Real radiation at NNLO

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Outline

- Motivation
- Isolation of infrared poles
  - general procedure (NLO)
  - double real radiation at NNLO
    - analytical subtraction scheme
    - sector decomposition
- Summary and Outlook

Motivation

precision measurements led to successful predictions (e.g. top mass) and stringent tests of Standard Model

only possible in combination with **"precision calculations"** $\Rightarrow$ RADCOR

future International Linear Collider will reach precision at the per mille level

measurement of $e^+e^- \to 3$ jet observables offers possibility for determination of strong coupling constant α_s with unseen precision

Jet production

α_s world average:

$\alpha_s(M_Z) = 0.1179 \pm 0.003$ (stat. and sys.)   S. Bethke 04

theory error on determination of α_s from jet observables using NLO calculations:

scale dependence: +0.007, –0.005

hadronisation etc: ± 0.0009

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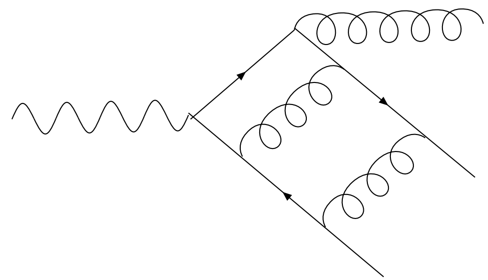
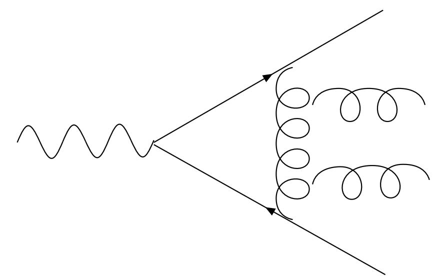
$\Rightarrow$ NNLO necessary to match experimental precision !

partonic ingredients for $e^+e^- \to 3$ jets at NNLO:

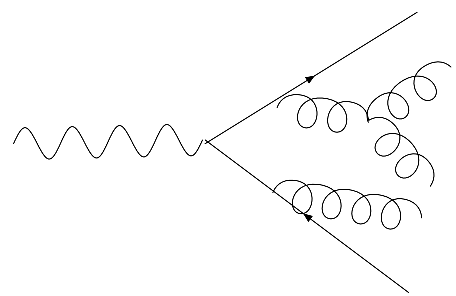
- 2-loop virtual



- one-loop plus single real emission



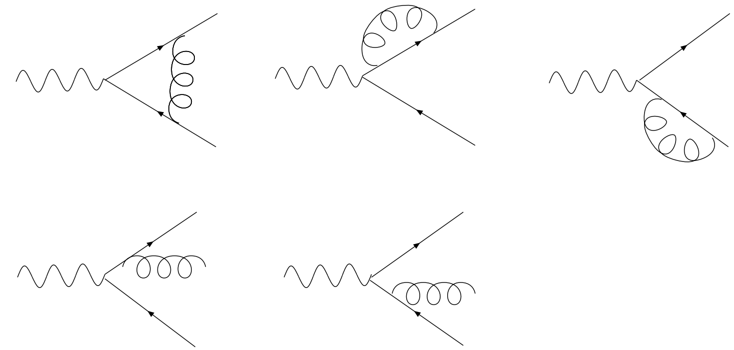
- double real emission



5 partons in final state
up to 2 theoretically unresolved (soft or collinear)

difficulty: subtraction of infrared poles in phase space integrals

Subtraction of Infrared Poles: NLO



virtual: $d\sigma^V = P_2/\epsilon^2 + P_1/\epsilon + P_0$

real: integration of subtraction terms $d\sigma^S$ over singular regions of phase space

$$\Rightarrow \int_{\text{sing}} d\sigma^S = -P_2/\epsilon^2 - P_1/\epsilon + Q_0$$

$$\sigma^{NLO} = \underbrace{\int_{m+1} \left[d\sigma^R - d\sigma^S \right]_{\epsilon=0}}_{\text{numerically}} + \underbrace{\int_m \left[\underbrace{d\sigma^V}_{\text{analytically}} + \underbrace{\int_1 d\sigma^S}_{\text{analytically}} \right]_{\epsilon=0}}_{\text{numerically}}$$

Subtraction of Infrared Poles: NNLO

two conceptually different approaches:

- manual construction of a subtraction scheme and analytic integration over subtraction terms in $D = 4 - 2\epsilon$ dimensions

