1^{(10)}$ is **12672** .

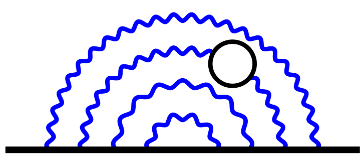
- They are classified into 32 gauge invariant groups within 6 distinct sets.

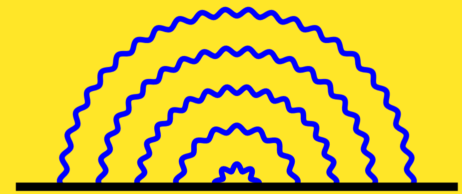
Classification of 10th order diagrams

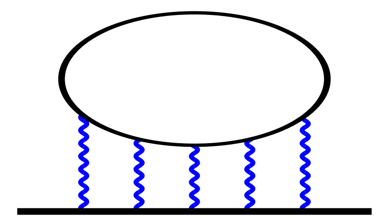
Set I  208

Set II  600

Set III  1140

Set IV  2072

Set V  6354

Set VI  2298

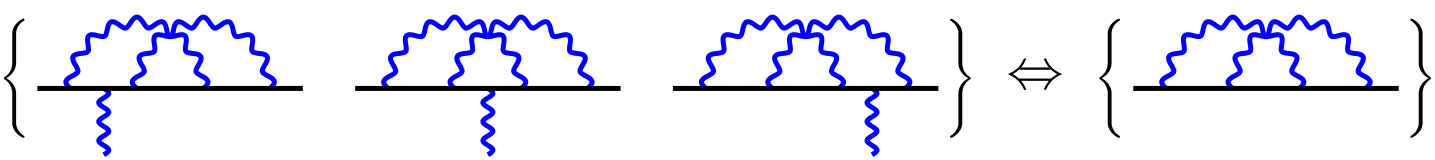
Obstacles in set V diagrams

Set V consists of 6354 Feynman diagrams that have no closed lepton loops.

- 9 vertex diagrams are related to 1 self-energy diagram by the Ward-Takahashi identity:

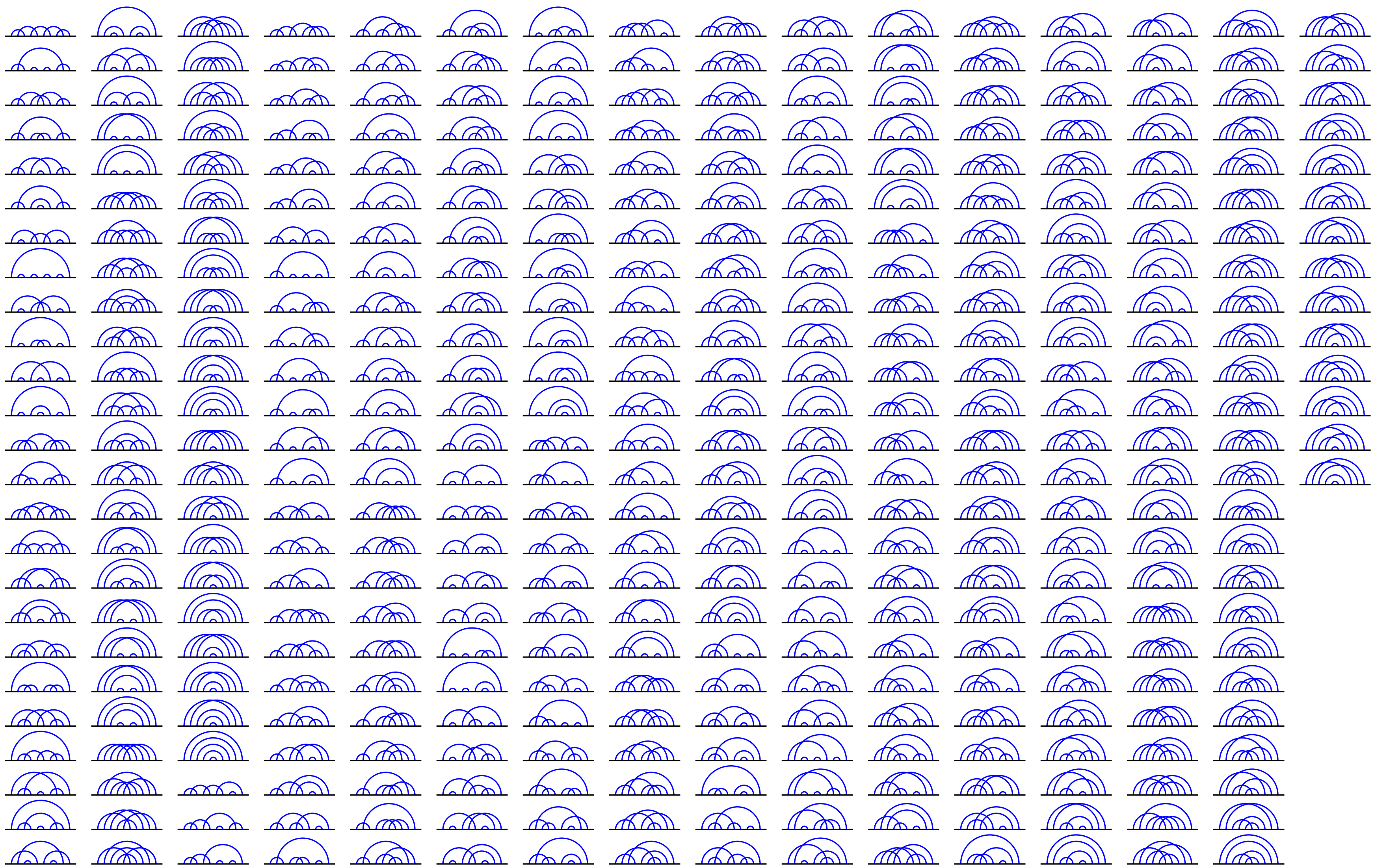
$$\Lambda^{\nu}(p,q) \simeq -q_{\mu}\left[\frac{\partial \Lambda^{\mu}(p,q)}{\partial q_{\nu}}\right]_{q\to 0} - \frac{\partial \Sigma(p)}{\partial p_{\nu}}$$

