Two-Loop Bhabha Scattering in QED

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Plan of the Talk

- Introduction
- Laporta Algorithm and Differential Equations Method
- Differential Cross-Section: Virtual Contribution + Soft-Photon Emission
  - $N_F = 1$

    R. B., A. Ferroglia, P. Mastrolia, E. Remiddi, and J. J. van der Bij, '04-'05
  - Vertex corrections

    R. B. and A. Ferroglia, '05
- Some Numerics
- Summary

Why Bhabha Scattering?

- Bhabha Scattering is a fundamental process for e^+e^- collider physics because it is chosen for the precise evaluation of the Luminosity:

$$L = \frac{N}{\sigma_{th}}$$

  where N is the measured number of events and σ_{th} is the theoretical cross section.
  L enters as a normalization factor in the cross section measurements $\Rightarrow$ a process in which δL is as small as possible is required.

- It is a process with a large cross section and it is QED-dominated $\Rightarrow$ precise experimental measurements and precise theoretical calculation of the cross section.

MOREOVER:

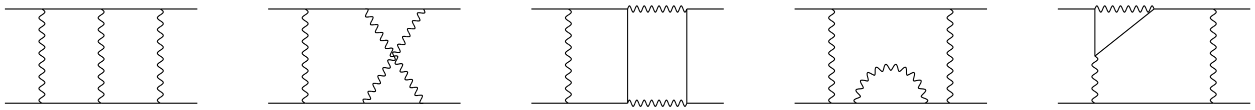
- Precision required by experiments (DaΦne, PEP-II, KEKB, BEPC, VEPP-2M and also ILC) $10^{-3} - 10^{-4}$ $\Rightarrow$ evaluation of the NNLO corrections as one of the ingredients to match the precision with a Monte Carlo event generator

Bhabha Scattering Cross Section in the Literature

- One-loop corrections:
  - QED and EW corrections (Consoli '79, Böhm-Denner-Hollik '88, Greco '88)
- Two-loop QED corrections:
  - Log-enhanced corrections (virtual and real) for the small- and large-angle Bhabha scattering (Faldt-Osland '94, Arbuzov-Fadin-Kuraev-Lipatov-Merenkov-Trentadue '95-'97)
  - Virtual corrections to the cross section with $m = 0$ (Bern-Dixon-Ghinkulov '00)
  - Log-enhanced photonic contributions (Glover-Tausk-van der Bij '01)
  - $N_F = 1$ set of corrections (virtual + soft-photon) with $m \neq 0$ (B.-Ferroglia-Mastrolia-Remiddi-van der Bij '04-'05)
  - Reduction to the master integrals for the massive QED (Czakon-Gluza-Riemann '04)
  - Constant term of photonic corrections to large-angle Bhabha scattering not suppressed by the ratio m^2/s (Penin '05)
  - Photonic vertex corrections with $m \neq 0$ (B.-Ferroglia '05)
- Two-loop EW corrections:
  - Log-enhanced corrections in the annihilation channel (Fadin-Lipatov-Martin-Melles '00, Jantzen-Kühn-Moch-Penin-Smirnov '01-'05)

Missing parts

- Approximated ($m^2/s \to 0$) results:
  - KNOWN: Full QED photonic corrections (Faldt-Osland, Arbuzov-Fadin-Kuraev-Lipatov-Merenkov-Trentadue, Glover-Tausk-van der Bij, Penin), the full $N_F = 1$ virtual and soft-photon corrections and the log terms of the soft-pair production (Arbuzov-Fadin-Kuraev-Lipatov-Merenkov-Trentadue, B.-Ferroglia-Mastrolia-Remiddi-van der Bij)
  - ... but is still missing the constant term (i.e. not suppressed by the ratio m^2/s) of the soft-pair production and a complete evaluation of the EW sector
- Exact ($m^2 = 0$ and $m^2 \neq 0$) results:
  - KNOWN: Reduction to the MIs and MIs for the massless case (Smirnov , Tausk , Anastasiou-Glover-Oleari, Gehrmann-Remiddi, Bern-Dixon-Ghinkulov), reduction to the MIs, MIs for the Vertex corrections, some MIs for the box corrections (B.-Ferroglia-Mastrolia-Remiddi-van der Bij, Czakon-Gluza-Riemann, Smirnov-Heinrich), for the massive case,
  - ... but still missing the full set of MIs for the two-loop massive photonic boxes:



$\longrightarrow$ see Riemann's talk

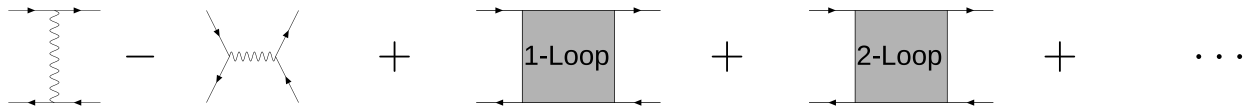
Differential Cross Section

We calculate the differential Cross Section:

$$\frac{d\sigma}{d\Omega} \sim \|\mathcal{M}\|^2$$

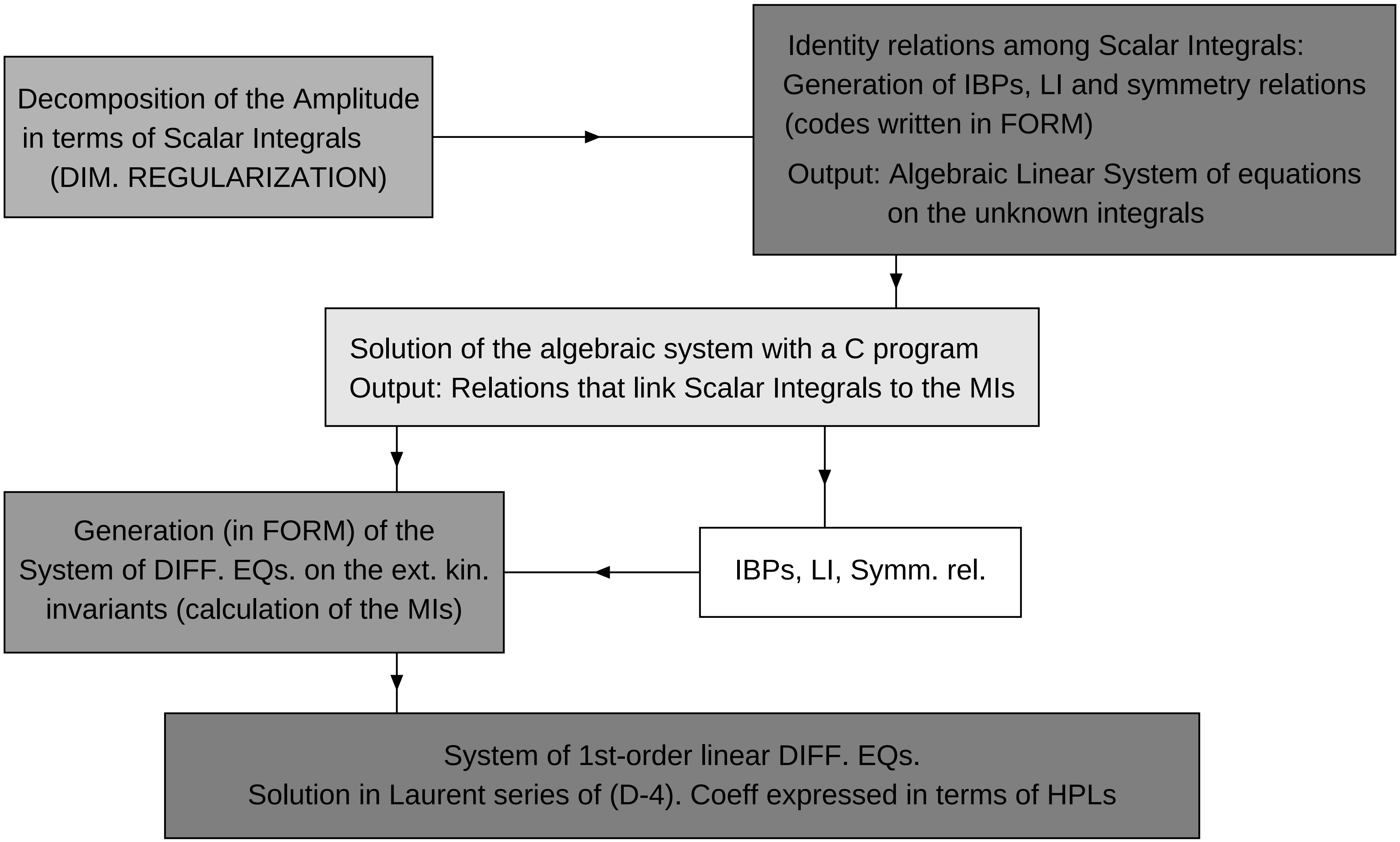
where:

$\mathcal{M} =$

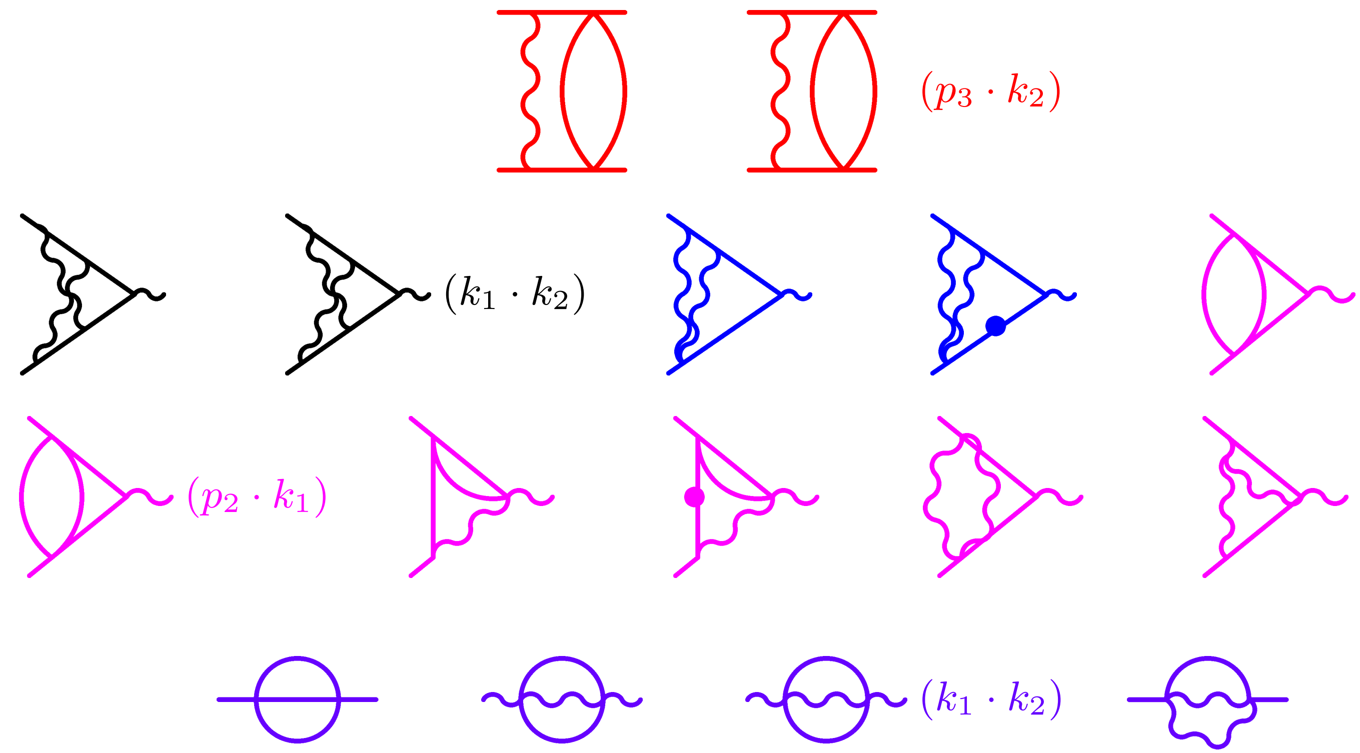


- We evaluate directly the squared matrix element. It appears as a combination of a huge number of scalar integrals, regularized in Dimensional Regularization (the Dirac algebra in performed in D dimensions as well)
- The scalar integrals are evaluated with the Laporta algorithm for their reduction to the MIs and the Differential Equations method for the computation of the MIs

Laporta Algorithm and Differential Equations



Two-Loop Master Integrals



R. B., P. Mastrolia and E. Remiddi, *Nucl. Phys.* **B661** (2003) 289.
R. B., A. Ferroglia, P. Mastrolia, E. Remiddi, and J. van der Bij, *Nucl. Phys.* **B681** (2004) 261.
M. Czakon, J. Gluza and T. Riemann, *Nucl. Phys. Proc. Suppl.* **135** (2004) 83.