  [A. Gehrmann-De Ridder, T. Gehrmann, N. Glover], [Del Duca, Somogyi, Trocsanyi], [Frixione, Grazzini], [Kilgore]

  $\longrightarrow$ see talks of T. Gehrmann, V. Del Duca

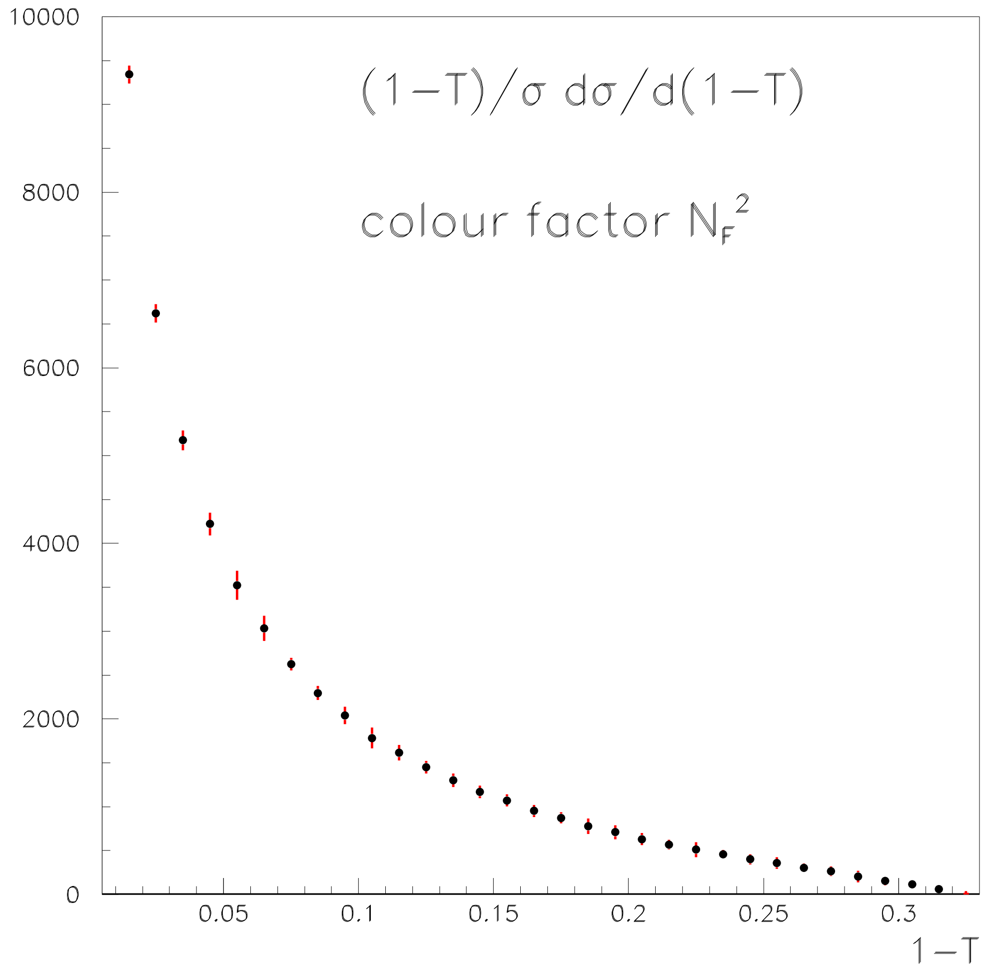
- sector decomposition: automated isolation of IR poles in parameter space and numerical integration of pole coefficients

  [GH, Binoth], [Anastasiou, Melnikov, Petriello]

analytical subtraction scheme

[T. Gehrmann, A. Gehrmann-De Ridder, N. Glover]

example: average thrust distribution for N_F^2 colour factor



subtraction of poles

difficulty in general: singularities are entangled in parameter space (overlapping structure)

usual subtraction procedure:

$$\int_0^1 dx\, dy\, x^{-1-\epsilon} f(x,y) \;=\; -\frac{1}{\epsilon}\int_0^1 dy\;\; f(0,y) + \int_0^1 dx\, dy\, x^{-\epsilon}\,\frac{f(x,y)-f(0,y)}{x}$$