e.g. 4th order case:



By this, the number of diagrams reduces to 706.

- Time reversal invariance reduces further to 389.



Each diagram is known to have a large number of UV divergent parts, and is difficult to construct.

Maximally 47 UV subtraction terms are required.

e.g.

$$
\begin{aligned}
\Delta M_{\mathrm{X001}} = M_{\mathrm{X001}} \\
& - L_{2\mathrm{v}} M_{\mathrm{m01}}(\ell_3,\ell_4,\ell_5,\ell_6,\ell_7,\ell_8,\ell_9) - L_{2\mathrm{v}} M_{\mathrm{m01}}(\ell_1,\ell_2,\ell_3,\ell_4,\ell_5,\ell_6,\ell_7) - L_{4\mathrm{a1v}} M_{6\mathrm{f}}(\ell_5,\ell_6,\ell_7,\ell_8,\ell_9) - L_{4\mathrm{a1v}} M_{6\mathrm{f}}(\ell_1,\ell_2,\ell_3,\ell_4,\ell_5) \\
& - L_{6\mathrm{f1v}} M_{4\mathrm{a}}(\ell_1,\ell_2,\ell_3) - L_{6\mathrm{f1v}} M_{4\mathrm{a}}(\ell_7,\ell_8,\ell_9) - L_{\mathrm{m011v}} M_2(\ell_1) - L_{\mathrm{m011v}} M_2(\ell_9) \\
& + L_{2\mathrm{v}} L_{2\mathrm{v}} M_{6\mathrm{f}}(\ell_5,\ell_6,\ell_7,\ell_8,\ell_9) + L_{2\mathrm{v}} L_{2\mathrm{v}} M_{6\mathrm{f}}(\ell_3,\ell_4,\ell_5,\ell_6,\ell_7) + L_{2\mathrm{v}} L_{2\mathrm{v}} M_{6\mathrm{f}}(\ell_1,\ell_2,\ell_3,\ell_4,\ell_5) \\
& + L_{2\mathrm{v}} L_{4\mathrm{a1v}} M_{4\mathrm{a}}(\ell_7,\ell_8,\ell_9) + L_{2\mathrm{v}} L_{4\mathrm{a1v}} M_{4\mathrm{a}}(\ell_3,\ell_4,\ell_5) + L_{2\mathrm{v}} L_{4\mathrm{a1v}} M_{4\mathrm{a}}(\ell_1,\ell_2,\ell_3) \\
& + L_{2\mathrm{v}} L_{4\mathrm{a1v}} M_{4\mathrm{a}}(\ell_7,\ell_8,\ell_9) + L_{2\mathrm{v}} L_{4\mathrm{a1v}} M_{4\mathrm{a}}(\ell_5,\ell_6,\ell_7) + L_{2\mathrm{v}} L_{4\mathrm{a1v}} M_{4\mathrm{a}}(\ell_1,\ell_2,\ell_3) \\
& + L_{2\mathrm{v}} L_{6\mathrm{f1v}} M_2(\ell_3) + L_{2\mathrm{v}} L_{6\mathrm{f1v}} M_2(\ell_9) + L_{2\mathrm{v}} L_{6\mathrm{f1v}} M_2(\ell_1) + L_{2\mathrm{v}} L_{6\mathrm{f1v}} M_2(\ell_9) + L_{2\mathrm{v}} L_{6\mathrm{f1v}} M_2(\ell_1) + L_{2\mathrm{v}} L_{6\mathrm{f1v}} M_2(\ell_7) \\