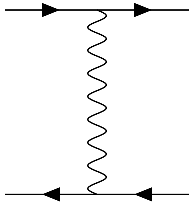
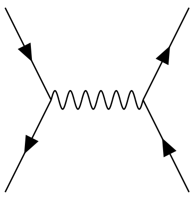
Differential Cross-Section:

Virtual Contribution

The Differential Cross-Section can be expanded in series of α/π as follows:

$$\frac{d\sigma(s,t,m^2)}{d\Omega} = \frac{d\sigma_0(s,t,m^2)}{d\Omega} + \left(\frac{\alpha}{\pi}\right)\frac{d\sigma_1(s,t,m^2)}{d\Omega} + \left(\frac{\alpha}{\pi}\right)^2 \frac{d\sigma_2(s,t,m^2)}{d\Omega} + \mathcal{O}\left(\left(\frac{\alpha}{\pi}\right)^3\right)$$

The Tree-Level:

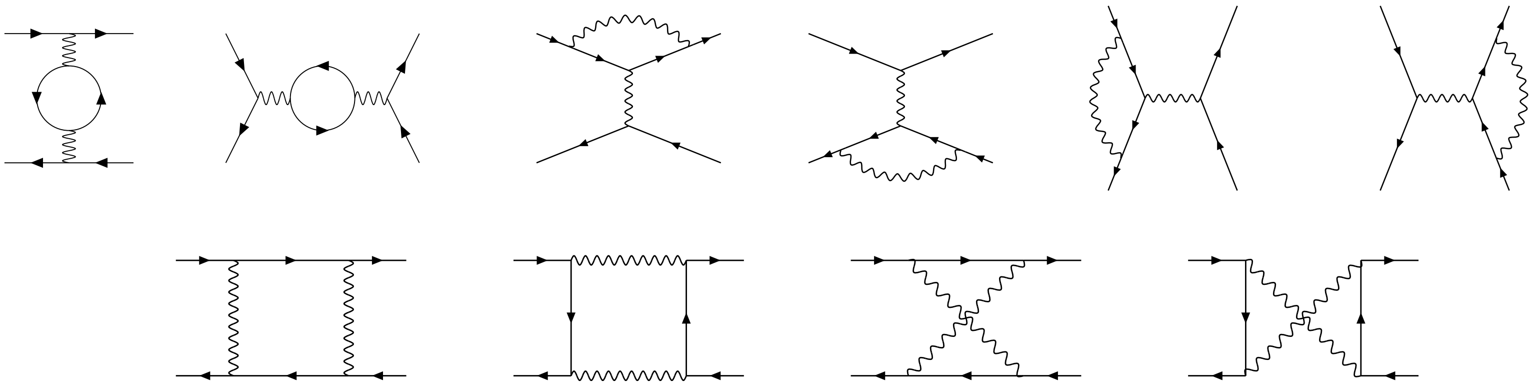
$\mathcal{M}_0 =$  $-$ 

Performing the traces over the Dirac indices of $\|\mathcal{M}_0\|^2$ in D dimensions we have

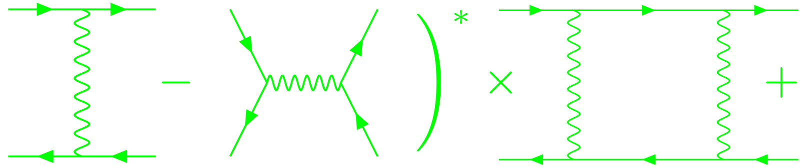
$$\begin{aligned}\frac{d\sigma_0(s,t,m^2)}{d\Omega} = \frac{\alpha^2}{s}\Bigg\{ & \frac{1}{s^2}\left[st + \frac{s^2}{2} + (t-2m^2)^2\right] + \frac{1}{t^2}\left[st + \frac{t^2}{2} + (s-2m^2)^2\right] \\ & + \frac{1}{st}\left[(s+t)^2 - 4m^4\right] \\ & + (D-4)\left\{\frac{1}{s^2}\left[\frac{s^2}{4}\right] + \frac{1}{t^2}\left[\frac{t^2}{4}\right] + \frac{1}{st}\left[\frac{1}{2}(s+t)^2 - \frac{1}{2}st - m^2(s+t)\right]\right\} \\ & + (D-4)^2\left\{\frac{1}{st}\left[-\frac{st}{4}\right]\right\} \Bigg\}\end{aligned}$$

$$\frac{d\sigma(s,t,m^2)}{d\Omega} = \frac{d\sigma_0(s,t,m^2)}{d\Omega} + \left(\frac{\alpha}{\pi}\right)\frac{d\sigma_1(s,t,m^2)}{d\Omega} + \left(\frac{\alpha}{\pi}\right)^2\frac{d\sigma_2(s,t,m^2)}{d\Omega} + \mathcal{O}\left(\left(\frac{\alpha}{\pi}\right)^3\right)$$

At one loop the following virtual diagrams contribute:

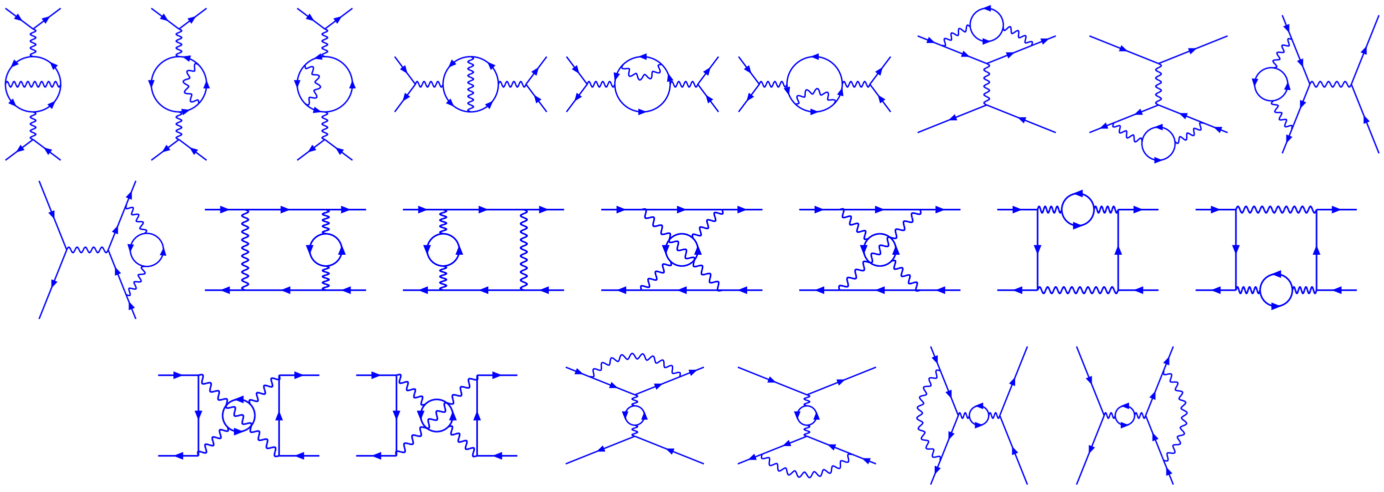


$$\left(\frac{\alpha}{\pi}\right)\frac{d\sigma_1^V(s,t,m^2)}{d\Omega} = \frac{s}{16}\sum_{\text{spin}}\Bigg\{ \Big(\quad - \quad \Big)^* \times \quad + \text{c.c.} + \cdots \Bigg\}$$

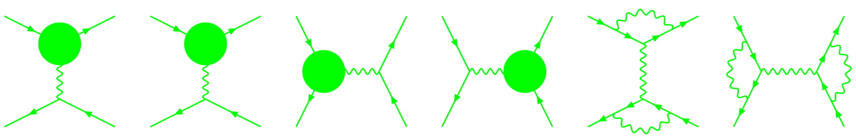


The $\mathcal{O}(\alpha^4)$ contributions that we consider come from the following Two-Loop diagrams:

$N_F = 1$



Vertices

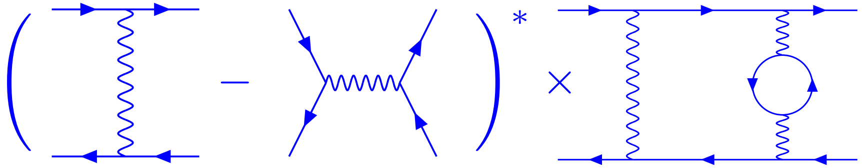


as well as from the product of one-loop diagrams.

$$\frac{d\sigma(s,t,m^2)}{d\Omega} = \frac{d\sigma_0(s,t,m^2)}{d\Omega} + \left(\frac{\alpha}{\pi}\right)\frac{d\sigma_1(s,t,m^2)}{d\Omega} + \left(\frac{\alpha}{\pi}\right)^2 \frac{d\sigma_2(s,t,m^2)}{d\Omega} + \mathcal{O}\left(\left(\frac{\alpha}{\pi}\right)^3\right)$$

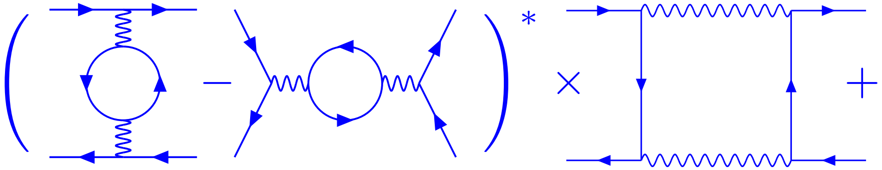
where

$$\left(\frac{\alpha}{\pi}\right)^2 \frac{d\sigma_2^V(s,t,m^2)}{d\Omega} = \frac{s}{16}\sum_{\text{spin}}\Bigg\{$$



$$+ \text{c.c.}$$

$$+$$



$$+ \text{c.c.} + \cdots \Bigg\}$$

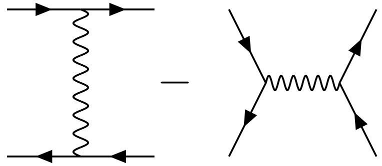
Differential Cross-Section:

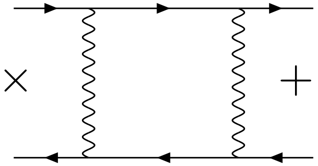
Real Soft-Photon Emission

One-Loop

As an example consider

$$\left(\frac{\alpha}{\pi}\right)\frac{d\sigma_1^V(s,t,m^2)}{d\Omega} = \frac{s}{16}\sum_{\text{spin}}\Big\{\Big(\quad - \quad\Big)^{*} \times \quad + \text{c.c.}\Big\}$$





$$= \frac{\alpha^3}{4\pi s}\left[\frac{m^2}{s}\text{Re}B_1^{(1l)}(s,t) + \frac{m^2}{t}\text{Re}B_2^{(1l)}(s,t)\right]$$

where

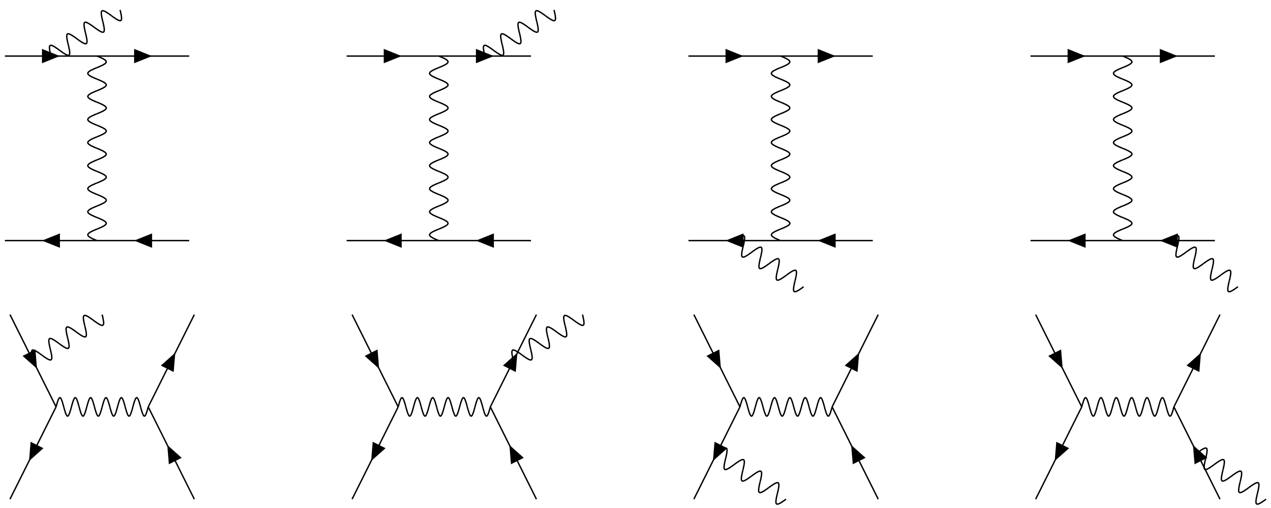
$$B_i^{(1l)}(s,t) = \frac{1}{(D-4)}B_i^{(1l,-1)}(s,t) + B_i^{(1l,0)}(s,t) + \mathcal{O}\Big((D-4)\Big)$$

and the IR pole is

$$B_1^{(1l,-1)}(-P^2,-Q^2) = \Bigg(-48 - \frac{8}{x^2(1-y)^2} + \frac{8}{x^2(1-y)} + \frac{32}{x(1-y)^2} - \frac{32}{x(1-y)} - \frac{16}{x} - \frac{32x}{(1-y)^2} + \frac{32x}{(1-y)} + 16x + \frac{8x^2}{(1-y)^2}$$
$$- \frac{8x^2}{(1-y)} - \frac{8}{y(1+x)} - \frac{8}{y(1-x)} + \frac{8}{y} - \frac{8y}{(1+x)} - \frac{8y}{(1-x)} + 8y - \frac{96}{(1+x)(1-y)^2} + \frac{96}{(1+x)(1-y)} + \frac{80}{(1+x)}$$
$$+ \frac{32}{(1-x)(1-y)^2} - \frac{32}{(1-x)(1-y)} + \frac{16}{(1-x)} + \frac{32}{(1-y)^2} - \frac{32}{(1-y)}\Bigg)H(0;x)$$

In order to get a finite differential cross-section we have to add the real soft-photon emission, which cancels the IR poles coming from the virtual cross-section.

