Two-Loop Corrections to Bhabha Scattering

Alexander Penin

TTP Karlsruhe & INR Moscow

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- Test ground for new methods of multiloop calculations
- Classical problem of perturbative QED

Born approximation

$$\frac{d\sigma^{(0)}}{d\Omega} = \frac{\alpha^2}{s}\left(\frac{1-x+x^2}{x}\right)^2 + O\left(m_e^2/s\right), \qquad x = \frac{1-\cos\theta}{2}$$

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Phenomenologycally interesting:

- High energy region $s,\ t,\ u \gg m_e^2$
- Small angle Bhabha scattering $t \ll s,\ x \sim 0$
- Large angle Bhabha scattering $t \sim s,\ x \sim 1$

Radiative corrections

Only inclusive processes are IR finite and observable

$$\sigma = \sum_{n=0}^{\infty} \left(\frac{\alpha}{\pi}\right)^n \sigma^{(n)}, \qquad \sigma^{(1)} = \sigma_v^{(1)} + \sigma_r^{(1)}, \qquad \sigma^{(2)} = \sigma_{vv}^{(2)} + \sigma_{rv}^{(2)} + \sigma_{rr}^{(2)}, \ldots$$

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Two types of IR divergences:

- Soft divergences, regulated by λ or ε. Disappear in soft-photon-inclusive cross section with the energy cutoff $\mathcal{E}_{cut}$ on the emitted photons
- Collinear divergences, regulated by m_e or ε. Disappear in collinear-photon-inclusive cross section with the angular cutoff θ_{cut} on the emitted photons

Inclusive cross section

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- Put $m_e = 0$
  - Define QED "structure function" for initial states
  - Define QED 'jets" with angular resolution $\theta_{cut} \gg \sqrt{m_e^2/s}$ for final states

Inclusive cross section

Two ways to include the photon bremsstrahlung

- Put $m_e = 0$
  - Define QED “structure function” for initial states
  - Define QED ‘jets” with angular resolution $\theta_{cut} \gg \sqrt{m_e^2/s}$ for final states
- Keep $m_e \neq 0$
  - Split real radiation into “soft” and “hard” by $\mathcal{E}_{cut} \ll m_e$
  - Compute the virtual+soft real part analytically
  - Compute the hard real part with actual experimental cuts by means of Monte Carlo

Structure of the corrections

$$\frac{d\sigma^{(1)}}{d\sigma^{(0)}} = \delta_1^{(1)} \ln\left(\frac{s}{m_e^2}\right) + \delta_0^{(1)} + \mathcal{O}\left(m_e^2/s\right)$$

$$\delta_1^{(1)} = 4\ln\left(\frac{\mathcal{E}_{cut}}{\mathcal{E}}\right) + \ldots, \qquad \mathcal{E} = \sqrt{s}/2$$

$$\frac{d\sigma^{(2)}}{d\sigma^{(0)}} = \delta_2^{(2)} \ln^2\left(\frac{s}{m_e^2}\right) + \delta_1^{(2)} \ln\left(\frac{s}{m_e^2}\right) + \delta_0^{(2)} + \mathcal{O}\left(m_e^2/s\right)$$

$$\delta_2^{(2)} = 8\ln^2\left(\frac{\mathcal{E}_{cut}}{\mathcal{E}}\right) + 12\ln\left(\frac{\mathcal{E}_{cut}}{\mathcal{E}}\right) + \ldots$$

$$\delta_1^{(2)} = -16\left[1 + \ln\left(\frac{1-x}{x}\right)\right] \ln^2\left(\frac{\mathcal{E}_{cut}}{\mathcal{E}}\right) + \ldots$$

History and current status of two-loop calculations

- Logarithmic corrections to SA scattering

  A.B. Arbuzov, V.S. Fadin, E.A. Kuraev, L.N. Lipatov, N.P. Merenkov, L. Trentadue

- Logarithmic corrections to LA scattering

  E.W. Glover, J.B. Tausk, J.J. van der Bij

- Full massless result for virtual correction

  Z. Bern, L. Dixon, A. Ghinculov

- $m_e \neq 0$, fermion loop insertions

  R. Bonciani, A. Ferroglia, P. Mastrolia, E. Remiddi, J.J. van der Bij

- Photonic corrections, leading order in m_e^2/s

  A. Penin

Framework of calculation

- Purely photonic corrections
- Nonzero photon mass $\lambda \ll m_e$
- Leading order in m_e^2/s

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Systematic expansion of Feynman integrals in m_e^2/s:

Expansion by regions (V.Smirnov)

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Systematic expansion of Feynman integrals in m_e^2/s:

Expansion by regions (V.Smirnov)

In the leading order in m_e^2/s the massless and massive results are related by change of IR regularization scheme:

Infrared subtractions

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- For a given amplitude $\mathcal{A}$ construct an auxilary amplitude $\bar{\mathcal{A}}$ with the same structure of IR singularities.