& + L_{4\mathrm{a1v}} L_{4\mathrm{a1v}} M_2(\ell_1) + L_{4\mathrm{a1v}} L_{4\mathrm{a1v}} M_2(\ell_5) + L_{4\mathrm{a1v}} L_{4\mathrm{a1v}} M_2(\ell_9) \\
& - L_{2\mathrm{v}} L_{2\mathrm{v}} L_{2\mathrm{v}} M_{4\mathrm{a}}(\ell_7,\ell_8,\ell_9) - L_{2\mathrm{v}} L_{2\mathrm{v}} L_{2\mathrm{v}} M_{4\mathrm{a}}(\ell_5,\ell_6,\ell_7) - L_{2\mathrm{v}} L_{2\mathrm{v}} L_{2\mathrm{v}} M_{4\mathrm{a}}(\ell_3,\ell_4,\ell_5) - L_{2\mathrm{v}} L_{2\mathrm{v}} L_{2\mathrm{v}} M_{4\mathrm{a}}(\ell_1,\ell_2,\ell_3) \\
& - L_{2\mathrm{v}} L_{2\mathrm{v}} L_{4\mathrm{a1v}} M_2(\ell_5) - L_{2\mathrm{v}} L_{2\mathrm{v}} L_{4\mathrm{a1v}} M_2(\ell_9) - L_{2\mathrm{v}} L_{2\mathrm{v}} L_{4\mathrm{a1v}} M_2(\ell_3) - L_{2\mathrm{v}} L_{2\mathrm{v}} L_{4\mathrm{a1v}} M_2(\ell_9) \\
& - L_{2\mathrm{v}} L_{2\mathrm{v}} L_{4\mathrm{a1v}} M_2(\ell_3) - L_{2\mathrm{v}} L_{2\mathrm{v}} L_{4\mathrm{a1v}} M_2(\ell_7) - L_{2\mathrm{v}} L_{2\mathrm{v}} L_{4\mathrm{a1v}} M_2(\ell_1) - L_{2\mathrm{v}} L_{2\mathrm{v}} L_{4\mathrm{a1v}} M_2(\ell_1) \\
& - L_{2\mathrm{v}} L_{2\mathrm{v}} L_{4\mathrm{a1v}} M_2(\ell_9) - L_{2\mathrm{v}} L_{2\mathrm{v}} L_{4\mathrm{a1v}} M_2(\ell_7) - L_{2\mathrm{v}} L_{2\mathrm{v}} L_{4\mathrm{a1v}} M_2(\ell_1) - L_{2\mathrm{v}} L_{2\mathrm{v}} L_{4\mathrm{a1v}} M_2(\ell_5) \\
& + L_{2\mathrm{v}} L_{2\mathrm{v}} L_{2\mathrm{v}} L_{2\mathrm{v}} M_2(\ell_9) + L_{2\mathrm{v}} L_{2\mathrm{v}} L_{2\mathrm{v}} L_{2\mathrm{v}} M_2(\ell_7) + L_{2\mathrm{v}} L_{2\mathrm{v}} L_{2\mathrm{v}} L_{2\mathrm{v}} M_2(\ell_5) + L_{2\mathrm{v}} L_{2\mathrm{v}} L_{2\mathrm{v}} L_{2\mathrm{v}} M_2(\ell_3) + L_{2\mathrm{v}} L_{2\mathrm{v}} L_{2\mathrm{v}} L_{2\mathrm{v}} M_2(\ell_1)
\end{aligned}
$$

- Some automated scheme is required to get rid of human errors.