At $\mathcal{O}(\alpha^3)$ the diagrams contributing are the following



Due to the iconalization of the vertex, the real emission contribution at $\mathcal{O}(\alpha^3)$ is given by

$$\left(\frac{\alpha}{\pi}\right) \frac{d\sigma_1^S(s,t,m^2)}{d\Omega} = \left(\frac{\alpha}{\pi}\right) \frac{d\sigma_0^D(s,t,m^2)}{d\Omega} \sum_{i,j=1}^{4} J_{ij}$$

with

$$J_{ij} = \epsilon_i \epsilon_j \, (p_i \cdot p_j) \, I_{ij},$$

$$\epsilon_i = +1 \text{ for } i = 1,4 \text{ and } \epsilon_i = -1 \text{ for } i = 2,3$$

and

$$I_{ij} = \frac{1}{\Gamma\left(3 - \frac{D}{2}\right) \pi^{(D-4)/2}} \frac{m^{D-4}}{4\pi^2} \int^{\omega} \frac{d^D k}{k_0} \frac{1}{(p_i \cdot k)(p_j \cdot k)}$$

- ω is the cut-off on the unobserved photon energy
- because $J_{ij} = J_{ji}$ we have in fact:

$$\left(\frac{\alpha}{\pi}\right) \frac{d\sigma_1^S(s,t,m^2)}{d\Omega} = \left(\frac{\alpha}{\pi}\right) \frac{d\sigma_0^D(s,t,m^2)}{d\Omega} \, 4 \sum_{j=1}^{4} J_{1j}$$

where $d\sigma_0^D(s,t,m^2)/d\Omega$ is the Born cross-section exact in D.

$$I_{1j} = \frac{\rho_{1j}}{2} \left[\left(\frac{1}{D-4} + \frac{1}{2} \ln \left(\frac{\omega^2}{m^2} \right) + \ln 2 \right) I_{1j}^{(0)} - \Delta I_{1j} \right] + \mathcal{O}\left(D-4\right)$$

$$\rho_{11} = 1 \, , \quad \rho_{12} = \frac{1}{x} \, , \quad \rho_{13} = \frac{1}{y} \, , \quad \rho_{14} = \frac{1}{z}$$

$$x = \frac{\sqrt{s} - \sqrt{s-4m^2}}{\sqrt{s} + \sqrt{s-4m^2}} \, , \quad y = \frac{\sqrt{4m^2-t} - \sqrt{-t}}{\sqrt{4m^2-t} + \sqrt{-t}} \, , \quad z = \frac{\sqrt{4m^2-u} - \sqrt{-u}}{\sqrt{4m^2-u} + \sqrt{-u}}$$

$$I_{1j}^{(0)} = \frac{2}{m^2} \int_0^1 dr \left[1 - 2r \left(1 + \rho_{1j} \frac{p_1 \cdot p_j}{m^2} \right) \right]^{-1}$$

$$\Delta I_{1j} = \int_0^1 dr \frac{1}{(P_{1j})^2} \frac{(P_{1j})_0}{|\vec{P_{1j}}|} \ln \frac{(P_{1j})_0 - |\vec{P_{1j}}|}{(P_{1j})_0 + |\vec{P_{1j}}|}$$

$$P_{1j}^{\mu} = p_j^{\mu} + r \left(\rho_{1j} p_1^{\mu} - p_j^{\mu} \right)$$

$$I_{11}^{(0)} = \frac{2}{m^2}\,, \qquad I_{12}^{(0)} = -\frac{4}{m^2}\frac{x^2}{1-x^2}\ln x$$

$$I_{13}^{(0)} = -\frac{4}{m^2}\frac{y^2}{1-y^2}\ln y\,, \qquad I_{14}^{(0)} = -\frac{4}{m^2}\frac{z^2}{1-z^2}\ln z$$

$$\Delta I_{11} = -\frac{1}{m^2}\frac{1+x}{1-x}\ln x$$

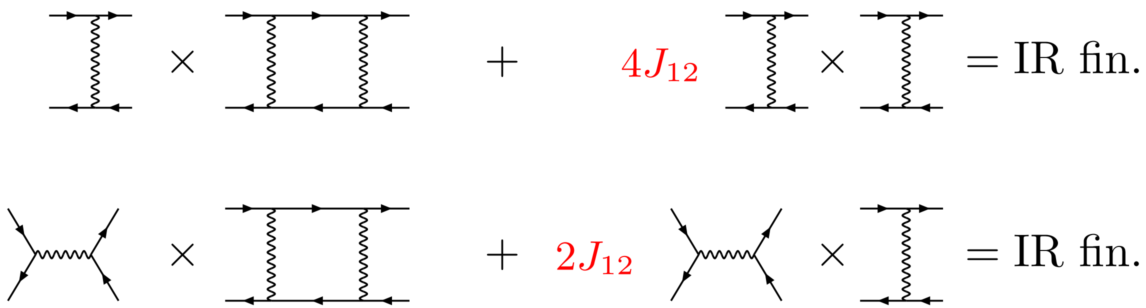
$$\Delta I_{1l} = -\frac{2}{m^2(\rho_{1l}^2-1)}\Big[\mathrm{Li}_2(a_l^{(1)}) + \mathrm{Li}_2(a_l^{(2)}) - \mathrm{Li}_2(a_l^{(3)}) - \mathrm{Li}_2(a_l^{(4)})\Big]$$

$$a_l^{(1)} = \frac{1-x\rho_{1l}}{1+\rho_{1l}}\,, \quad a_l^{(2)} = \frac{x-\rho_{1l}}{x(1+\rho_{1l})}\,, \quad a_l^{(3)} = \frac{\rho_{1l}-x}{1+\rho_{1l}}\,, \quad a_l^{(4)} = -\frac{1-x\rho_{1l}}{x(1+\rho_{1l})}$$

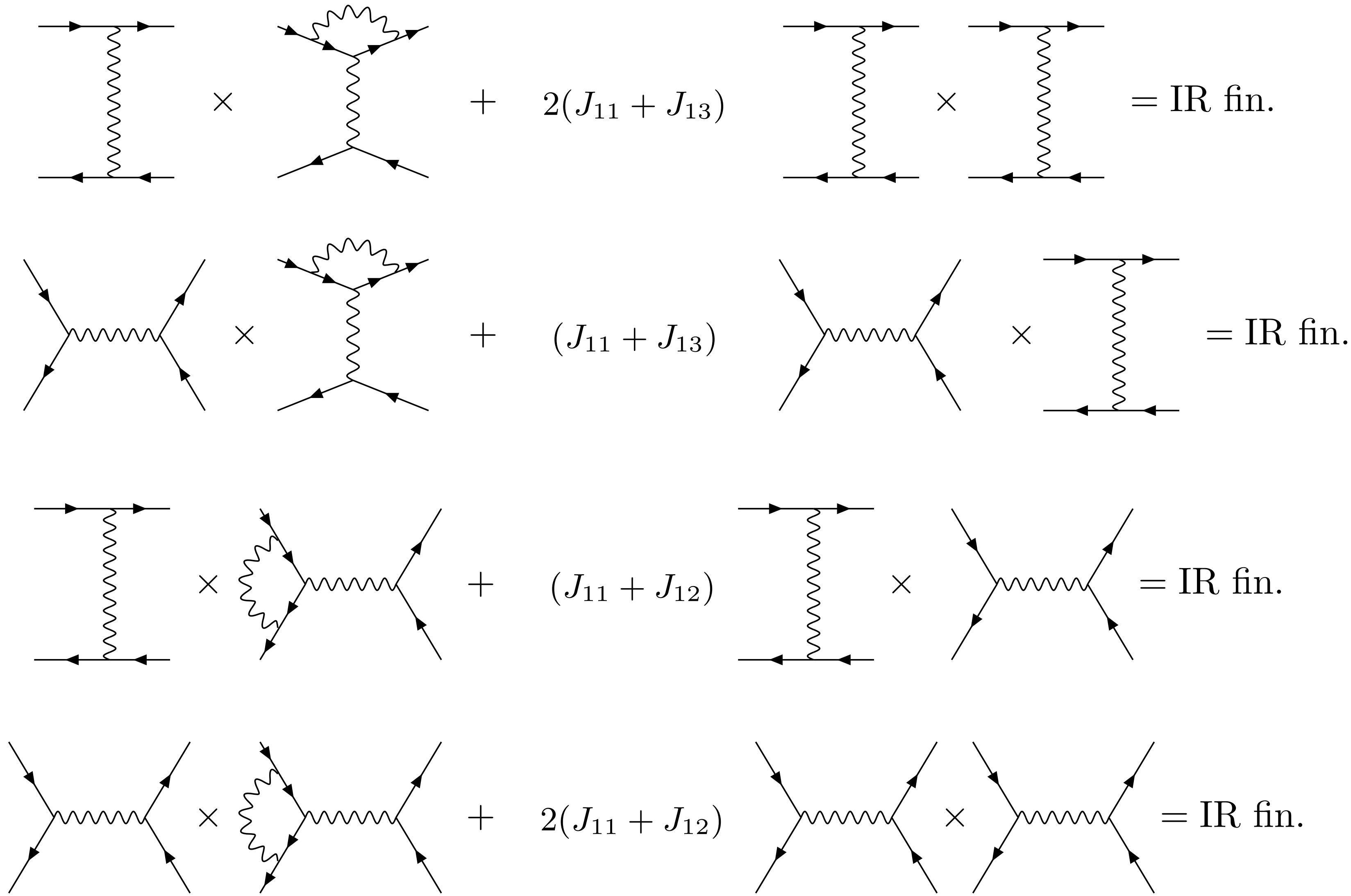
Finally, the contribution UV-renormalized and IR finite at $\mathcal{O}(\alpha^3)$ is given by

$$
\begin{aligned}
\left(\frac{\alpha}{\pi}\right) \frac{d\sigma_1(s,t,m^2)}{d\Omega} &= \left(\frac{\alpha}{\pi}\right) \left(\frac{d\sigma_1^V(s,t,m^2)}{d\Omega} + \frac{d\sigma_1^S(s,t,m^2)}{d\Omega} \right) \\
&= \left(\frac{\alpha}{\pi}\right) \left(\frac{d\sigma_1^V(s,t,m^2)}{d\Omega} + \frac{d\sigma_0^D(s,t,m^2)}{d\Omega} \, 4 \sum_{j=1}^{4} J_{1j} \right)
\end{aligned}
$$

In particular, for our example we find



Note the fact that we need the $\mathcal{O}(D-4)$ of the Born cross-section in order to get its correct finite part!


×
+ $2(J_{11} + J_{13})$
×
= IR fin.
×
+ $(J_{11} + J_{13})$
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= IR fin.
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+ $(J_{11} + J_{12})$
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= IR fin.
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+ $2(J_{11} + J_{12})$
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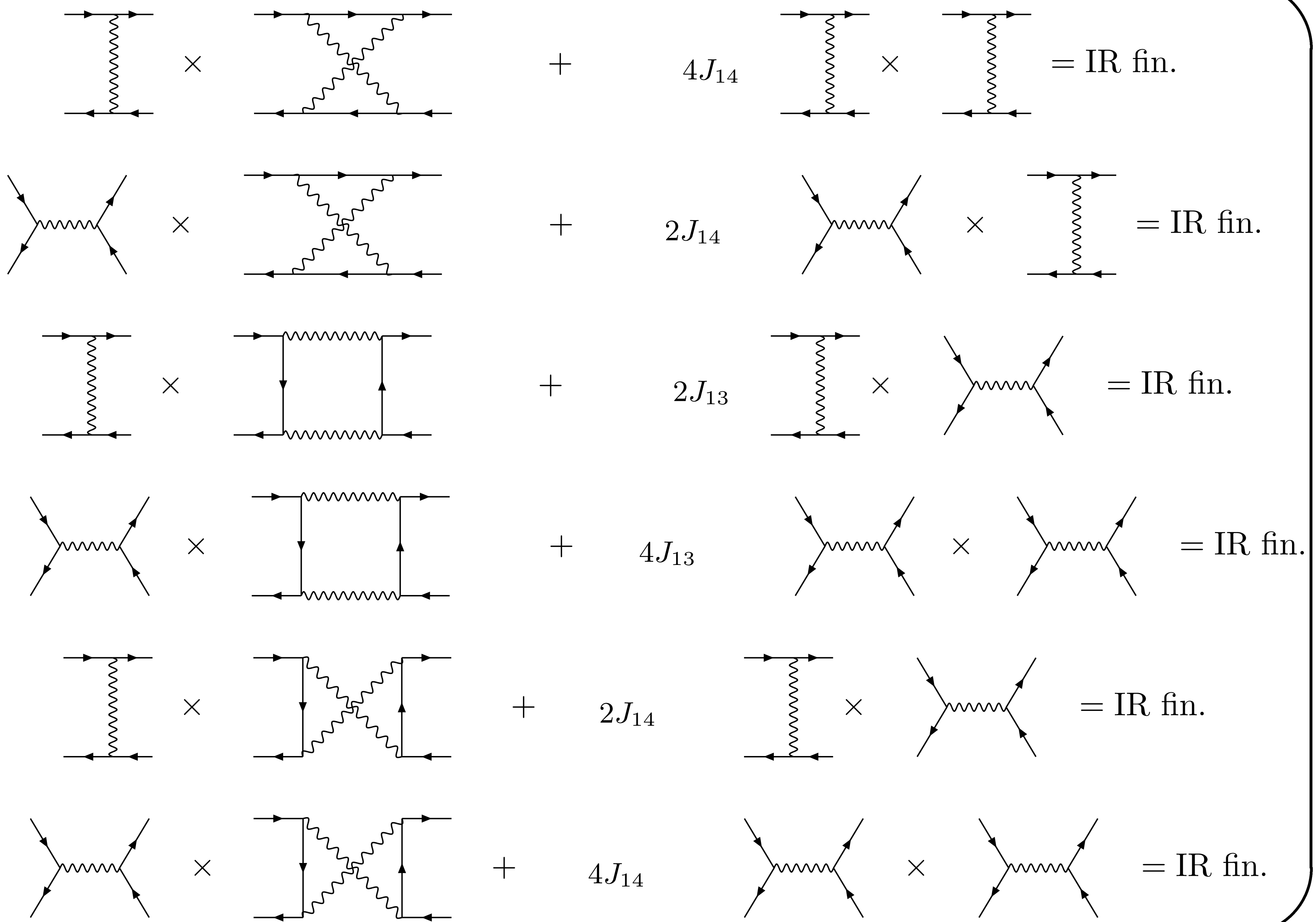
= IR fin.