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$$\delta\mathcal{A} = \left[\mathcal{A}(\varepsilon) - \bar{\mathcal{A}}(\varepsilon)\right]_{\varepsilon\to 0}$$

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- For a given amplitude $\mathcal{A}$ construct an auxilary amplitude $\bar{\mathcal{A}}$ with the same structure of IR singularities.
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$$\delta\mathcal{A} = \left[\mathcal{A}(\varepsilon) - \bar{\mathcal{A}}(\varepsilon)\right]_{\varepsilon\to 0}$$

- Compute the auxilary amplitude $\bar{\mathcal{A}}$ for $\lambda,\ m_e \longrightarrow 0$
- The amplitude $\mathcal{A}$ in the limit $\lambda,\ m_e \longrightarrow 0$ is given by

$$\mathcal{A}(\lambda, m_e)\Big|_{\lambda,\ m_e\to 0} = \bar{\mathcal{A}}(\lambda, m_e)\Big|_{\lambda,\ m_e\to 0} + \delta\mathcal{A}$$

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Example: vector form factor $\mathcal{F} = \sum_{n=0}^{\infty} \left(\frac{\alpha}{\pi}\right)^n f^{(n)}$, $(Q^2 = -s)$

How to construct $\bar{\mathcal{A}}$?

IR singularities factorize and exponentiate

Example: vector form factor $\mathcal{F} = \sum_{n=0}^{\infty}\left(\frac{\alpha}{\pi}\right)^n f^{(n)}$, $(Q^2=-s)$

$\lambda,\ m_e = 0$:

$$f^{(1)} = \left[-\frac{1}{2\varepsilon^2}-\frac{3}{4\varepsilon}-2+\frac{\pi^2}{24}+\left(-4+\frac{\pi^2}{16}+\frac{7}{6}\zeta(3)\right)\varepsilon+\left(-8+\frac{\pi^2}{6}+\frac{7}{4}\zeta(3)+\frac{47}{2880}\pi^4\right)\varepsilon^2\right]\frac{1}{Q^{2\varepsilon}}$$

$\lambda \ll m_e \ll Q$:

$$f^{(1)} = -\frac{1}{4}\ln^2\left(\frac{Q^2}{m_e^2}\right)+\left[\frac{1}{2}\ln\left(\frac{\lambda^2}{m_e^2}\right)+\frac{3}{4}\right]\ln\left(\frac{Q^2}{m_e^2}\right)-\frac{1}{2}\ln\left(\frac{\lambda^2}{m_e^2}\right)-1+\frac{\pi^2}{12}$$

$m_e \ll \lambda \ll Q$:

$$f^{(1)} = -\frac{1}{4}\ln^2\left(\frac{Q^2}{\lambda^2}\right)+\frac{3}{4}\ln\left(\frac{Q^2}{\lambda^2}\right)-\frac{7}{8}-\frac{\pi^2}{6}$$

Exponentiation

$$\mathcal{F} \sim \exp\left\{\frac{\alpha}{2\pi}\left[\ln\left(\frac{Q^2}{m_e^2}\right)-1\right]\ln\left(\frac{\lambda^2}{m_e^2}\right)\right\}$$

(D.R.Yennie, S.C.Frautschi, H.Suura)

$$\frac{\partial}{\partial \ln\left(Q^2\right)}\mathcal{F} = \left[-\frac{\alpha}{2\pi}\ln\left(Q^2\right)+\phi(m_e,\lambda,\varepsilon,\alpha)\right]\mathcal{F}$$

(A.Mueller, J.Collins)

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$\lambda,\ m_e = 0$:

$$\mathcal{F} = \left(1+\mathcal{O}(\alpha)\right)\exp\left\{-\frac{\alpha}{2\pi}\left(\frac{1}{\varepsilon^2}+\left(\frac{3}{2}+\mathcal{O}(\alpha)\right)\frac{1}{\varepsilon}\right)\left(\frac{\mu^2}{Q^2}\right)^{\varepsilon}\right\}$$

$\lambda \ll m_e \ll Q$:

$$\mathcal{F} = \left(1+\mathcal{O}(\alpha)\right)\exp\left\{\frac{\alpha}{4\pi}\left[-\ln^2\left(\frac{Q^2}{m_e^2}\right)+2\left[\ln\left(\frac{Q^2}{m_e^2}\right)-1\right]\ln\left(\frac{\lambda^2}{m_e^2}\right)+\left(3+\mathcal{O}(\alpha)\right)\ln\left(\frac{Q^2}{m_e^2}\right)\right]\right\}$$

$m_e \ll \lambda \ll Q$:

$$\mathcal{F} = \left(1+\mathcal{O}(\alpha)\right)\exp\left\{\frac{\alpha}{4\pi}\left[-\ln^2\left(\frac{Q^2}{\lambda^2}\right)+\left(3+\mathcal{O}(\alpha)\right)\ln\left(\frac{Q^2}{\lambda^2}\right)\right]\right\}$$

Two-loop form factor

$\lambda,\ m_e = 0$ (T. Matsuura, S.C. van der Marck, W.L. van Neerven):

$$f^{(2)} = \frac{1}{2}\left(f^{(1)}\right)^2 - \left(\frac{3}{32} - \frac{\pi^2}{8} + \frac{3}{2}\zeta(3)\right)\frac{1}{2\varepsilon}\left(\frac{\mu^2}{Q^2}\right)^{2\varepsilon} - \frac{1}{128} + \frac{29}{96}\pi^2 - \frac{15}{8}\zeta(3) - \frac{2}{45}\pi^4$$