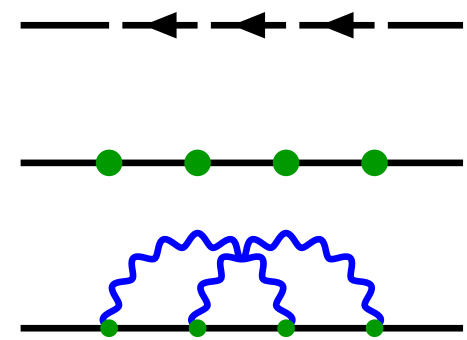
Diagrams without lepton loops

Our subjects are 1PI self-energy diagrams without closed lepton loops. They have quite simple structure:

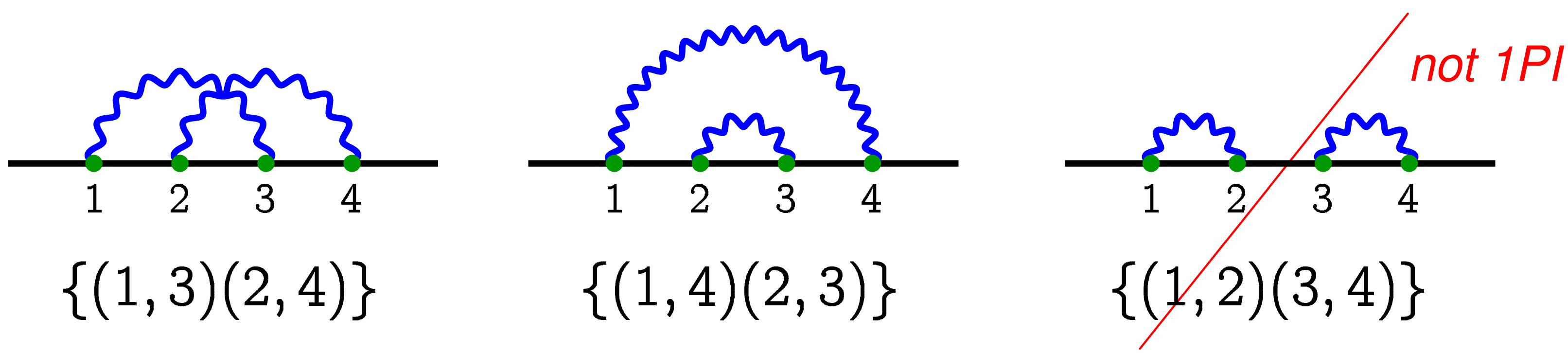
(1) All lepton propagators form a single path.

(2) All vertices lie on the lepton path.

(3) Photon propagators contract pairs of vertices.



A diagram is represented by "pattern of contraction".



$\{(1,3)(2,4)\}$ $\{(1,4)(2,3)\}$ $\{(1,2)(3,4)\}$

Everything about a diagram is contained in this simple expression.

Therefore,

- We can generate all diagrams by combinatorics of contractions.
- Independent set of closed paths on a diagram are easily identified. (they are used for constructing Feynman integrals.)
- Subdiagrams relevant for UV divergence are easily identified.

$\Longrightarrow$ Automated procedure will readily be implemented.

General formalism

Evaluating a diagram:

- Amplitude

  Integration over loop momentum k_r is converted into Feynman parametric integrals over $\{z_i\}$.

$$\frac{1}{i}\Sigma_{\mathrm{G}} = (ie)^{2n}\left[\prod_{r=1}^{n}\int\frac{d^4k_r}{(2\pi)^4}\right]\gamma^{\mu_1}\frac{i}{\not{p}_1-m}\cdots\frac{i}{\not{p}_{2n-1}-m}\gamma^{\mu_{2n}}\prod_{r=1}^{n}\frac{-ig_{\mu_i\mu_j}}{{k_r}^2}$$

$$= \left(\frac{\alpha}{\pi}\right)^n\frac{1}{4^n}\Gamma(n-1)\int(dz)_G\,\mathbb{F}\,\frac{1}{U^2V^{n-1}}$$

- Subtracting divergences
  - UV divergence
  - IR divergence

Amplitude

Feynman parametric integral over 13 dimensional space:

$$\frac{1}{i}\Sigma_{\mathrm{G}} = \left(\frac{\alpha}{\pi}\right)^n \frac{1}{4^n}\Gamma(n-1)\int (dz)_G \left[\frac{F_0(B_{ij},A_j)}{U^2V^{n-1}} + \frac{F_1(B_{ij},A_j)}{U^3V^{n-2}} + \cdots\right]$$

- Integrand is expressed by B_{ij}, A_i, U, V
- Building blocks B_{ij}, A_i, U, V are homogeneous forms of Feynman parameters $\{z_i\}$.