× + $4J_{14}$ × = IR fin.
× + $2J_{14}$ × = IR fin.
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× + $4J_{13}$ × = IR fin.
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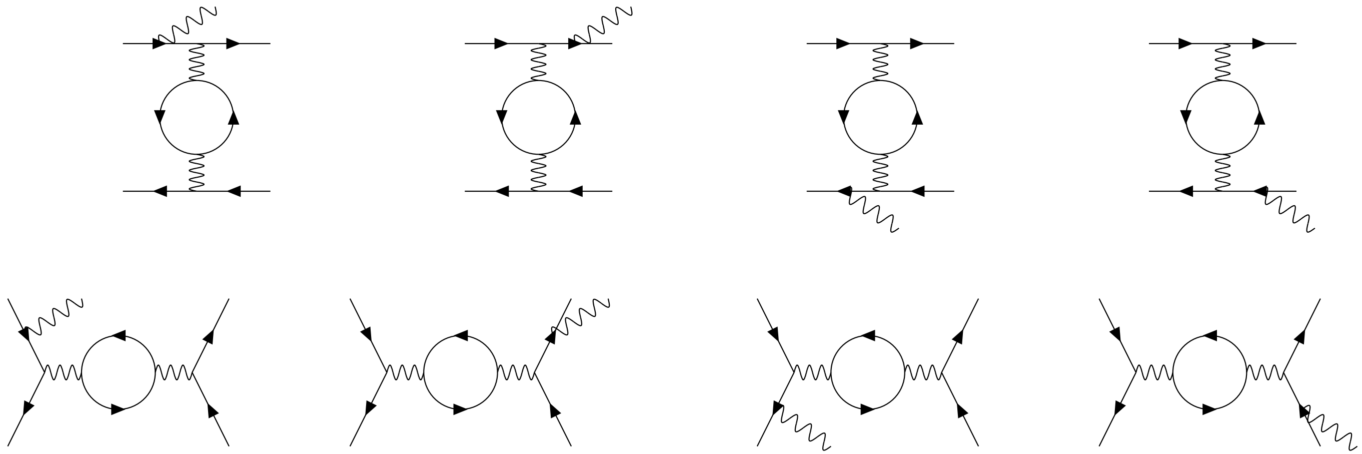
Real photon emission for the $\mathcal{O}(\alpha^4 N_F = 1)$



$$\left(\frac{\alpha}{\pi}\right)^2 \frac{d\sigma_2^S(s,t,m^2)}{d\Omega}\bigg|_{N_F=1} = \left(\frac{\alpha}{\pi}\right)^2 \frac{d\sigma_1^D(s,t,m^2)}{d\Omega}\bigg|_{N_F=1} 4 \sum_{j=1}^{4} J_{1j}$$

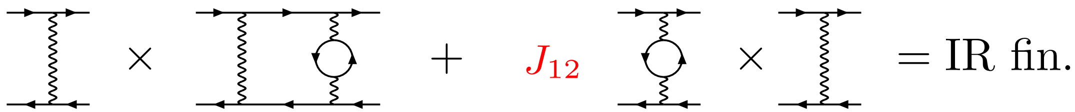
- $d\sigma_1^D(s,t,m^2)/d\Omega|_{N_F=1}$ is the one-loop $N_F = 1$ cross-section as a series in $(D-4)$.
- Note the fact that we need the $\mathcal{O}(D-4)$ of the $N_F = 1$ cross-section in order to get the correct finite part of the real emission, but again we DO NOT need the $\mathcal{O}(D-4)$ of the integrals J_{1j}, because $d\sigma_1^D(s,t,m^2)/d\Omega|_{N_F=1}$ is finite!

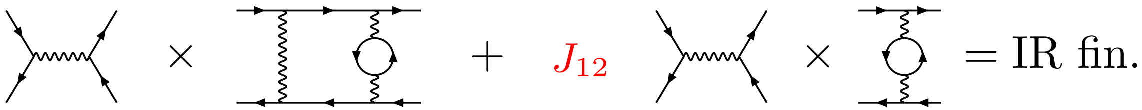
Finally, the contribution UV-renormalized and IR finite at $\mathcal{O}(\alpha^4 N_F = 1)$ is given by

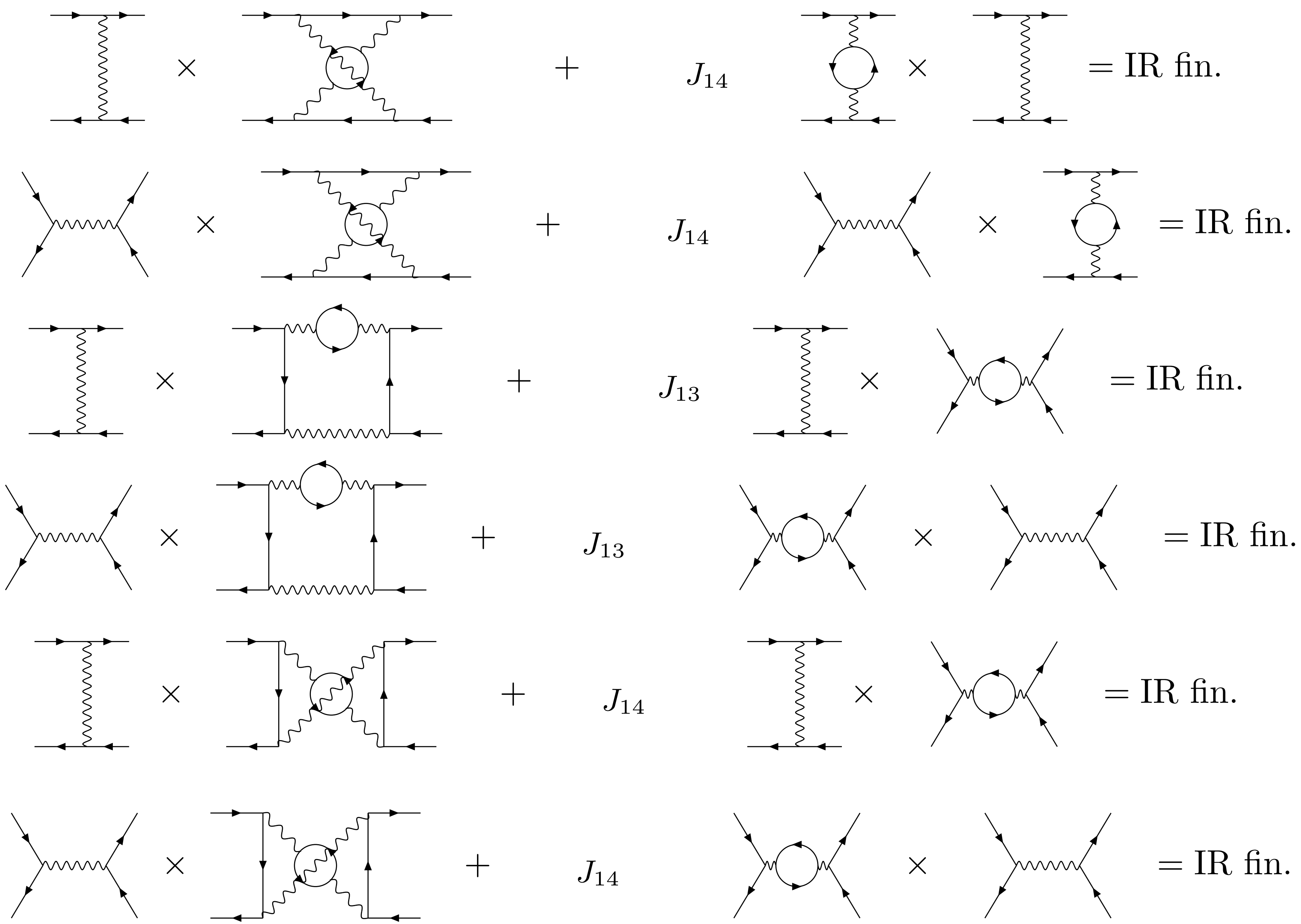
$$\left(\frac{\alpha}{\pi}\right) \frac{d\sigma_2(s,t,m^2)}{d\Omega}\bigg|_{N_F=1} = \left(\frac{\alpha}{\pi}\right)\left(\frac{d\sigma_2^V(s,t,m^2)}{d\Omega}\bigg|_{N_F=1} + \frac{d\sigma_1^S(s,t,m^2)}{d\Omega}\bigg|_{N_F=1} \right)$$

$$= \left(\frac{\alpha}{\pi}\right)\left(\frac{d\sigma_2^V(s,t,m^2)}{d\Omega}\bigg|_{N_F=1} + \frac{d\sigma_1^D(s,t,m^2)}{d\Omega}\bigg|_{N_F=1} 4\sum_{j=1}^{4} J_{1j} \right)$$

In particular, for a two-loop box we find






× + J_{14} × = IR fin.
× + J_{14} × = IR fin.
× + J_{13} × = IR fin.
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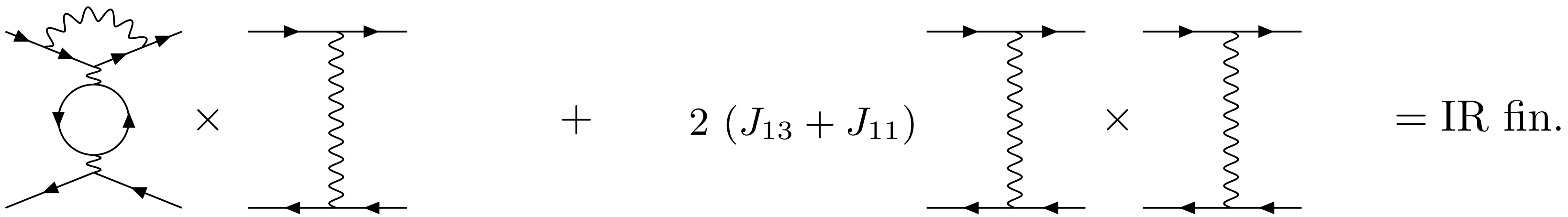
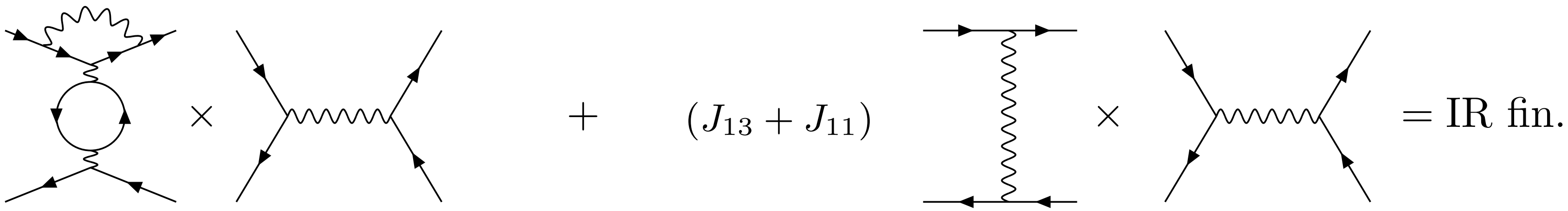
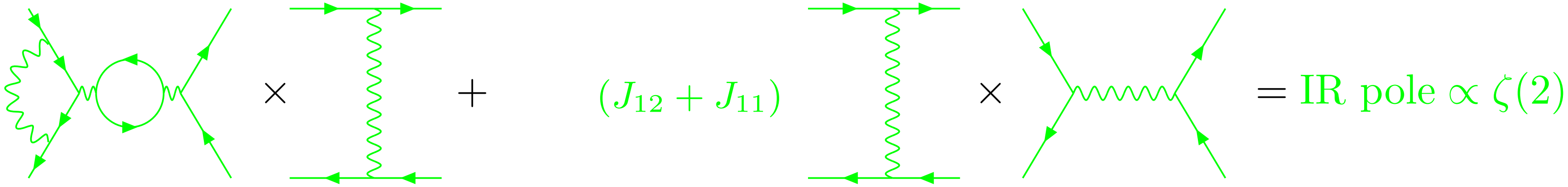
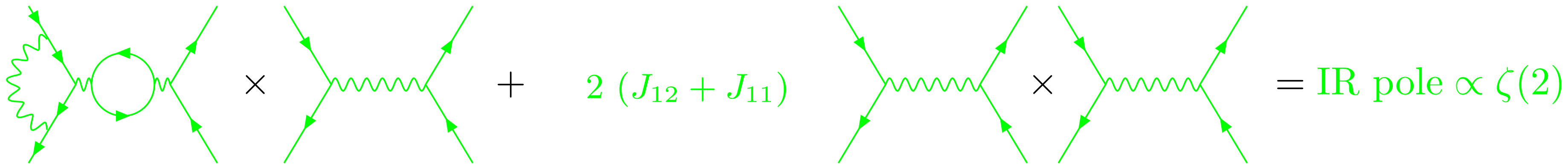
Reducible Diagrams $\mathcal{O}(\alpha^4 N_F = 1)$

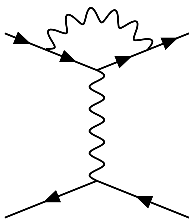
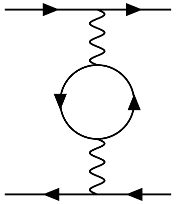
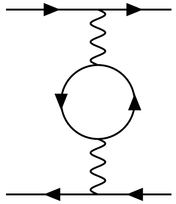
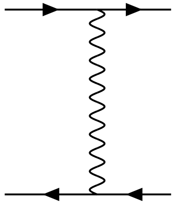


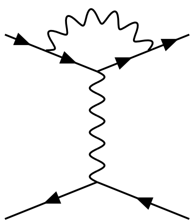
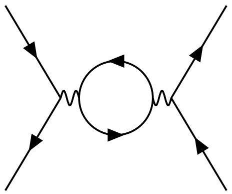
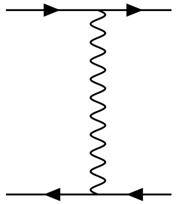
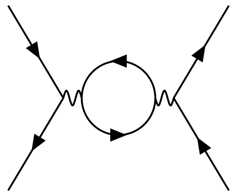


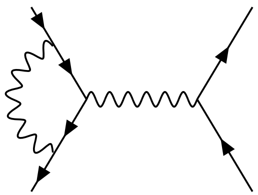
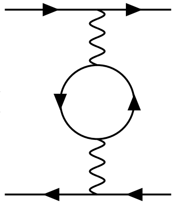
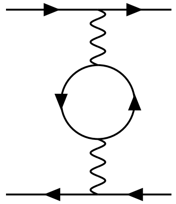
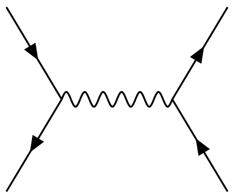


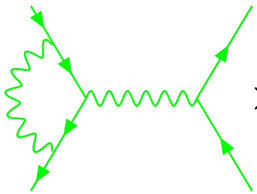
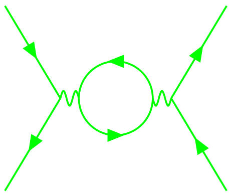
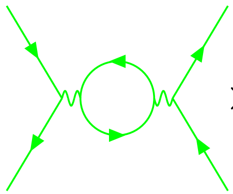
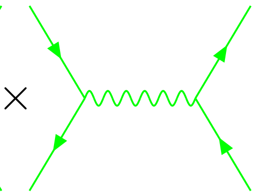