$\lambda \ll m_e \ll Q$ (P. Mastrolia, E. Rmiddi):

$$f^{(2)} = \frac{1}{2}\left(f^{(1)}\right)^2 + \left(\frac{3}{32} - \frac{\pi^2}{8} + \frac{3}{2}\zeta(3)\right)\ln\left(\frac{Q^2}{m_e^2}\right) + \frac{11}{8} + \frac{17}{32}\pi^2 - \frac{9}{4}\zeta(3) - \frac{7}{240}\pi^4 - \frac{\pi^2\ln(2)}{2}$$

$m_e \ll \lambda \ll Q$ (B. Feucht, J.H. Kühn, A.A. Penin, V.A. Smirnov):

$$f^{(2)} = \frac{1}{2}\left(f^{(1)}\right)^2 + \left(\frac{3}{32} - \frac{\pi^2}{8} + \frac{3}{2}\zeta(3)\right)\ln\left(\frac{Q^2}{\lambda^2}\right) + \frac{51}{128} + \frac{15}{16}\pi^2 + 5\zeta(3) - \frac{83}{360}\pi^4 - \frac{2}{3}\pi^2\ln^2(2) + \frac{2}{3}\ln^4(2) + 16\,\mathrm{Li}_4\left(\frac{1}{2}\right)$$

$e^+e^- \rightarrow e^+e^-$ amplitude

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$$\mathcal{A} = \sum_{n=0}^{\infty} \left(\frac{\alpha}{\pi}\right)^n A^{(n)}$$

- $A^{(n)}$ are two-component vectors in the chiral basis

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$$\mathcal{A} = \mathcal{F}^{\,2}\tilde{\mathcal{A}}$$

- Reduced amplitude $\tilde{\mathcal{A}}$ is free of collinear logs

Exponentiation

$$\frac{\partial}{\partial \ln(Q^2)} \tilde{\mathcal{A}} = \frac{\alpha}{\pi} \ln\left(\frac{1-x}{x}\right) \tilde{\mathcal{A}} \qquad (\text{A.Sen; G.Sterman})$$

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$\lambda = 0$:

$$\tilde{\mathcal{A}} = \left(1 + \mathcal{O}(\alpha)\right) \exp\left[-\frac{\alpha}{\pi} \ln\left(\frac{1-x}{x}\right) \frac{1}{\varepsilon} \left(\frac{\mu^2}{Q^2}\right)^{\varepsilon}\right]$$

$\lambda \neq 0$:

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The auxilary amplitude

$$\overline{A}^{(2)} = \frac{1}{2}\left(A^{(1)}\right)^2 + 2\left[f^{(2)} - \frac{1}{2}\left(f^{(1)}\right)^2\right]$$

Two-loop infrared divergences

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Our prediction

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Catani's formula

$$\begin{aligned} A^{(1)} &= \mathbf{I}^{(1)} + A^{(1)}_{\text{fin}} \\ A^{(2)} &= \left[-\frac{1}{2}\left(\mathbf{I}^{(1)}\right)^2 + \mathbf{H}^{(2)}\right] + \mathbf{I}^{(1)} A^{(1)} + A^{(2)}_{\text{fin}} \end{aligned}$$

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Scheme invariance

$$\mathbf{I}'^{(1)} = \mathbf{I}^{(1)} + G\,, \qquad A'^{(1)}_{\text{fin}} = A^{(1)}_{\text{fin}} - G$$

$$\mathbf{H}'^{(2)} = \mathbf{H}^{(2)} + F\,, \qquad A'^{(2)}_{\text{fin}} = A^{(2)}_{\text{fin}} - \left(\frac{1}{2}G^2 + F\right)$$

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$$\mathbf{H}'^{(2)} = \mathbf{H}^{(2)} + F\,, \qquad A'^{(2)}_{\text{fin}} = A^{(2)}_{\text{fin}} - \left(\frac{1}{2}G^2 + F\right)$$

$$\boxed{G = A^{(1)}_{\text{fin}}\,, \qquad F = 2f^{(2)} - (f^{(1)})^2 - \mathbf{H}^{(2)}}$$

Infrared matching

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- Change of the scheme

$$\mathbf{I}'^{(1)} \equiv \text{one-loop correction to } \mathcal{A}\,,$$

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$$\mathbf{I}'^{(1)}(\varepsilon),\ \mathbf{H}'^{(1)}(\varepsilon)$$

$$\downarrow$$

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- Inverse change of the scheme

$$\boxed{\mathbf{I}^{(1)}(\lambda, m_e), \mathbf{H}^{(1)}(\lambda, m_e)}$$

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Catani's operators in massless case

$$
\begin{aligned}
\mathbf{I}^{(1)} &= \frac{e^{-\varepsilon\gamma_E}}{\Gamma(1-\varepsilon)}\left(\frac{1}{\varepsilon^2}+\frac{3}{2\varepsilon}\right)\left[-\left(\frac{\mu^2}{-s}\right)^{\varepsilon}-\left(\frac{\mu^2}{-t}\right)^{\varepsilon}+\left(\frac{\mu^2}{-u}\right)^{\varepsilon}\right] \\
\mathbf{H}^{(2)} &= \frac{e^{-\varepsilon\gamma_E}}{\Gamma(1-\varepsilon)}\frac{1}{\varepsilon}\left(\frac{3}{32}-\frac{\pi^2}{8}+\frac{3}{2}\zeta(3)\right)\left[-\left(\frac{\mu^2}{-s}\right)^{\varepsilon}-\left(\frac{\mu^2}{-t}\right)^{\varepsilon}+\left(\frac{\mu^2}{-u}\right)^{\varepsilon}\right]
\end{aligned}
$$