  B_{ij}: Related to loop momenta. They are determined by the topology of diagram.

  A_i: Related to flow of external momenta. They are the currents satisfying “Kirchhoff law”.

Expressions of integrand and building blocks are obtained analytically by Computer Algebra System, FORM, Maple, *etc*.

Subtracting divergences

The original integral is divergent and must be renormalized.

- Requirements:
  - Numerical approach is taken.
    $\longrightarrow$ must be a finite value.
  - Diagram-by-diagram evaluation.
- Our strategy:
  - Intermediate renormalization scheme in 3 steps:
    (1) K-operation for UV divergence.
    (2) I-operation for IR divergence.
    (3) residual renormalization to realize on-shell renormalization.
  - Numerical point-wise subtraction.
    Prepare subtraction term as an integral defined on the same parameter space as the original

UV subtraction

- We employ Zimmermann's forest formula to subtract UV divergent parts X_f associated with forest f, and obtain finite part ΔM_G.

$$\Delta M_G = M_G - \sum_{f \in \mathfrak{F}} X_f$$

$$\Rrightarrow \int (dz)_G \left[J_G - \sum_{f \in \mathfrak{F}} \mathbb{K}_f J_G \right]$$

- We prepare UV subtraction terms $\mathbb{K}_f J_G$ in the same Feynman parameter space as the original integrand J_G, so that they cancel out singularities of J_G point-by-point.
- This setup is crucial for numerical integration.

K-operation and Forest formula

- Subtraction term $\mathbb{K}J_G$ associated with a divergent subdiagram S is obtained by *K-operation* acted on J_G,
via simple power counting in the limit:

$$z_i \sim O(\epsilon) \qquad z_i \in S$$

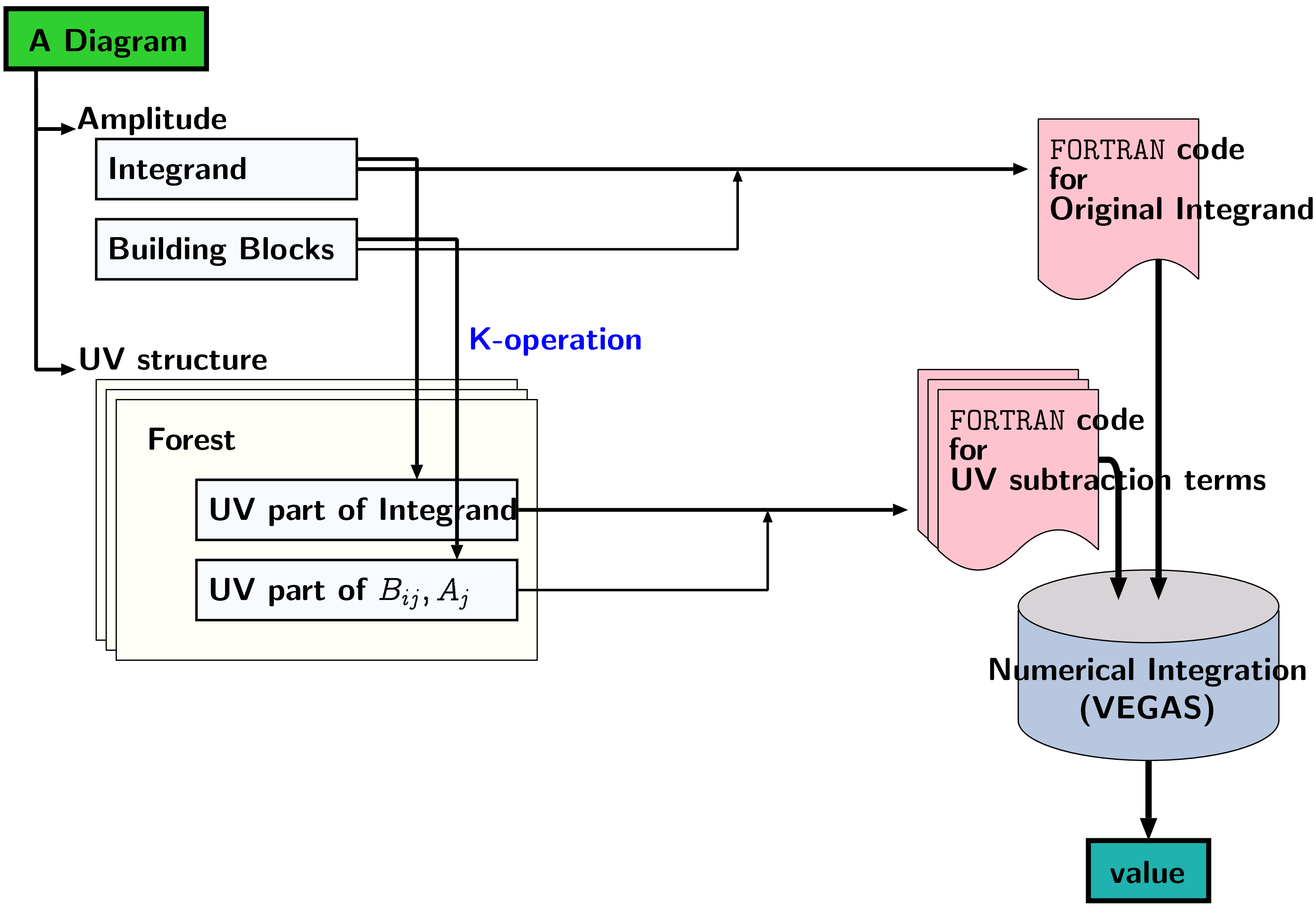
Thus UV-divergent part associated with S is extracted ($M_G^{\rm UV}$).