One-Loop Vertices times One-Loop $\mathcal{O}(\alpha^4 N_F = 1)$

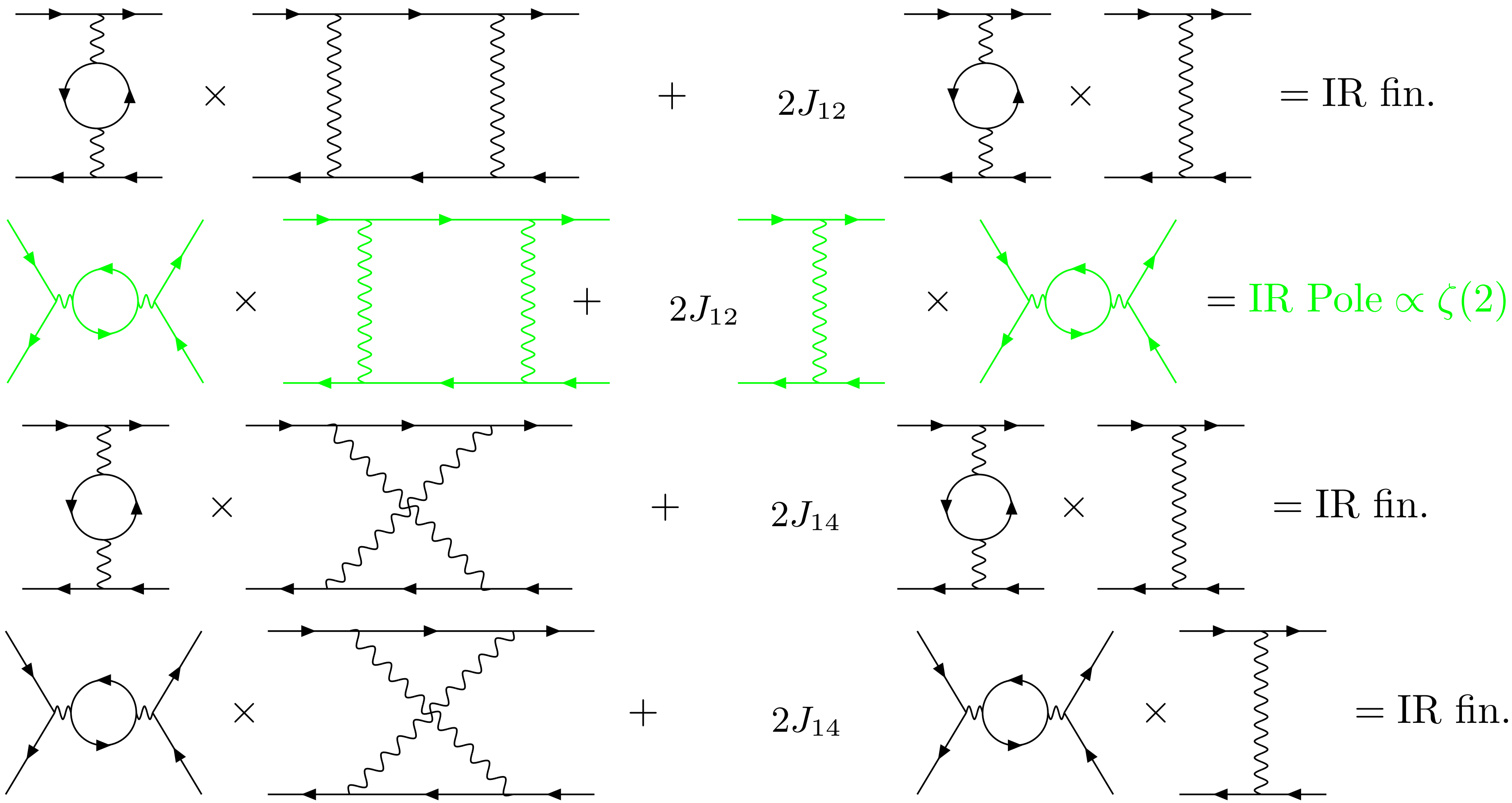
 $\times$  $+$ $(J_{13} + J_{11})$  $\times$  $=$ IR fin.

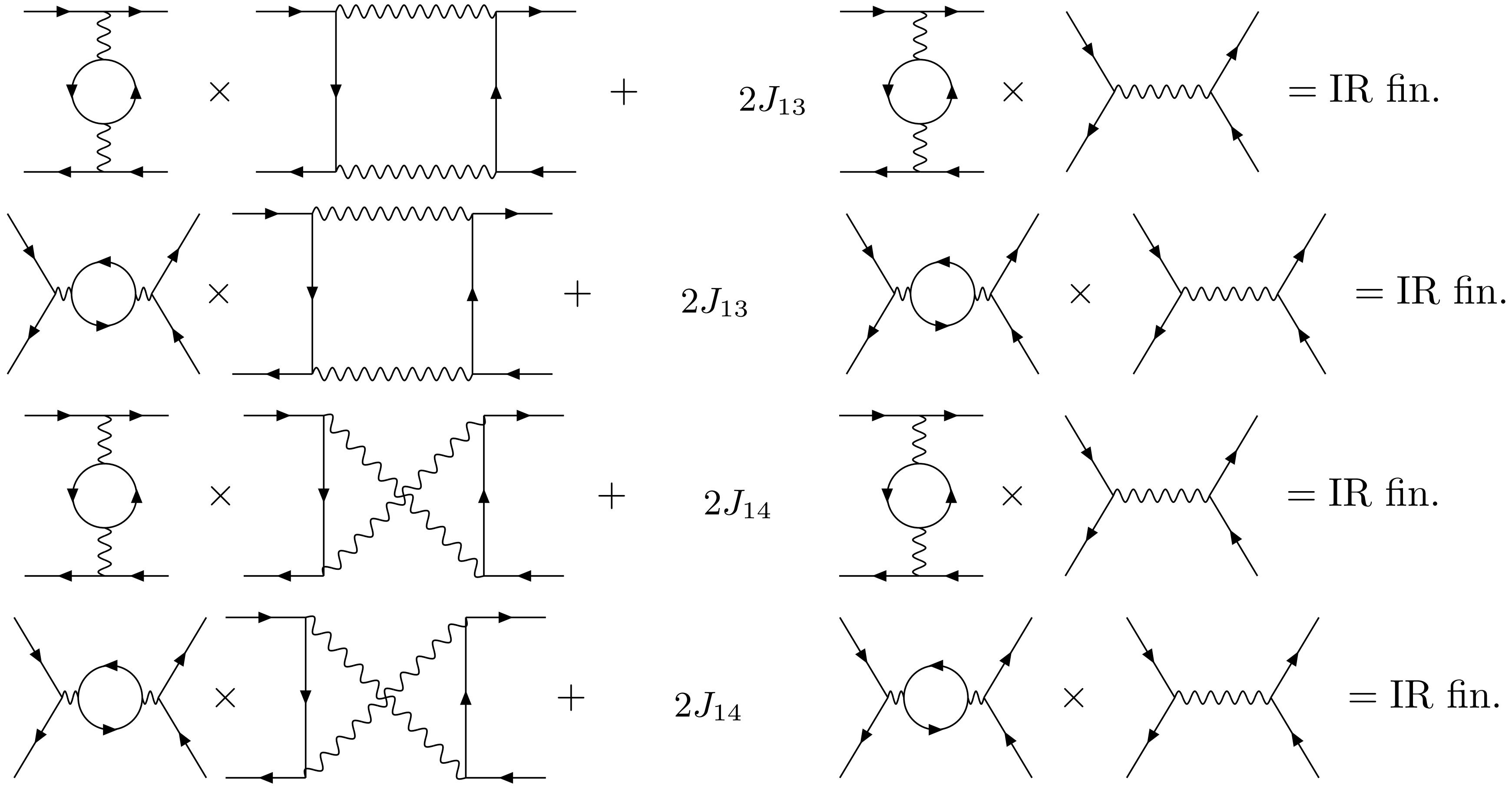
 $\times$  $+$ $(J_{13} + J_{11})$  $\times$  $=$ IR fin.

 $\times$  $+$ $(J_{12} + J_{11})$  $\times$  $=$ IR fin.

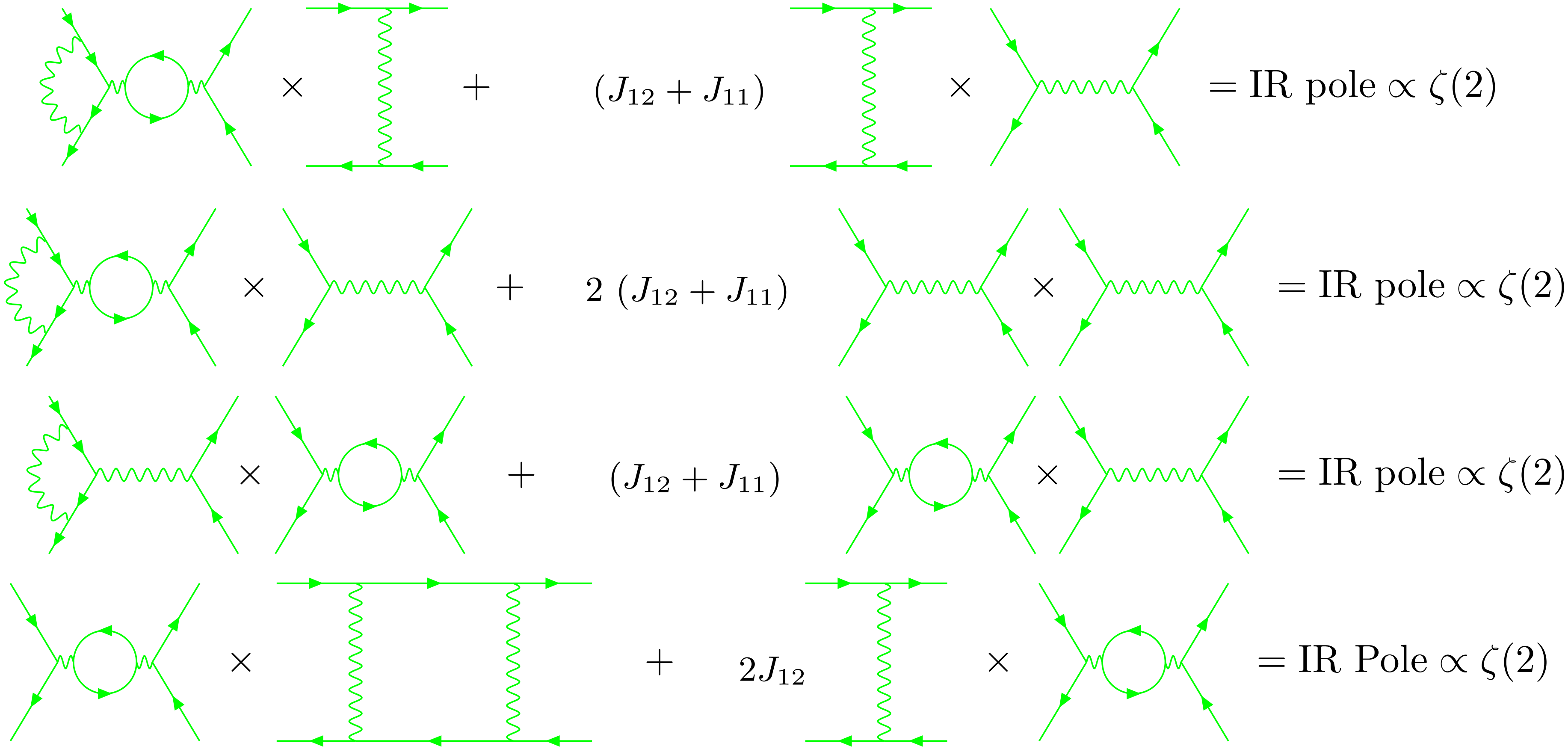
 $\times$  $+$ $(J_{12} + J_{11})$  $\times$  $=$ IR pole $\propto \zeta(2)$

One-Loop boxes times One-loop $\mathcal{O}(\alpha^4 N_F = 1)$




× + 2J13 × = IR fin.
× + 2J13 × = IR fin.
× + 2J14 × = IR fin.
× + 2J14 × = IR fin.

Single IR Poles proportional to $\zeta(2)$

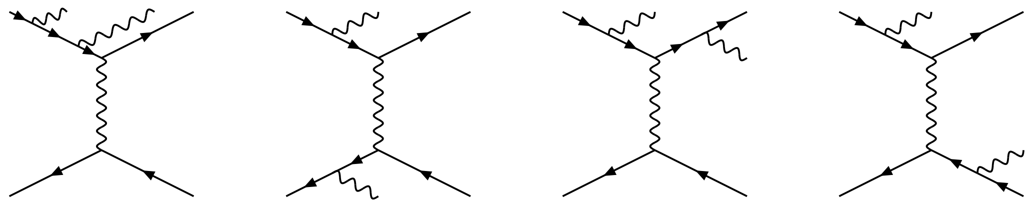


These poles cancel in the final sum!

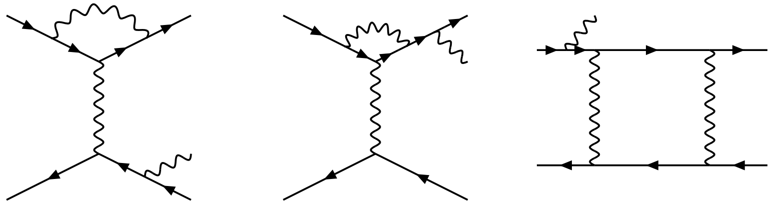
Real photon emission at $\mathcal{O}(\alpha^4)$

The inclusion of the two-loop vertex corrections requires the following real soft-emission diagrams

Tree-Level with the emission of two photons



One-Loop diagrams with the emission of one photon



....

Since the soft-photon corrections in QED exponentiate, the contribution due to the emission of two photons from the tree-level and the one due to the emission of one photon from the one-loop amplitude are given respectively by:

$$\left(\frac{\alpha}{\pi}\right)^2 \frac{d\sigma_2^{S,2-ph}(s,t,m^2)}{d\Omega} = \frac{1}{2}\left(\frac{\alpha}{\pi}\right)^2 \frac{d\sigma_0^{D}(s,t,m^2)}{d\Omega} \, (4 \sum_{j=1}^{4} J_{1j})^2$$

$$\left(\frac{\alpha}{\pi}\right)^2 \frac{d\sigma_2^{S,1-ph}(s,t,m^2)}{d\Omega} = \left(\frac{\alpha}{\pi}\right)^2 \frac{d\sigma_1^{(V,D)}(s,t,m^2)}{d\Omega} \, 4 \sum_{j=1}^{4} J_{1j}$$

Note that:

- the term $(4\sum_{j=1}^{4} J_{1j})^2$ in the first equation has a double IR pole: therefore also the $\mathcal{O}(D-4)^2$ of $d\sigma_0^D/d\Omega$ is required;
- because $d\sigma_1^{(V,D)}/d\Omega$ is IR-divergent, in principle we need also the $\mathcal{O}(D-4)$ of the integrals J_{1j}, BUT it is possible to prove that actually we DO NOT need it.