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Catani's operators in massive case

$$\mathbf{I}^{(1)} = -\frac{1}{2}\ln^2\left(\frac{s}{m_e^2}\right)+\left[\ln\left(\frac{\lambda^2}{m_e^2}\right)+\frac{3}{2}-\ln\left(\frac{x}{1-x}\right)+i\pi\right]\ln\left(\frac{s}{m_e^2}\right)+\left[-1+\ln\left(\frac{x}{1-x}\right)-i\pi\right]$$
$$\ln\left(\frac{\lambda^2}{m_e^2}\right)+2-\frac{2}{3}\pi^2+\frac{3}{2}\ln\left(\frac{x}{1-x}\right)-\frac{1}{2}\ln^2(x)+\frac{1}{2}\ln^2(1-x)-\frac{3}{2}i\pi$$

$$\mathbf{H}^{(2)} = \left(\frac{3}{16}-\frac{\pi^2}{4}+3\zeta(3)\right)\left[\ln\left(\frac{s}{m_e^2}\right)+\ln\left(\frac{x}{1-x}\right)-i\pi\right]+\frac{177}{64}+\frac{11}{24}\pi^2-\frac{3}{4}\zeta(3)-\frac{7}{120}\pi^4-\pi^2\ln(2)$$

Result *(page 1 of 2)*

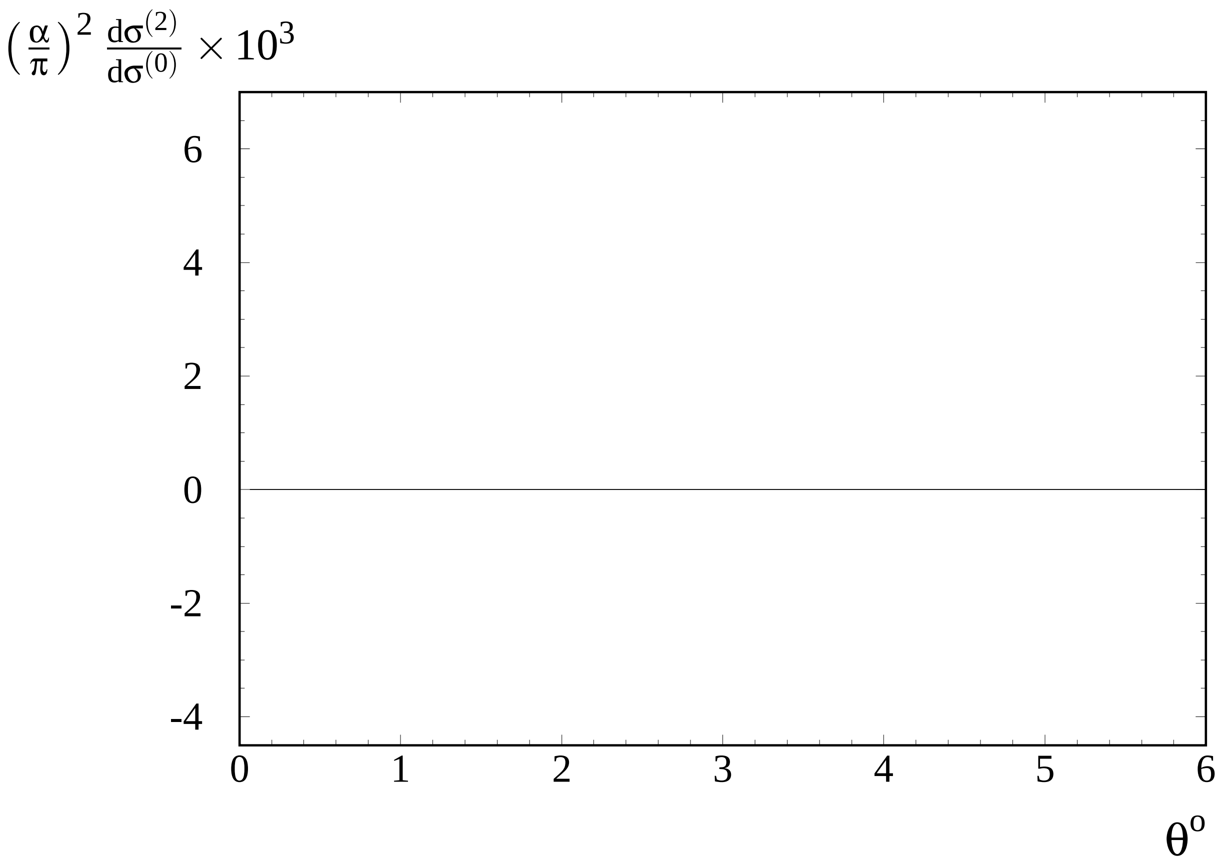
$$
\begin{aligned}
\delta_0^{(2)} &= 8\mathcal{L}_\varepsilon^2 + \left(1-x+x^2\right)^{-2}\left[\left(\frac{4}{3}-\frac{8}{3}x-x^2+\frac{10}{3}x^3-\frac{8}{3}x^4\right)\pi^2+\left(-12+16x-18x^2+6x^3\right)\ln(x)\right.\\
&+\left(2x+2x^3\right)\ln(1-x)+\left(-3x+x^2+3x^3-4x^4\right)\ln^2(x)+\left(-8+16x-14x^2+4x^3\right)\ln(x)\\
&\times\ln(1-x)+\left(4-10x+14x^2-10x^3+4x^4\right)\ln^2(1-x)+\left(1-x+x^2\right)^2(16+8\mathrm{Li}_2(x)\\
&\left.-8\mathrm{Li}_2(1-x))\right]\mathcal{L}_\varepsilon+\frac{27}{2}-2\pi^2\ln(2)+\left(1-x+x^2\right)^{-2}\left(\left(\frac{83}{24}-\frac{125}{24}x+\frac{13}{4}x^2+\frac{19}{24}x^3-\frac{25}{24}x^4\right)\right.\\
&\times\pi^2+\left(-9+\frac{43}{2}x-34x^2+22x^3-9x^4\right)\zeta(3)+\left(-\frac{11}{90}-\frac{5}{24}x+\frac{29}{180}x^2+\frac{23}{180}x^3-\frac{49}{480}x^4\right)\pi^4\\