- By construction, $M_G^{\rm UV}$ analytically factorizes into lower order term $M_{G/S}$ exactly and counter term $\hat{L}_S$ by:

$$M_{G/S} \times \hat{L}_S$$

A forest with multiple of UV divergent subdiagrams is handled by successive operation of *K-operation*.

Automated flow

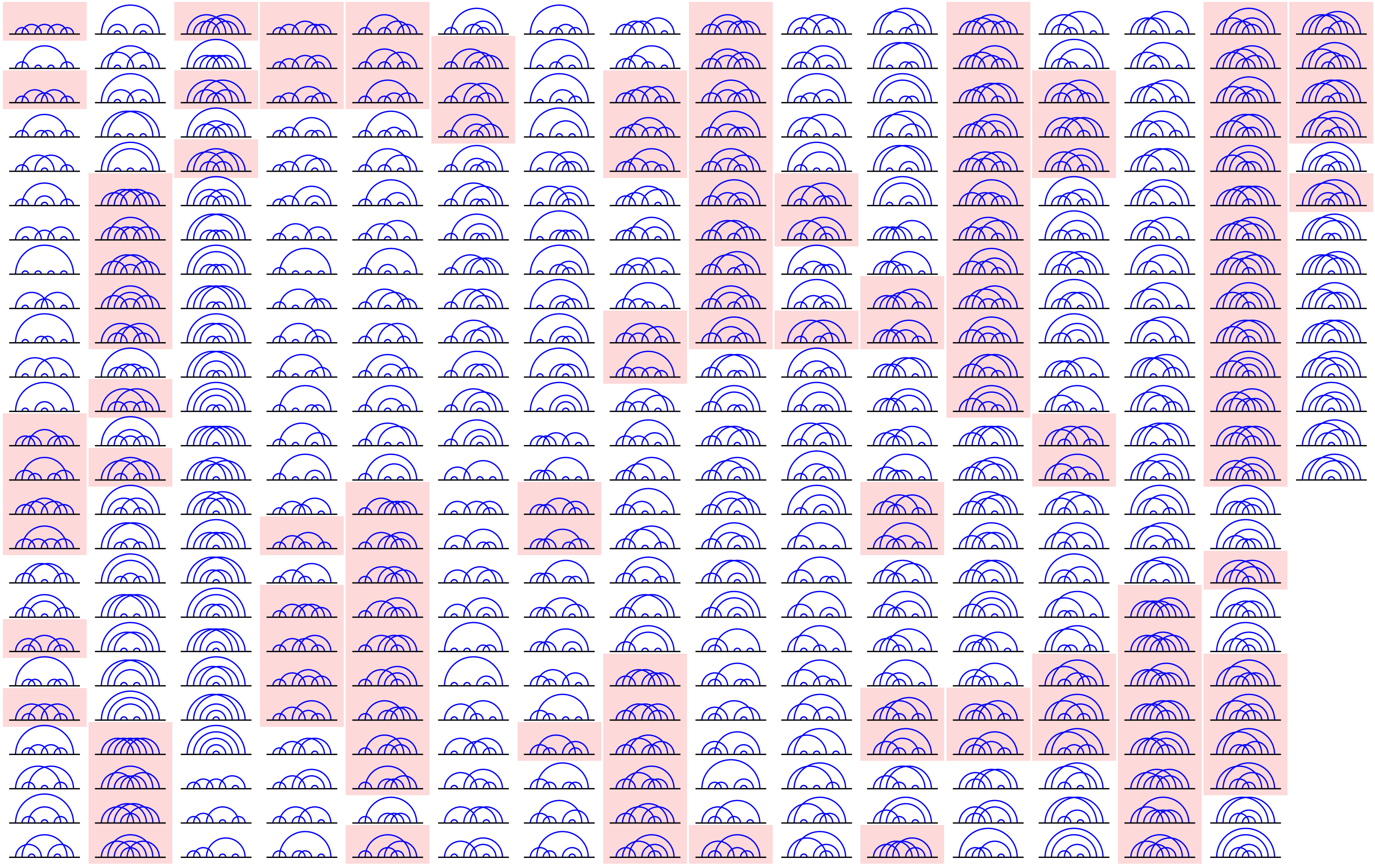


Current status

- We obtained a program to list up all the topologically distinct diagrams without lepton loops.
- Program to generate Feynman parametric integral for each diagram is obtained.
  - Integrand
  - Building blocks, B_{ij}, A_j.
- Program for vertex renormalization is obtained.
  Program including self-energy subdiagrams is almost done.
- IR subtraction and residual renormalization step are in progress.
- All the steps are applicable to arbitrary order.

- All diagrams which contain only vertex renormalization are being processed by numerical integration (2232 diagrams).

Diagrams which contains only vertex renormalization are shown below. They correspond to 2232 diagrams of set V.



Crude estimates of those diagrams are presented below, just to confirm that renormalization is working.