Cancellation of $\mathcal{O}(D-4)$ of J_{1j}

To prove the fact that we do not need to know the $\mathcal{O}(D-4)$ of the integrals J_{1j}, consider that

- $d\sigma_1^{(V,D)}/d\Omega$ includes the contribution of the vertices and the boxes

$$\frac{d\sigma_1^{(V,D)}(s,t,m^2)}{d\Omega} = \left.\frac{d\sigma_1^{(V,D)}(s,t,m^2)}{d\Omega}\right|_{(1l,V)} + \left.\frac{d\sigma_1^{(V,D)}(s,t,m^2)}{d\Omega}\right|_{(1l,B)}$$

- These two contributions are IR divergent:

$$\left.\frac{d\sigma_1^{V}(s,t,m^2)}{d\Omega}\right|_{(1l,j)} = \frac{1}{(D-4)}\left.\frac{d\sigma_1^{(V,-1)}(s,t,m^2)}{d\Omega}\right|_{(1l,j)} + \left.\frac{d\sigma_1^{(V,0)}(s,t,m^2)}{d\Omega}\right|_{(1l,j)}$$

- If we write $4\sum_{j=1}^{4} J_{1j} = S_{\rm IR}$, we also have:

$$S_{\rm IR} = \frac{S_{\rm IR}^{(-1)}}{(D-4)} + S_{\rm IR}^{(0)} + (D-4)S_{\rm IR}^{(1)} + \mathcal{O}\left((D-4)^2\right)$$

- The cancellation of the IR divergences in the $\mathcal{O}(\alpha^3)$ cross-section guarantees that

$$\left.\frac{d\sigma_1^{(V,-1)}(s,t,m^2)}{d\Omega}\right|_{(1l,B)} + \left.\frac{d\sigma_1^{(V,-1)}(s,t,m^2)}{d\Omega}\right|_{(1l,V)} + \frac{d\sigma_0(s,t,m^2)}{d\Omega} S_{\rm IR}^{(-1)} = 0$$

- From the double emission cross-section we find that the term proportional to $S_{\rm IR}^{(1)}$ is

$$\frac{d\sigma_2^{(S,\texttt{double})}(s,t,m^2)}{d\Omega} \rightarrow \frac{d\sigma_0(s,t,m^2)}{d\Omega} S_{\rm IR}^{(-1)} S_{\rm IR}^{(1)}$$

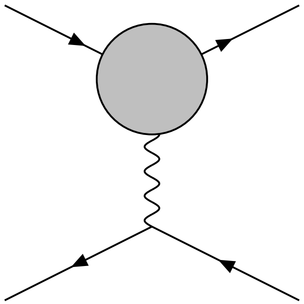
- The term proportional to $S_{\rm IR}^{(1)}$ appearing in the single photon emission cross-sections is

$$\frac{d\sigma_2^{(S,\texttt{single})}(s,t,m^2)}{d\Omega} \rightarrow \left[\left.\frac{d\sigma_1^{(V,-1)}(s,t,m^2)}{d\Omega}\right|_{(1l,B)} + \left.\frac{d\sigma_1^{(V,-1)}(s,t,m^2)}{d\Omega}\right|_{(1l,V)} \right] S_{\rm IR}^{(1)}$$

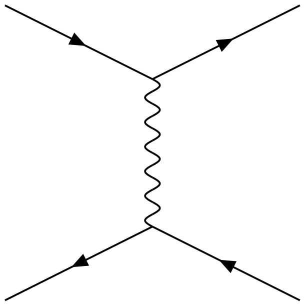
We conclude that the terms proportional to $S_{\rm IR}^{(1)}$ cancel out in the total real-emission cross-section at order $\mathcal{O}(\alpha^4)$!

Diagrammatic IR subtraction

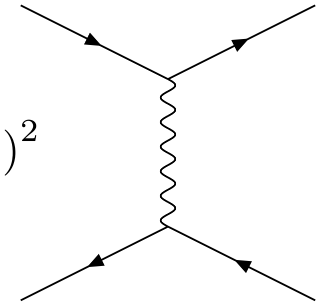
As for the previous contributions, we have particular combinations of diagrams that are IR finite:



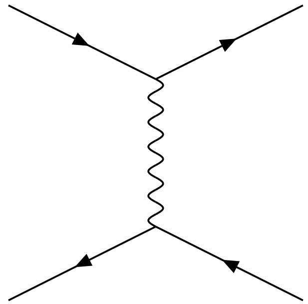
$\times$



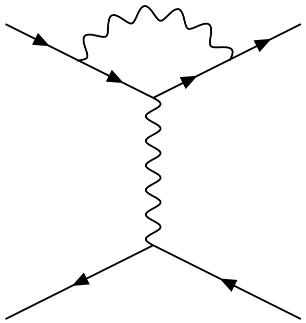
$+ \frac{1}{16} \left(J_{11} + J_{13}\right)^2$



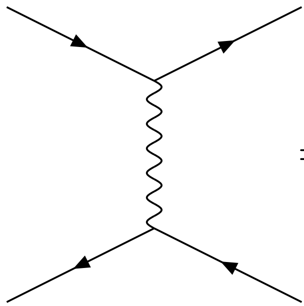
$\times$



$+ \frac{1}{4} \left(J_{11} + J_{13}\right)$

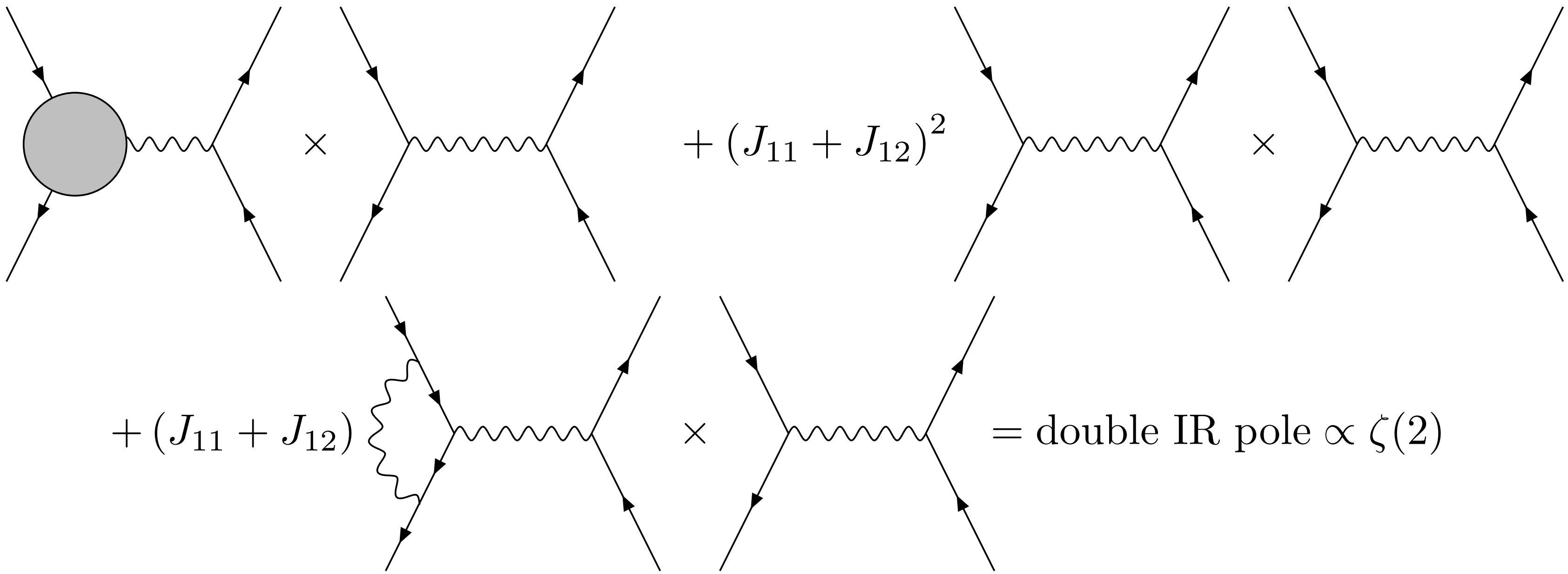


$\times$



$= \text{IR finite}$

We have also residual IR poles proportional to $\zeta(2)$:



These poles need, in order to be canceled, the contributions due to the photonic box diagrams

What we have with $m^2 \neq 0$

Basically we have everything except the contribution of the photonic double boxes.

- Two-loop $N_F = 1$ contribution. It is UV and IR finite
- Two-loop photonic vertex and One-loop times One-loop contributions. They are UV finite but they still contain IR poles that have to cancel with the ones coming from the photonic double boxes

The $m^2/s \to 0$ Approximation

In order to do checks against know results in the literature and to provide some phenomenological considerations we expanded our exact results retaining terms that do not vanish in the limit $m^2/s \to 0$.

- We write the small electron mass approximation of the cross section as

$$\frac{d\sigma_2}{d\sigma_0} = \delta_2^{(2)}(\xi)\ln^2\left(\frac{s}{m_e^2}\right) + \delta_2^{(1)}(\xi)\ln\left(\frac{s}{m_e^2}\right) + \delta_2^{(0)}(\xi) + \mathcal{O}\left(\frac{m^2}{s}\right)$$

  where

$$\xi = \frac{1-\cos\theta}{2}$$

- This approximation is not valid in the almost-forward ($|t| < m^2$) and in the almost-backward ($|u| < m^2$) directions, where terms of order m^2/t and m^2/u become important

- The functions $\delta_2^{(i)}(\xi)$ are known for the photonic corrections

$\longrightarrow$ see Penin's talk

Two-Loop $N_F = 1$: $m^2/s \to 0$ Approximation

$$\frac{d\sigma_{N_F=1}}{d\sigma_0} = \delta^{(2)}_{N_F=1,3} \ln^3\left(\frac{s}{m^2}\right) + \delta^{(2)}_{N_F=1,2} \ln^2\left(\frac{s}{m^2}\right) + \delta^{(2)}_{N_F=1,1} \ln\left(\frac{s}{m^2}\right) + \delta^{(2)}_{N_F=1,0}$$

where:

$$\delta^{(2)}_{N_F=1,3} = \frac{1}{(1-\xi+\xi^2)^2}\left\{-\frac{1}{9}+\frac{2}{9}\xi-\frac{1}{3}\xi^2+\frac{2}{9}\xi^3-\frac{1}{9}\xi^4\right\}$$

$$\delta^{(2)}_{N_F=1,2} = \frac{1}{(1-\xi+\xi^2)^2}\left\{\ln\left(\frac{4w^2}{s}\right)\left(-\frac{4}{3}+\frac{8}{3}\xi-4\xi^2+\frac{8}{3}\xi^3-\frac{4}{3}\xi^4\right)-\left(\frac{17}{18}-\frac{17}{9}\xi+\frac{17}{6}\xi^2-\frac{17}{9}\xi^3+\frac{17}{18}\xi^4\right)-\left(\frac{1}{3}-\frac{2}{3}\xi+\xi^2 -\frac{2}{3}\xi^3+\frac{1}{3}\xi^4\right)\ln(1-\xi)-\left(\frac{1}{3}-\frac{1}{3}\xi+\frac{1}{3}\xi^3-\frac{1}{3}\xi^4\right)\ln(\xi)\right\}$$

$$\delta^{(2)}_{N_F=1,1} = \frac{1}{(1-\xi+\xi^2)^2}\left\{\ln\left(\frac{4w^2}{s}\right)\left[\left(\frac{4}{3}-\frac{8}{3}\xi+4\xi^2-\frac{8}{3}\xi^3+\frac{4}{3}\xi^4\right)\ln(1-\xi)+\left(\frac{32}{9}-\frac{64}{9}\xi+\frac{32}{3}\xi^2-\frac{64}{9}\xi^3+\frac{32}{9}\xi^4\right) -\left(\frac{8}{3}-\frac{14}{3}\xi+6\xi^2-\frac{10}{3}\xi^3+\frac{4}{3}\xi^4\right)\ln(\xi)\right]+\frac{43}{27}-\frac{2}{3}\zeta(2)\xi-\frac{86}{27}\xi-\frac{7}{2}\zeta(2)\xi^2+\frac{43}{9}\xi^2+\frac{13}{3}\zeta(2)\xi^3-\frac{86}{27}\xi^3-\frac{8}{3}\zeta(2)\xi^4 +\frac{43}{27}\xi^4-\frac{2}{3}\zeta(2)+\left(\frac{2}{3}-\frac{11}{6}\xi+\frac{5}{2}\xi^2-\frac{11}{6}\xi^3+\frac{2}{3}\xi^4\right)\ln^2(1-\xi)+\left(\frac{10}{9}-\frac{31}{18}\xi+\frac{10}{3}\xi^2-\frac{31}{18}\xi^3+\frac{10}{9}\xi^4\right)\ln(1-\xi) +\left(\frac{8}{3}-\frac{16}{3}\xi+8\xi^2-\frac{16}{3}\xi^3+\frac{8}{3}\xi^4\right)Li_2(\xi)-\left(\frac{2}{3}-\frac{4}{3}\xi-\frac{1}{2}\xi^2+\frac{5}{3}\xi^3-\frac{4}{3}\xi^4\right)\ln(\xi)\ln(1-\xi)-\left(\frac{17}{9}-\frac{26}{9}\xi+\frac{9}{2}\xi^2 -\frac{47}{18}\xi^3+\frac{10}{9}\xi^4\right)\ln(\xi)-\left(\frac{1}{3}+\frac{5}{12}\xi-\frac{1}{4}\xi^2-\frac{5}{12}\xi^3+\frac{2}{3}\xi^4\right)\ln^2(\xi)\right\}$$

A. B. Arbuzov, E. A. Kuraev, N. P. Merenkov and L. Trentadue, Phys. Atom. Nucl. **60** (1997) 591 [Yad. Fiz. **60N4** (1997) 673]