&+\left[-\frac{93}{8}+\frac{231}{16}x-\frac{279}{16}x^2+\frac{93}{16}x^3+\left(-\frac{3}{2}+\frac{13}{4}x-\frac{7}{12}x^2-\frac{11}{8}x^3\right)\pi^2+\left(12-12x+8x^2\right.\right.\\
&\left.\left.-x^3\right)\zeta(3)\right]\ln(x)+\left[\frac{9}{2}-\frac{43}{8}x+\frac{17}{8}x^2+\frac{29}{8}x^3-\frac{9}{2}x^4+\left(\frac{x}{4}+\frac{x^2}{2}+\frac{5}{24}x^3+\frac{19}{48}x^4\right)\pi^2\right]\ln^2(x)\\
&+\left(\frac{67}{24}x-\frac{5}{4}x^2-\frac{2}{3}x^3\right)\ln^3(x)+\left(\frac{7}{48}x+\frac{5}{96}x^2-\frac{x^3}{12}+\frac{43}{96}x^4\right)\ln^4(x)+\left\{3x+3x^3+\left(\frac{7}{6}x\right.\right.\\
&\left.-\frac{73}{24}x^2+\frac{15}{8}x^3\right)\pi^2+\left(-6+6x-x^2-4x^3\right)\zeta(3)+\left[-8+\frac{21}{2}x-\frac{45}{4}x^2+x^4+\left(1-\frac{x}{6}+\frac{x^2}{12}\right.\right.\\
&\left.\left.\left.-\frac{x^3}{3}-\frac{x^4}{8}\right)\pi^2\right]\ln(x)+\left(6-11x+\frac{35}{4}x^2-\frac{15}{8}x^3\right)\ln^2(x)+\left(\frac{2}{3}+\frac{x}{12}-\frac{x^3}{3}+\frac{5}{24}x^4\right)\ln^3(x)\right\}\\
&\times\ln(1-x)+\left[\frac{7}{2}-6x+\frac{45}{4}x^2-6x^3+\frac{7}{2}x^4+\left(-\frac{17}{24}+\frac{7}{6}x-\frac{25}{24}x^2-\frac{13}{48}x^4\right)\pi^2+\left(-3+\frac{23}{4}x\right.\right.\\
&\left.\left.-\frac{23}{4}x^2+\frac{9}{8}x^3\right)\ln(x)+\left(\frac{7}{2}-\frac{41}{8}x+\frac{31}{8}x^2+\frac{3}{8}x^3-\frac{13}{16}x^4\right)\ln^2(x)\right]\ln^2(1-x)+\left[\frac{3}{8}x+\frac{1}{6}x^2\right.\\
&\left.+\frac{3}{8}x^3+\left(-4+\frac{29}{6}x-\frac{49}{12}x^2+\frac{5}{6}x^3+\frac{7}{8}x^4\right)\ln(x)\right]\ln^3(1-x)+\left(\frac{1}{32}-\frac{3}{4}x+\frac{71}{48}x^2-\frac{29}{24}x^3\right.\\
&\left.+\frac{9}{32}x^4\right)\ln^4(1-x)+\left\{8-16x+24x^2-16x^3+8x^4+\left(\frac{7}{3}-3x+\frac{3}{4}x^2+\frac{5}{6}x^3-\frac{2}{3}x^4\right)\pi^2\right.
\end{aligned}
$$