X001	-0.34042	(0.04943)	**X116**	1.79114	(0.00882)	**X209**	0.14436	(0.00400)	**X322**	0.91017	(0.00572)
X003	-0.11310	(0.01383)	**X117**	0.32171	(0.00533)	**X210**	0.75169	(0.00852)	**X343**	3.87608	(0.00390)
X013	-1.35322	(0.00620)	**X118**	-3.18650	(0.01925)	**X225**	0.27706	(0.01599)	**X344**	3.41470	(0.00367)
X014	0.75314	(0.02153)	**X119**	-0.12694	(0.01957)	**X231**	-0.72760	(0.00967)	**X345**	-1.00102	(0.00288)
X015	2.10198	(0.00195)	**X120**	1.74905	(0.02757)	**X232**	0.37427	(0.01842)	**X346**	0.28443	(0.00367)
X016	-0.96093	(0.00192)	**X121**	-0.86533	(0.00484)	**X235**	0.67593	(0.01684)	**X347**	-2.67776	(0.00335)
X019	1.17519	(0.02029)	**X122**	-0.74104	(0.00672)	**X259**	0.01791	(0.00639)	**X348**	-0.48587	(0.00376)
X021	-0.29674	(0.00489)	**X123**	-3.32503	(0.01328)	**X260**	-0.40509	(0.00424)	**X349**	2.08073	(0.00619)
X031	2.29316	(0.00288)	**X125**	0.73918	(0.03300)	**X265**	-0.67469	(0.00388)	**X350**	1.45479	(0.00230)
X032	-0.24265	(0.00127)	**X127**	1.13048	(0.00985)	**X266**	0.11937	(0.00592)	**X351**	0.24490	(0.00340)
X033	-1.37714	(0.00143)	**X128**	0.57693	(0.02218)	**X271**	0.24188	(0.00872)	**X352**	-0.13189	(0.00252)
X034	1.25388	(0.00205)	**X129**	1.41734	(0.02232)	**X272**	-0.73345	(0.01469)	**X353**	0.18836	(0.00252)
X035	-0.58384	(0.00142)	**X165**	-2.10910	(0.01990)	**X275**	-0.74340	(0.00445)	**X354**	-2.03177	(0.00439)
X037	-0.74165	(0.00199)	**X166**	-2.27775	(0.02188)	**X276**	-0.55445	(0.00283)	**X355**	-1.05668	(0.00554)
X039	0.31638	(0.00441)	**X172**	1.36015	(0.03942)	**X277**	2.77770	(0.00265)	**X356**	2.06867	(0.00617)
X047	-4.45507	(0.00326)	**X178**	0.70338	(0.00485)	**X278**	-0.14964	(0.00737)	**X357**	0.36337	(0.00367)
X048	-0.80512	(0.00160)	**X179**	-0.43781	(0.00341)	**X279**	0.82134	(0.00439)	**X358**	0.03325	(0.00425)
X049	-0.02951	(0.00133)	**X180**	0.02543	(0.00567)	**X280**	-1.00961	(0.00464)	**X359**	-0.15207	(0.00467)
X050	-1.22223	(0.00176)	**X185**	-0.13128	(0.00497)	**X281**	-1.37236	(0.00407)	**X360**	-0.47233	(0.00563)
X051	-0.17333	(0.00202)	**X186**	1.14242	(0.00878)	**X282**	0.48596	(0.00385)	**X361**	2.52071	(0.01084)