R. B., A. Ferroglia, P. Mastrolia, E. Remiddi and J. J. van der Bij, Nucl. Phys. B716 (2005) 280

$$
\begin{aligned}
\delta^{(2)}_{N_F=1,0} \quad = \quad & \frac{1}{(1-\xi+\xi^2)^2}\Bigg\{\ln\left(\frac{4w^2}{s}\right)\Bigg[\Big(-\frac{20}{9}+\frac{40}{9}\xi-\frac{20}{3}\xi^2+\frac{40}{9}\xi^3-\frac{20}{9}\xi^4\Big)\ln(1-\xi)-\Big(\frac{20}{9}-\frac{40}{9}\xi+\frac{20}{3}\xi^2-\frac{40}{9}\xi^3+\frac{20}{9}\xi^4\Big) \\
& +\Big(\frac{4}{3}-2\xi+2\xi^2-\frac{2}{3}\xi^3\Big)\ln(\xi)\ln(1-\xi)+\Big(\frac{32}{9}-\frac{58}{9}\xi+\frac{26}{3}\xi^2-\frac{46}{9}\xi^3+\frac{20}{9}\xi^4\Big)\ln(\xi)-\Big(\frac{4}{3}-2\xi+2\xi^2-\frac{2}{3}\xi^3\Big)\ln^2(\xi)\Bigg] \\
& +\frac{1007}{108}+\frac{29}{6}\zeta(2)\xi-4\zeta(3)\xi-\frac{1007}{54}\xi-\frac{4}{3}\zeta(2)\xi^2+6\zeta(3)\xi^2+\frac{1007}{36}\xi^2-\frac{1}{2}\zeta(2)\xi^3-4\zeta(3)\xi^3-\frac{1007}{54}\xi^3-\frac{2}{3}\zeta(2)\xi^4+2\zeta(3)\xi^4 \\
& +\frac{1007}{108}\xi^4-\zeta(2)+2\zeta(3)-\Big(\frac{1}{6}\xi-\frac{5}{18}\xi^2+\frac{5}{18}\xi^3-\frac{1}{9}\xi^4\Big)\ln^3(1-\xi)-\Big(\frac{10}{9}-\frac{29}{9}\xi+\frac{9}{2}\xi^2-\frac{29}{9}\xi^3+\frac{10}{9}\xi^4\Big)\ln^2(1-\xi) \\
& -\Big(\frac{2}{3}-\frac{2}{3}\xi+\frac{2}{3}\xi^3-\frac{2}{3}\xi^4\Big)\ln(1-\xi)\mathrm{L}i_2(\xi)-\Big(\frac{56}{27}+\frac{19}{3}\zeta(2)\xi-\frac{161}{54}\xi-\frac{29}{3}\zeta(2)\xi^2+\frac{56}{9}\xi^2+\frac{17}{3}\zeta(2)\xi^3-\frac{161}{54}\xi^3 \\
& -\frac{4}{3}\zeta(2)\xi^4+\frac{56}{27}\xi^4-2\zeta(2)\Big)\ln(1-\xi)+\Big(\frac{1}{3}-\frac{5}{6}\xi+\frac{13}{12}\xi^2-\frac{2}{3}\xi^3+\frac{1}{3}\xi^4\Big)\ln(\xi)\ln^2(1-\xi)+\Big(\frac{10}{9}-\frac{37}{18}\xi-\frac{1}{2}\xi^2+\frac{25}{9}\xi^3 \\
& -\frac{20}{9}\xi^4\Big)\ln(\xi)\ln(1-\xi)+\Big(\frac{1}{3}-\frac{1}{3}\xi+\frac{4}{3}\xi^2-\xi^3+\frac{1}{3}\xi^4\Big)\ln^2(\xi)\ln(1-\xi)-\Big(\frac{1}{9}+\frac{19}{36}\xi-\frac{5}{18}\xi^2-\frac{1}{6}\xi^3+\frac{1}{9}\xi^4\Big)\ln^3(\xi) \\
& -\Big(\frac{17}{18}-\frac{35}{18}\xi+\frac{3}{2}\xi^2+\frac{11}{18}\xi^3-\frac{10}{9}\xi^4\Big)\ln^2(\xi)+\Big(4-\frac{19}{3}\xi+7\xi^2-3\xi^3+\frac{2}{3}\xi^4\Big)\ln(\xi)\mathrm{L}i_2(\xi)+\Big(\frac{43}{27}-2\zeta(2)\xi-\frac{61}{27}\xi \\
& -\frac{11}{6}\zeta(2)\xi^2+\frac{11}{2}\xi^2+4\zeta(2)\xi^3-\frac{211}{54}\xi^3-2\zeta(2)\xi^4+\frac{56}{27}\xi^4-\frac{2}{3}\zeta(2)\Big)\ln(\xi)-\Big(\frac{40}{9}-\frac{80}{9}\xi+\frac{40}{3}\xi^2-\frac{80}{9}\xi^3+\frac{40}{9}\xi^4\Big)\mathrm{L}i_2(\xi) \\
& -\Big(\frac{2}{3}-\frac{4}{3}\xi+2\xi^2-\frac{4}{3}\xi^3+\frac{2}{3}\xi^4\Big)\mathrm{L}i_3(\xi)-\Big(\frac{1}{3}\xi-\xi^2+\xi^3-\frac{2}{3}\xi^4\Big)\mathrm{L}i_3(1-\xi)+\Big(\frac{2}{3}-\xi+\xi^2-\frac{1}{3}\xi^3\Big)\mathrm{L}i_3\left(-\frac{\xi}{1-\xi}\right)\Bigg\}
\end{aligned}
$$

Two-Loop Soft-Pair: $m^2/s \to 0$ Approximation

$$\frac{d\sigma_{Pair}}{d\sigma_0} = \delta^{(2)}_{Pair,3} \ln^3\left(\frac{s}{m^2}\right) + \delta^{(2)}_{Pair,2} \ln^2\left(\frac{s}{m^2}\right) + \delta^{(2)}_{Pair,1} \ln\left(\frac{s}{m^2}\right) + \delta^{(2)}_{Pair,0}$$

where:

$$\delta^{(2)}_{Pair,3} = \frac{1}{(1-\xi+\xi^2)^2}\Big(\frac{1}{9} - \frac{2}{9}\xi + \frac{1}{3}\xi^2 - \frac{2}{9}\xi^3 + \frac{1}{9}\xi^4\Big)$$

$$\begin{aligned}\delta^{(2)}_{Pair,2} = {} & \frac{1}{(1-\xi+\xi^2)^2}\Bigg\{\ln\left(\frac{4w^2}{s}\right)\Big(\frac{1}{3} - \frac{2}{3}\xi + \xi^2 - \frac{2}{3}\xi^3 + \frac{1}{3}\xi^4\Big) - \Big(\frac{1}{3} - \frac{2}{3}\xi + \xi^2 - \frac{2}{3}\xi^3 + \frac{1}{3}\xi^4\Big)\ln(1-\xi) - \Big(\frac{5}{9} - \frac{10}{9}\xi \\ & + \frac{5}{3}\xi^2 - \frac{10}{9}\xi^3 + \frac{5}{9}\xi^4\Big) + \Big(\frac{1}{3} - \frac{2}{3}\xi + \xi^2 - \frac{2}{3}\xi^3 + \frac{1}{3}\xi^4\Big)\ln(\xi)\Bigg\}\end{aligned}$$

$$\begin{aligned}\delta^{(2)}_{Pair,1} = {} & \frac{1}{(1-\xi+\xi^2)^2}\Bigg\{\ln^2\left(\frac{4w^2}{s}\right)\Big(\frac{1}{3} - \frac{2}{3}\xi + \xi^2 - \frac{2}{3}\xi^3 + \frac{1}{3}\xi^4\Big) + \ln\left(\frac{4w^2}{s}\right)\Bigg[\Big(-\frac{2}{3} + \frac{4}{3}\xi - 2\xi^2 + \frac{4}{3}\xi^3 - \frac{2}{3}\xi^4\Big)\ln(1-\xi) \\ & - \Big(\frac{10}{9} - \frac{20}{9}\xi + \frac{10}{3}\xi^2 - \frac{20}{9}\xi^3 + \frac{10}{9}\xi^4\Big) + \Big(\frac{2}{3} - \frac{4}{3}\xi + 2\xi^2 - \frac{4}{3}\xi^3 + \frac{2}{3}\xi^4\Big)\ln(\xi)\Bigg] + \frac{56}{27} + \frac{4}{3}\zeta(2)\xi - \frac{112}{27}\xi - 2\zeta(2)\xi^2 \\ & + \frac{56}{9}\xi^2 + \frac{4}{3}\zeta(2)\xi^3 - \frac{112}{27}\xi^3 - \frac{2}{3}\zeta(2)\xi^4 + \frac{56}{27}\xi^4 - \frac{2}{3}\zeta(2) - \Big(\frac{1}{3} - \frac{2}{3}\xi + \xi^2 - \frac{2}{3}\xi^3 + \frac{1}{3}\xi^4\Big)\ln^2(1-\xi) + \Big(\frac{10}{9} - \frac{20}{9}\xi \\ & + \frac{10}{3}\xi^2 - \frac{20}{9}\xi^3 + \frac{10}{9}\xi^4\Big)\ln(1-\xi) - \Big(\frac{4}{3} - \frac{8}{3}\xi + 4\xi^2 - \frac{8}{3}\xi^3 + \frac{4}{3}\xi^4\Big)\mathrm{L}i_2(\xi) - \Big(\frac{2}{3} - \frac{4}{3}\xi + 2\xi^2 - \frac{4}{3}\xi^3 + \frac{2}{3}\xi^4\Big)\ln(\xi)\ln(1-\xi) \\ & - \Big(\frac{10}{9} - \frac{20}{9}\xi + \frac{10}{3}\xi^2 - \frac{20}{9}\xi^3 + \frac{10}{9}\xi^4\Big)\ln(\xi) + \Big(\frac{1}{3} - \frac{2}{3}\xi + \xi^2 - \frac{2}{3}\xi^3 + \frac{1}{3}\xi^4\Big)\ln^2(\xi)\Bigg\}\end{aligned}$$

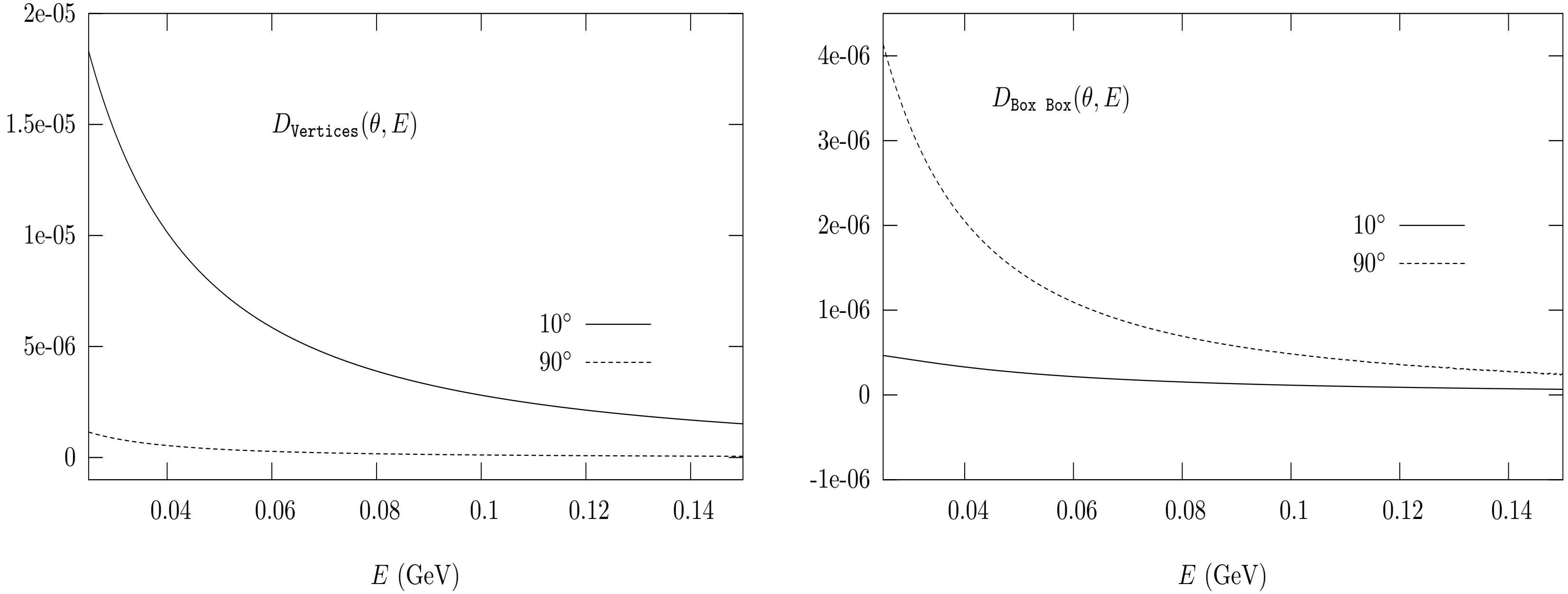
A. B. Arbuzov, E. A. Kuraev, N. P. Merenkov and L. Trentadue, Phys. Atom. Nucl. **60** (1997) 591 [Yad. Fiz. **60N4** (1997) 673]

$$
\begin{aligned}
\delta^{(2)}_{Pair,0} \quad = \quad & \frac{1}{(1-\xi+\xi^2)^2}\Big\{\ln^2\left(\frac{4w^2}{s}\right)\Big[\Big(-\frac{1}{3}+\frac{2}{3}\xi-\xi^2+\frac{2}{3}\xi^3-\frac{1}{3}\xi^4\Big)\ln(1-\xi)+\Big(\frac{1}{3}-\frac{2}{3}\xi+\xi^2-\frac{2}{3}\xi^3+\frac{1}{3}\xi^4\Big)\ln(\xi)\Big] \\
& \ln\left(\frac{4w^2}{s}\right)\Big[\Big(-\frac{1}{3}+\frac{2}{3}\xi-\xi^2+\frac{2}{3}\xi^3-\frac{1}{3}\xi^4\Big)\ln^2(1-\xi)+\Big(\frac{10}{9}-\frac{20}{9}\xi+\frac{10}{3}\xi^2-\frac{20}{9}\xi^3+\frac{10}{9}\xi^4\Big)\ln(1-\xi)-\Big(\frac{10}{9}-\frac{20}{9}\xi \\
& +\frac{10}{3}\xi^2-\frac{20}{9}\xi^3+\frac{10}{9}\xi^4\Big)\ln(\xi)+\Big(\frac{1}{3}-\frac{2}{3}\xi+\xi^2-\frac{2}{3}\xi^3+\frac{1}{3}\xi^4\Big)\ln^2(\xi)\Big]-\Big(\frac{1}{9}-\frac{2}{9}\xi+\frac{1}{3}\xi^2-\frac{2}{9}\xi^3+\frac{1}{9}\xi^4\Big)\ln^3(1-\xi) \\
& +\Big(\frac{5}{9}-\frac{10}{9}\xi+\frac{5}{3}\xi^2-\frac{10}{9}\xi^3+\frac{5}{9}\xi^4\Big)\ln^2(1-\xi)-\Big(\frac{2}{3}-\frac{4}{3}\xi+2\xi^2-\frac{4}{3}\xi^3+\frac{2}{3}\xi^4\Big)\ln(1-\xi)\mathrm{L}i_2(\xi)-\Big(\frac{56}{27}+\frac{8}{3}\zeta(2)\xi \\
& -\frac{112}{27}\xi-4\zeta(2)\xi^2+\frac{56}{9}\xi^2+\frac{8}{3}\zeta(2)\xi^3-\frac{112}{27}\xi^3-\frac{4}{3}\zeta(2)\xi^4+\frac{56}{27}\xi^4-\frac{4}{3}\zeta(2)\Big)\ln(1-\xi)-\Big(\frac{2}{3}-\frac{4}{3}\xi+2\xi^2-\frac{4}{3}\xi^3 \\
& +\frac{2}{3}\xi^4\Big)\ln^2(\xi)\ln(1-\xi)+\Big(\frac{1}{9}-\frac{2}{9}\xi+\frac{1}{3}\xi^2-\frac{2}{9}\xi^3+\frac{1}{9}\xi^4\Big)\ln^3(\xi)-\Big(\frac{5}{9}-\frac{10}{9}\xi+\frac{5}{3}\xi^2-\frac{10}{9}\xi^3+\frac{5}{9}\xi^4\Big)\ln^2(\xi) \\
& -\Big(\frac{2}{3}-\frac{4}{3}\xi+\frac{2}{\xi}^2-\frac{4}{3}\xi^3+\frac{2}{3}\xi^4\Big)\ln(\xi)\mathrm{L}i_2(\xi)+\Big(\frac{56}{27}+\frac{4}{3}\zeta(2)\xi-\frac{112}{27}\xi-2\zeta(2)\xi^2+\frac{56}{9}\xi^2+\frac{4}{3}\zeta(2)\xi^3-\frac{112}{27}\xi^3 \\
& -\frac{2}{3}\zeta(2)\xi^4+\frac{56}{27}\xi^4-\frac{2}{3}\zeta(2)\Big)\ln(\xi)\Big\}
\end{aligned}
$$