Result *(page 2 of 2)*

$$
\begin{aligned}
&+\left[-6+\frac{11}{2}x-4x^2+x^3+\left(2-\frac{11}{4}x+\frac{7}{4}x^2+\frac{x^3}{4}-x^4\right)\ln(x)\right]\ln(x)+\left[\frac{3}{2}x-\frac{x^2}{4}+x^3\right.\\
&\left.+\left(-4+9x-\frac{15}{2}x^2+2x^3\right)\ln(x)+\left(-1-\frac{7}{2}x+\frac{25}{4}x^2-5x^3+2x^4\right)\ln(1-x)\right]\ln(1-x)+\Big(2\\
&-4x+6x^2-4x^3+2x^4\Big)\,\mathrm{Li}_2(x)\Big\}\mathrm{Li}_2\,(x)+\Big\{-8+16x-24x^2+16x^3-8x^4+\left[-\frac{2}{3}+\frac{4}{3}x\right.\\
&\left.+\frac{x^2}{2}-\frac{5}{3}x^3+\frac{2}{3}x^4\right]\pi^2+\left[6-8x+9x^2-3x^3+\left(\frac{3}{2}x-\frac{x^2}{2}-\frac{3}{2}x^3+2x^4\right)\ln(x)\right]\ln(x)+\Big[-x\\
&-\frac{x^2}{4}-\frac{x^3}{2}+\left(10-14x+9x^2\right)\ln(x)+\left(-8+11x-\frac{31}{4}x^2+\frac{x^3}{2}+x^4\right)\ln(1-x)\Big]\ln(1-x)\\
&+\left(-4+8x-12x^2+8x^3-4x^4\right)\mathrm{Li}_2(x)+\left(2-4x+6x^2-4x^3+2x^4\right)\mathrm{Li}_2(1-x)\Big\}\mathrm{Li}_2\,(1-x)\\
&+\left[\frac{5}{2}x-5x^2+2x^3+\left(-4-x+x^2+2x^3-2x^4\right)\ln(x)+(6-6x+x^2+4x^3)\ln(1-x)\right]\mathrm{Li}_3\,(x)\\
&+\left[\frac{x}{2}-\frac{x^3}{2}+(-6+5x+3x^2-5x^3)\ln(x)+\left(6-10x+10x^3-6x^4\right)\ln(1-x)\right]\mathrm{Li}_3\,(1-x)\\
&+\left(-2+\frac{17}{2}x-\frac{17}{2}x^3+2x^4\right)\mathrm{Li}_4\,(x)+\left(7x-\frac{9}{2}x^2-4x^3+6x^4\right)\mathrm{Li}_4\,(1-x)+\Big(-6+4x\\
&+\frac{9}{2}x^2-7x^3\Big)\,\mathrm{Li}_4\left(-\frac{x}{1-x}\right)\Big),
\end{aligned}
$$

$$\mathcal{L}_\varepsilon=\left[1-\ln\left(x/(1-x)\right)\right]\ln\left(\mathcal{E}_{cut}/\mathcal{E}\right).$$

Two-loop photonic corrections to SA Bhabha scattering

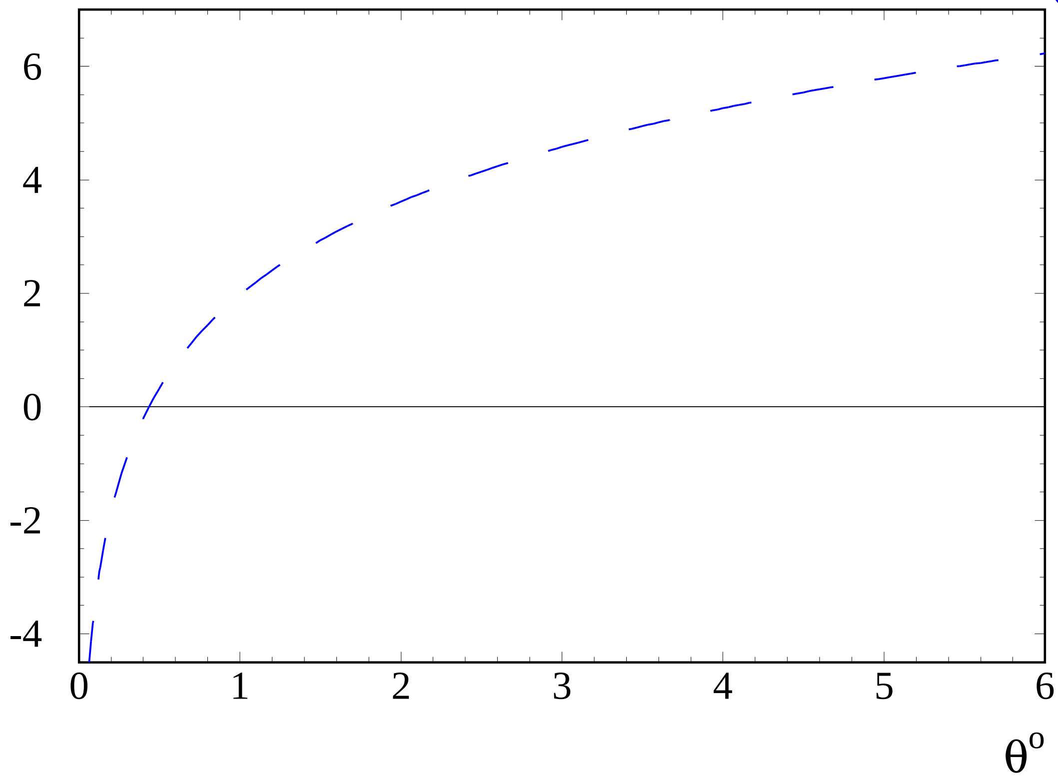


$$\sqrt{s} = 100 \text{ GeV}, \quad \ln\left(\frac{\mathcal{E}_{cut}}{\mathcal{E}}\right) = 0$$