X053	0.36460	(0.00153)	**X195**	-1.06649	(0.00450)	**X283**	-0.05080	(0.00561)	**X362**	-0.56599	(0.00358)
X055	-0.36339	(0.00142)	**X196**	-2.03753	(0.00288)	**X284**	-0.27114	(0.00320)	**X363**	-2.34078	(0.00262)
X076	-5.19446	(0.03379)	**X197**	-0.38704	(0.00222)	**X285**	0.01690	(0.00389)	**X364**	2.38344	(0.00337)
X077	3.18404	(0.06924)	**X198**	-2.33747	(0.00442)	**X286**	0.76614	(0.00587)	**X367**	-0.71804	(0.00490)
X078	0.82179	(0.07104)	**X199**	1.04594	(0.00455)	**X287**	0.17755	(0.01168)	**X370**	-1.47907	(0.00453)
X091	-1.85164	(0.07314)	**X200**	0.00793	(0.00703)	**X296**	0.54479	(0.00457)	**X371**	-0.00744	(0.00415)
X093	-1.75719	(0.00771)	**X201**	-0.48774	(0.00369)	**X297**	-0.47919	(0.00468)	**X372**	-1.28486	(0.00428)
X094	-1.05792	(0.01610)	**X202**	1.92431	(0.00297)	**X303**	0.32133	(0.00246)	**X373**	0.55778	(0.00697)
X095	0.57719	(0.00717)	**X203**	0.90371	(0.00233)	**X304**	-0.34223	(0.00489)	**X376**	1.03581	(0.00341)
X096	1.24779	(0.02784)	**X204**	-1.91907	(0.00671)	**X305**	0.46192	(0.00397)	**X377**	0.41220	(0.00524)
X101	-0.26275	(0.01629)	**X205**	-0.90380	(0.00489)	**X313**	0.94419	(0.00713)	**X378**	1.29109	(0.00583)
X102	-1.43773	(0.05228)	**X206**	1.62847	(0.01119)	**X314**	0.78814	(0.01293)	**X379**	-0.35067	(0.00901)
X103	0.76540	(0.03423)	**X207**	0.28937	(0.00418)	**X320**	0.55630	(0.00518)	**X381**	1.06166	(0.00659)
X115	-0.59498	(0.01112)	**X208**	0.52057	(0.00524)	**X321**	-0.92478	(0.01276)			

Concluding remarks

- Numerical estimates:
  - Typical integral is composed of $80,000$ lines of FORTRAN code.
  - Rough estimates of time-scale for each diagram:
    - 10 - 20 min. for code generation
    - 10^6 sampling points $\times$ 20 iterations take 5 - 7 hours on 32 CPU PC cluster
  - To evaluate within a few percent of accuracy, it will take:
    - a year for set V diagrams.
    - 2 - 3 years for full $A_1^{(10)}$ contributions.
- Theoretical issues:
  - To complete the automated procedure to include IR subtraction, and residual renormalization.
  - To extend to general diagrams with lepton loops.