Numerical Evaluation

- We have an almost-complete result for the QED corrections at $\mathcal{O}(\alpha^4)$ of the Bhabha scattering differential cross section: we have to estimate the phenomenological impact of retaining the mass of the electron in the calculations (at least for the pieces that are known also exactly):
  - We have two codes (in Mathematica and in Fortran, using the routines for the HPLs, Gehrmann-Remiddi '01) in order also to cross-check the results
  - What we find is that, for the $N_F = 1$ and known photonic corrections, the relative size of the $\mathcal{O}(m^2/s)$ terms with respect to the one-loop exact cross section is $\sim 10^{-6}$ already at $E \sim 100$ MeV
    $\Rightarrow$ we conclude that the $m^2/s \to 0$ limit is enough for phenomenology

Photonic Corrections: $m^2 \neq 0$ vs. $m^2/s \to 0$



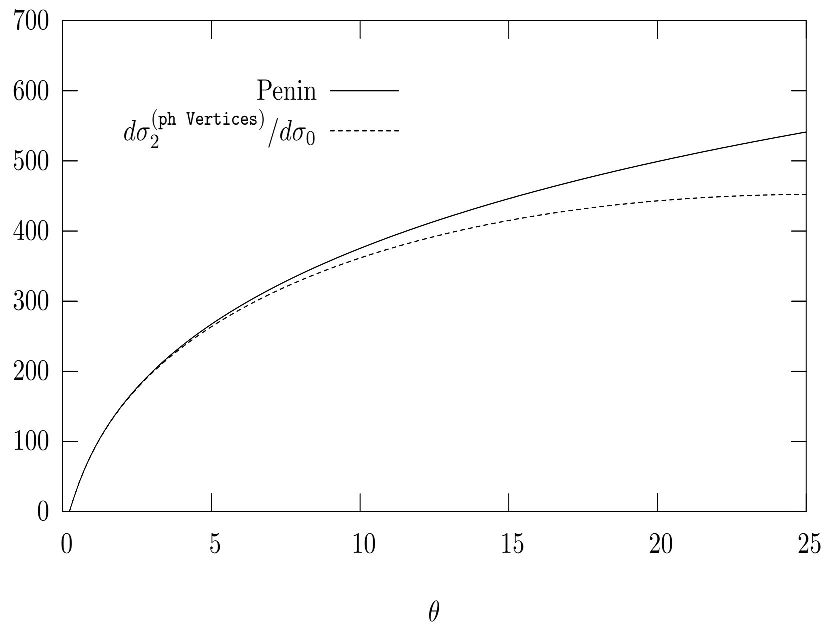
$$D_i(\theta, E) = \left(\frac{\alpha}{\pi}\right)^2 \left| \left(\frac{d\sigma_2^{(\mathtt{ph}\ i)}}{d\Omega} - \left.\frac{d\sigma_2^{(\mathtt{ph}\ i)}}{d\Omega}\right|_L \right) \right| \left(\frac{d\sigma_0}{d\Omega} + \left(\frac{\alpha}{\pi}\right) \frac{d\sigma_1}{d\Omega} \right)^{-1}$$

The soft-photon energy cut-off is set equal to the beam energy: $\omega = E$

Small-Angle limit

The Vertex corrections agree with the small-angle limit of the photonic corrections

$$\frac{d\sigma_2^{(\mathrm{ph})}}{d\sigma_0} \overset{\theta\to 0}{=} \frac{1}{(1-\xi+\xi^2)^2}\left\{\ln^2\left(\frac{s}{m^2}\right)\left[\frac{9}{2}+2\ln^2\left(\frac{4\omega^2}{s}\right)\right.\right.$$
$$\left.+6\ln\left(\frac{4\omega^2}{s}\right)\right]+\ln\left(\frac{s}{m^2}\right)\left[6\zeta(3)-3\zeta(2)-\frac{93}{8}+9\ln(\xi)\right.$$
$$\left.-4\ln^2\left(\frac{4\omega^2}{s}\right)[1-\ln(\xi)]-2\ln\left(\frac{4\omega^2}{s}\right)[7-6\ln(\xi)]\right]$$
$$-9\zeta(3)+\frac{51}{4}\zeta(2)-12\zeta(2)\ln(2)-\frac{32}{5}\zeta^2(2)$$
$$+\frac{27}{2}+6\zeta(3)\ln(\xi)-3\zeta(2)\ln(\xi)-\frac{93}{8}\ln(\xi)+\frac{9}{2}\ln(\xi)^2$$
$$+\ln^2\left(\frac{4\omega^2}{s}\right)\left[2-4\ln(\xi)+2\ln^2(\xi)\right]$$
$$\left.+\ln\left(\frac{4\omega^2}{s}\right)\left[8-14\ln(\xi)+6\ln^2(\xi)\right]+\mathcal{O}(\xi)\right\}$$



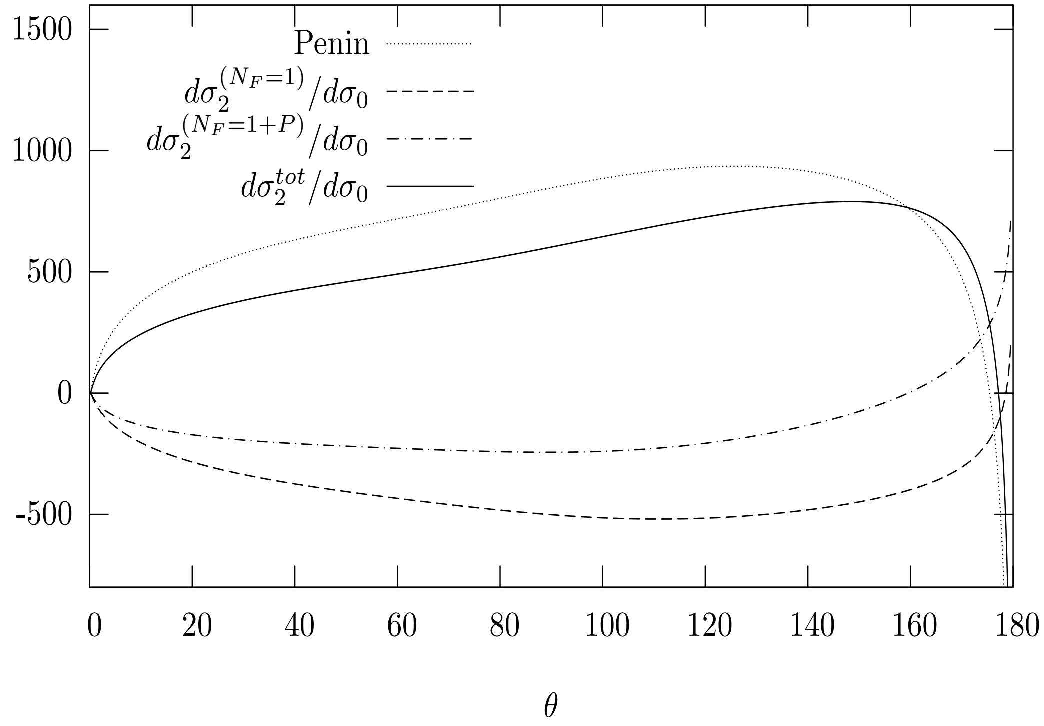
$E = 0.5$ GeV and $\omega = E$

The numbers are in units of $\alpha^2/\pi^2 \sim 5.4\cdot 10^{-6}$

V. S. Fadin, E. A. Kuraev, L. Trentadue, L. N. Lipatov and N. P. Merenkov, Phys. Atom. Nucl. **56** (1993) 1537 [Yad. Fiz. **56N11** (1993) 145]

Two-loop Logarithmic corrections

The full corrections in the limit $m^2/s \to 0$, for $\sqrt{s} = 1$ GeV:

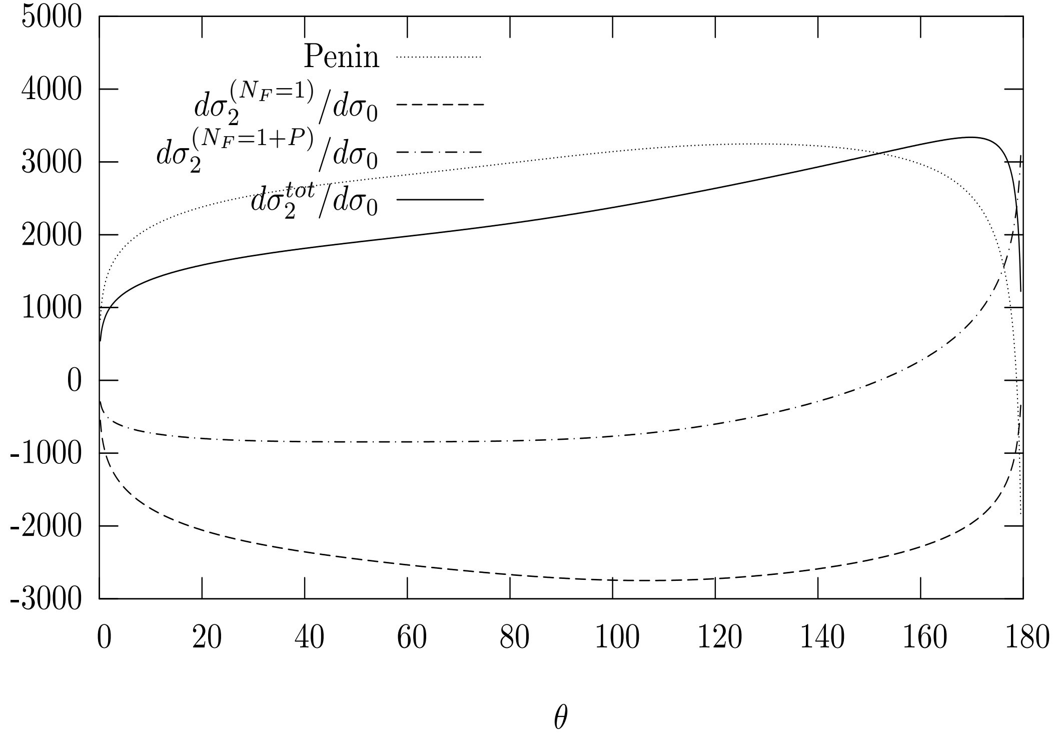


$E = 0.5$ GeV and $\Omega = \omega = E$

The numbers are in units of $\alpha^2/\pi^2 \sim 5.4 \cdot 10^{-6}$

Two-loop Logarithmic corrections

The full corrections in the limit $m^2/s \to 0$, for $\sqrt{s} = 500$ GeV:



$E = 0.5$ GeV and $\Omega = \omega = E$

The numbers are in units of $\alpha^2/\pi^2 \sim 5.4 \cdot 10^{-6}$

Checks

- One loop. Analytical checks with van Nieuwenhuizen (electron-muon scattering). Numerical checks with Consoli and Böhm-Denner-Hollik
- Two-loop $N_F = 1$. Analytical check of the MIs with Czakon-Gluza-Riemann. Indirect checks on the full result: cancellation of divergences in the final result; cancellation of the $\ln^3(s/m^2)$ against the soft-pair emission contribution
- Two-loop photonic contributions. Numerical checks of the vertex diagrams with S. Uccirati & TOPSIDE collaboration. Analytical check of the MIs with Czakon-Gluza-Riemann. Checks of the poles of the form factors with Barbieri-Mignaco-Remiddi ($\ln(\lambda/m) = -1/(D-4)$) and limits ($g-2$ of the electron). Direct check of the gauge independence of the photonic vertex corrections in the class of the covariant linear gauges. Checks of the small-angle region with Arbuzov et al. Check with A. Penin of the $m^2/s \to 0$ limit of vertices and one-loop times one-loop diagrams

Summary

- The calculation of the Two-Loop $N_F = 1$ and Two-Loop Vertex QED corrections to the Bhabha scattering differential cross-section was carried out keeping the electron mass different from zero, by means of the Laporta algorithm and the Differential Equations Method
- As a remark, we find that, from a phenomenological point of view, there is no difference between the exact calculation and the LL approximation (both for the $N_F = 1$ set and for the vertices)
- The numerical evaluation of the analytic expressions presented in the talk is implemented in two computer programs, written in Fortran and Mathematica. In the $m^2/s \to 0$ approximation, the full set of Two-Loop QED contributions are known and included in the program ($N_F = 1$ as well as Penin's formulas). The exact results, instead, include everything except the Two-Loop photonic-box corrections
- The Two-Loop photonic-box corrections are still missing (!?)