Two-loop photonic corrections to SA Bhabha scattering

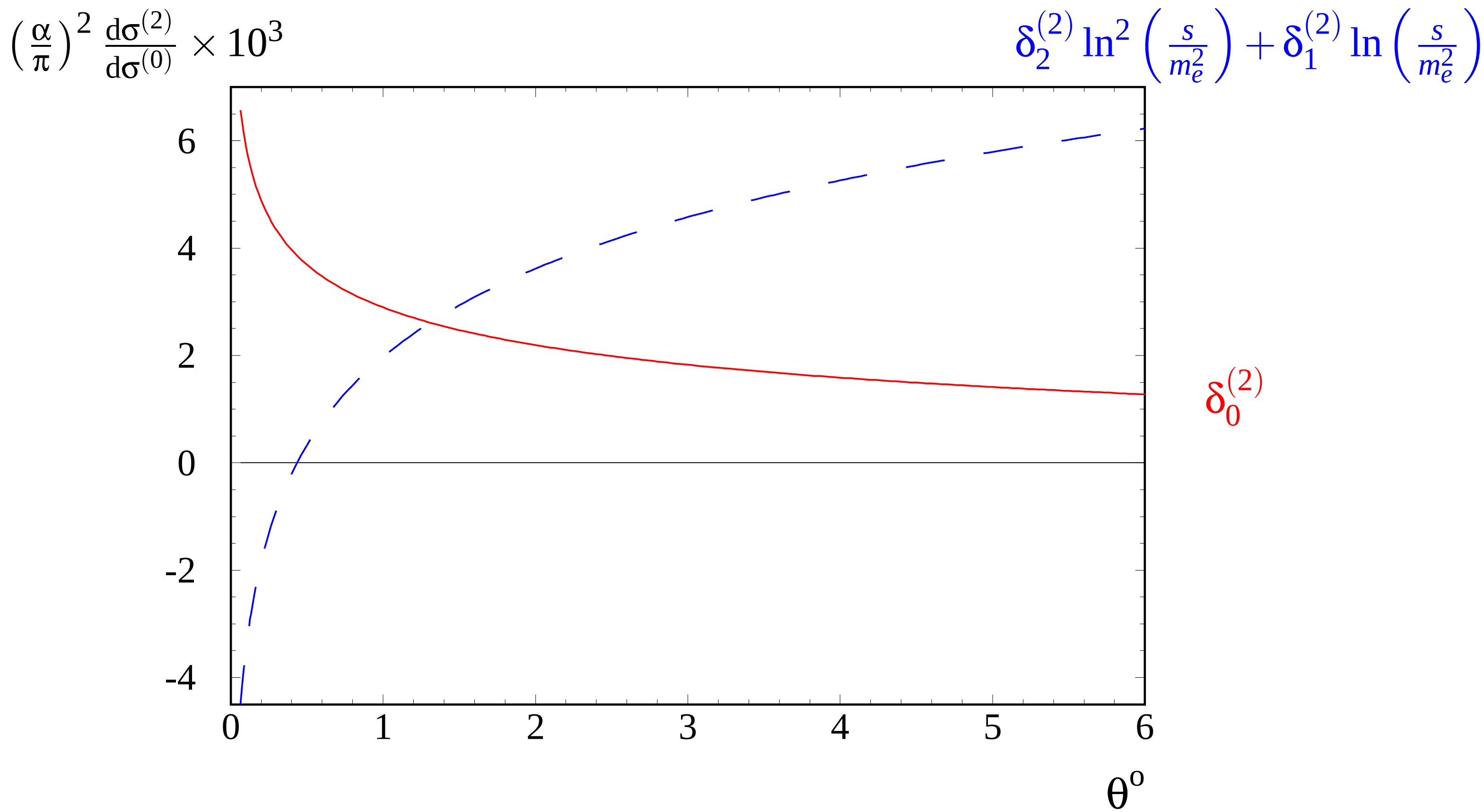
$\left(\frac{\alpha}{\pi}\right)^2 \frac{d\sigma^{(2)}}{d\sigma^{(0)}} \times 10^3$

$\delta_2^{(2)} \ln^2\left(\frac{s}{m_e^2}\right) + \delta_1^{(2)} \ln\left(\frac{s}{m_e^2}\right)$