$f(x,y) = 1/(x+y) \;\Rightarrow\; f(0,y) = 1/y \Rightarrow$ subtraction fails

solution: decompose into two sectors $x > y$ and $x < y$, remap integrations to unit cube

sector decomposition

$$I = \int_0^1 dx\, dy\, x^{-1-\epsilon} (x+y)^{-1}$$

$$= \int_0^1 dx\, dy\, x^{-1-\epsilon} (x+y)^{-1} [\underbrace{\Theta(x-y)}_{(1)} + \underbrace{\Theta(y-x)}_{(2)}]$$

subst. $(1)\; y = x\, t_2 \qquad (2)\; x = y\, t_1$

$$I = \int_0^1 dx\, x^{-1-\epsilon} \int_0^1 dt_2\, (1+t_2)^{-1}$$

$$+ \int_0^1 dy\, y^{-1-\epsilon} \int_0^1 dt_1\, t_1^{-1-\epsilon} (1+t_1)^{-1}$$

$\Rightarrow$ singularities are disentangled

general algorithm

- map parameter integrals to unit hypercube
- scan denominators for overlapping singularities

  automated subroutine: denominator $= 0$ for $\{x_1 \, \ldots \, x_k\} \to 0 \Rightarrow$ sector decomposition in $x_1 \, \ldots \, x_k$
- after disentangling of singularities:

  subtractions and expansion in ϵ (plus distributions)

$$x^{-1+\kappa\epsilon} = \frac{1}{\kappa\,\epsilon}\,\delta(x) + \sum_{n=0}^{\infty} \frac{(\kappa\epsilon)^n}{n!} \left[\frac{\ln^n(x)}{x}\right]_+$$

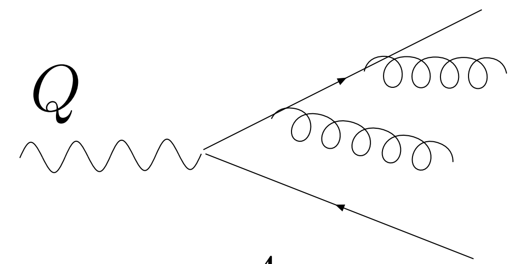
- result: Laurent series in ϵ

$$I = \sum_{k=-\mathrm{maxpole}}^{n} \epsilon^k \, C_k(x_i, m_i^2) \; + \; \mathcal{O}(\epsilon^{n+1})$$

- poles are isolated $\Rightarrow$ evaluate coefficients C_k numerically

application to $1 \to 4$ phase space

example: double real radiation part of $e^+e^- \to 2$ jets at NNLO ($1 \to 4$ partons)



$$\int d\Phi_4 = (2\pi)^{4-3D} \int \prod_{i=1}^{4} d^D p_i \, \delta^+(p_i^2) \, \delta(Q - \sum_{j=1}^{4} p_j)$$

define $x_1 = s_{12}/Q^2, \ldots, x_6 = s_{34}/Q^2$

$$\int d\Phi_4 \sim \int \prod_{j=1}^{6} dx_j \, \delta(1 - \sum_{i=1}^{6} x_i) \, [\lambda(x_1 x_6, x_2 x_5, x_3 x_4)]^{-1/2-\epsilon} \, \Theta(\lambda)$$

$$\lambda(x, y, z) = 2\,(xy + xz + yz) - (x^2 + y^2 + z^2)$$

Matrix element

$$|M_4|^2 \sim \frac{\mathcal{P}_1(x_i,\epsilon)}{x_2^2(x_2+x_4+x_6)^2} + \frac{\mathcal{P}_2(x_i,\epsilon)}{(x_2+x_4+x_6)(x_3+x_5+x_6)x_4x_5} + \ldots$$

$$(x_2+x_4+x_6) = s_{134}/Q^2, (x_3+x_5+x_6) = s_{234}/Q^2$$

important: choice of convenient parametrisation when mapping integrations to unit hypercube

(e.g. solve $\lambda = 0$ for variable not occurring in denominator)

minimises number of functions produced by iterated sector decomposition

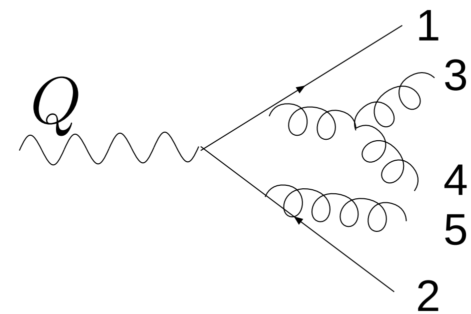
$e^+e^- \to 2$ jets: minimisation not vital as expressions are of moderate size sector decomposition for processes involving 5 particles at NNLO ($1 \to 4$ and $2 \to 3$) has seen many successful applications meanwhile:

- $e^+e^- \to 2$ jets at NNLO
  GH 03; Binoth, GH 04; Anastasiou, Melnikov, Petriello 03,04
- NNLO QED corrections to muon decay
  Anastasiou, Melnikov, Petriello 05
- NNLO Higgs production
  Anastasiou, Melnikov, Petriello 05

here for the first time:
application to process involving 6 particles:

$e^+e^- \to 3$ jets at NNLO ($1 \to 5$ process)

application to $1 \to 5$ phase space



phase space for $\gamma^* \to 5$ partons:

$Q = p_1 + \ldots + p_5$, 10 invariants $s_{12}, s_{13}, s_{23}, \ldots, s_{45}$

eliminate one s_{ij} by momentum conservation

in $D = 4$: remaining 9 invariants not independent:

nonlinear constraint from $\mathrm{Det}(G) = 0$

($G_{ij} = 2\, p_i \cdot p_j$ Gram matrix)

sector decomposition:

operates in $D \neq 4$ dimensions to isolate poles in $1/\epsilon$

$\Rightarrow$ work with 9 independent invariants

$1 \to 5$ phase space

use dimensionless invariants $x_1 = s_{12}/Q^2, \ldots, x_{10} = s_{45}/Q^2$

$$\int d\Phi_{1\to 5}^{D\neq 4} = C_\Gamma^{(5)} \int \prod_{j=1}^{10} dx_j \, \delta(1 - \sum_{i=1}^{10} x_i) \; [\Delta_5(\vec{x})]^{(D-6)/2} \, \Theta(\Delta_5)$$

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note: $C_\Gamma^{(5)} \sim V(D-4) = 2\pi^{-\epsilon}/\Gamma(-\epsilon) = \mathcal{O}(\epsilon)$

combines with

fake singularity in $[\Delta_5(\vec{x})]^{(D-6)/2} = [\Delta_5(\vec{x})]^{-1-\epsilon}$

$\Rightarrow$ after combination with $1 \to 5$ matrix element

up to $1/\epsilon^7$ poles in parameter integrals

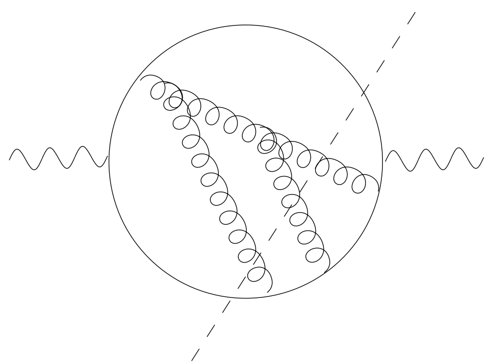
$$
\begin{aligned}
\Delta_5 \quad = \quad & x_{10}^2 x_1 x_2 x_3 + x_9^2 x_1 x_4 x_5 + x_8^2 x_2 x_4 x_6 + x_7^2 x_3 x_5 x_6 \\
& + x_6^2 x_1 x_7 x_8 + x_5^2 x_2 x_7 x_9 + x_4^2 x_3 x_8 x_9 + x_3^2 x_4 x_7 x_{10} \\
& + x_2^2 x_5 x_8 x_{10} + x_1^2 x_6 x_9 x_{10} \\
& + x_{10} \left[x_2 x_3 x_5 x_7 + x_1 x_3 x_6 x_7 + x_2 x_3 x_4 x_8 \right. \\
& \left. + x_1 x_2 x_6 x_8 + x_1 x_3 x_4 x_9 + x_1 x_2 x_5 x_9 \right] \\
& + x_9 \left[x_4 x_5 \left(x_3 x_7 + x_2 x_8 \right) + x_1 x_6 \left(x_5 x_7 + x_4 x_8 \right) \right] \\
& + x_6 x_7 x_8 \left(x_3 x_4 + x_2 x_5 \right)
\end{aligned}
$$

"real" $1/\epsilon^6$ poles come from regions where 3 particles become soft/collinear ("triple unresolved")

$\Rightarrow$ leads to 2-jet configuration

$\Rightarrow$ will be rejected by measurement function

example:



$$|\mathcal{M}|^2 \sim \mathcal{P}(s_{ij}, \epsilon)/(s_{1345}\, s_{2345}\, s_{345}\, s_{245}\, s_{34}\, s_{25})$$

$\{s_{34}, s_{345}, s_{1345}\} \rightarrow 0 \Rightarrow 1/\epsilon^6$ pole

measurement function

after ϵ expansion: finite part of $\int d\phi_5 \, |\mathcal{M}|^2 \sim$

$$\delta(s_{1345})\delta(s_{345})\delta(s_{34})\, G(s_{ij}) + \delta(s_{34}) \, \left[\frac{\ln(s_{25})}{s_{25}}\right]_+ F(s_{ij}) + \ldots$$

3-jet measurement function will enforce $s_{1345} > 0 \;\Rightarrow$
first term vanishes after inclusion of measurement function

problem:

- general procedure creates enormous number of terms

  (example graph: $\mathcal{O}(500)$ terms before, $\mathcal{O}(10^4)$ terms after iterated sector decomposition)

- many of them will be discarded later by measurement function
- would like to include measurement function only at very end in Fortran code to maintain flexibility

measurement function

solution:

make "preselection" already in ϵ expansion

(e.g. impose $s_{1345} > 0$) to discard 2-jet configurations

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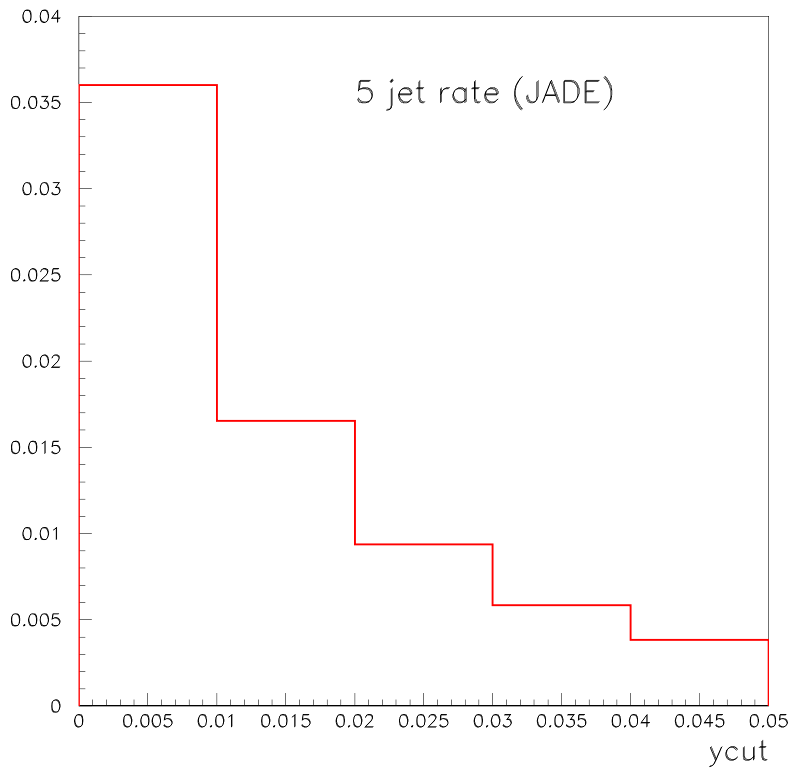
(e.g. impose $s_{1345} > 0$) to discard 2-jet configurations

- limits number of terms produced by ϵ expansion
- leaves freedom to define actual measurement function
  (e.g. jet algorithm, shape observables, $\ldots$)

  in Monte Carlo program only

example: JADE jet algorithm

5-jet rate for example graph:



y^{cut}: miminal separation between jets in terms of invariant mass

join bits and pieces

application to full $e^+e^- \to q\bar{q}ggg, q\bar{q}Q\bar{Q}g$ matrix elements

- choose different parametrisations, optimised for certain denominator structures

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all steps are fully automated

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- different for each colour structure
- analytical integration of subtraction terms may become impossible for processes involving several mass scales

advantages of sector decomposition:

- subtraction terms are integrated numerically
  $\Rightarrow$ no need to have simple subtraction terms
- procedure of isolating singularities is simple algorithm always the same for
  1. all colour structures of a given process
  2. different processes

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Monte Carlo programs for $e^+e^- \rightarrow 3$ jets at NNLO with both methods are under construction