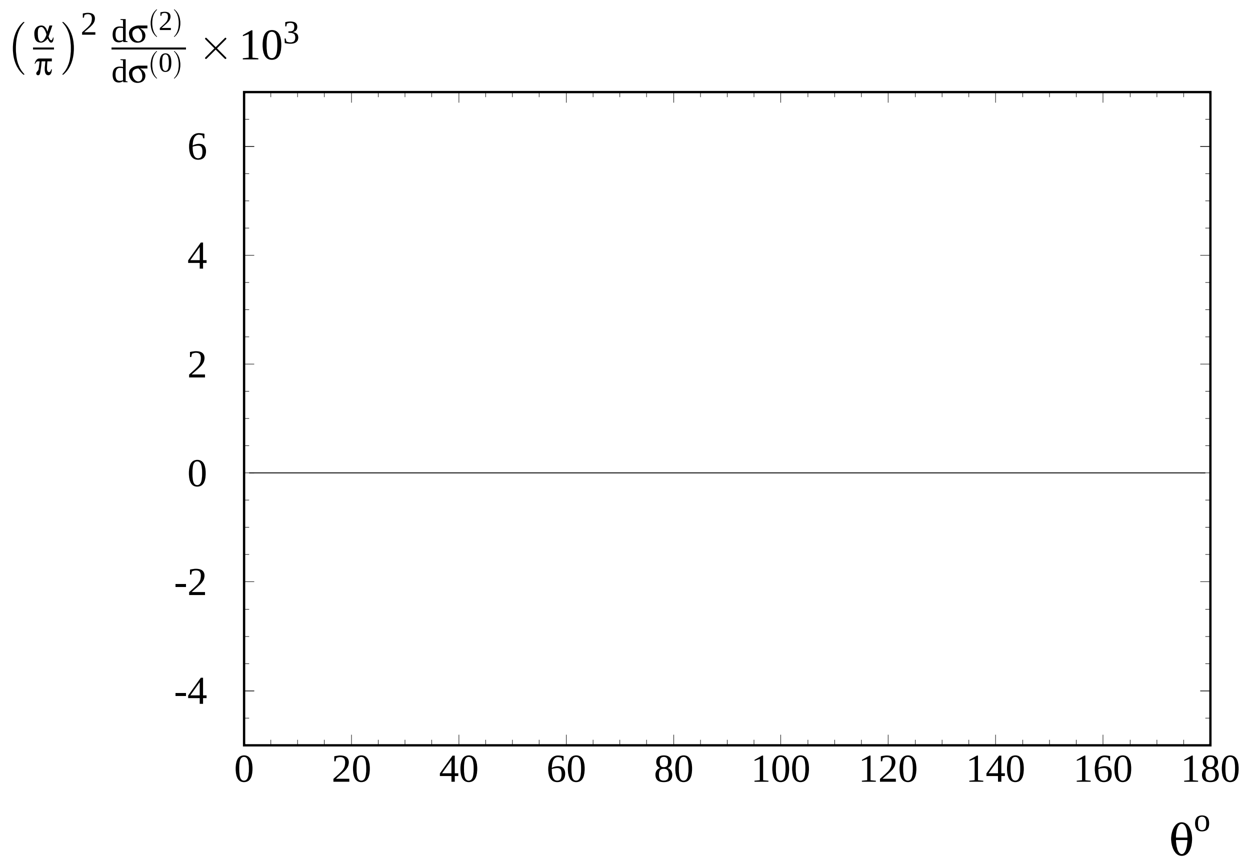
$\sqrt{s} = 100$ GeV, $\quad \ln\left(\frac{\mathcal{E}_{cut}}{\mathcal{E}}\right) = 0$

Two-loop photonic corrections to SA Bhabha scattering



$\sqrt{s} = 100$ GeV, $\quad \ln\left(\frac{\mathcal{E}_{cut}}{\mathcal{E}}\right) = 0$

Two-loop photonic corrections to LA Bhabha scattering

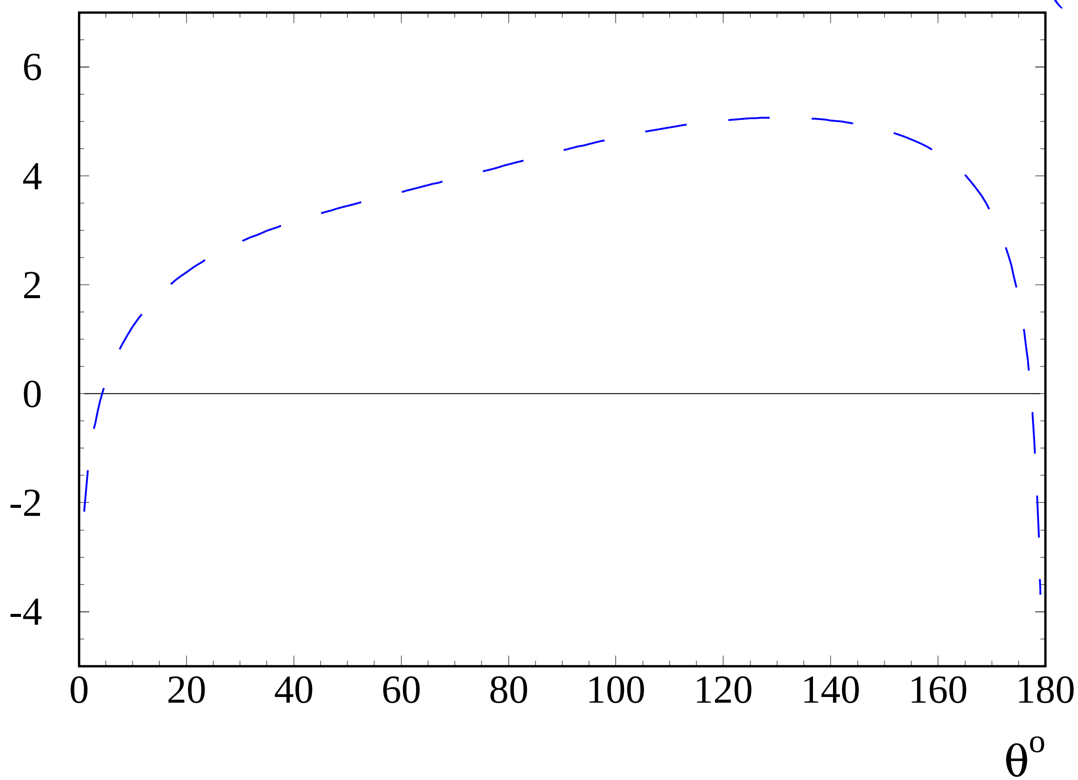


$$\sqrt{s} = 1\ \text{GeV}, \quad \ln\left(\frac{\mathcal{E}_{cut}}{\mathcal{E}}\right) = 0$$

Two-loop photonic corrections to LA Bhabha scattering

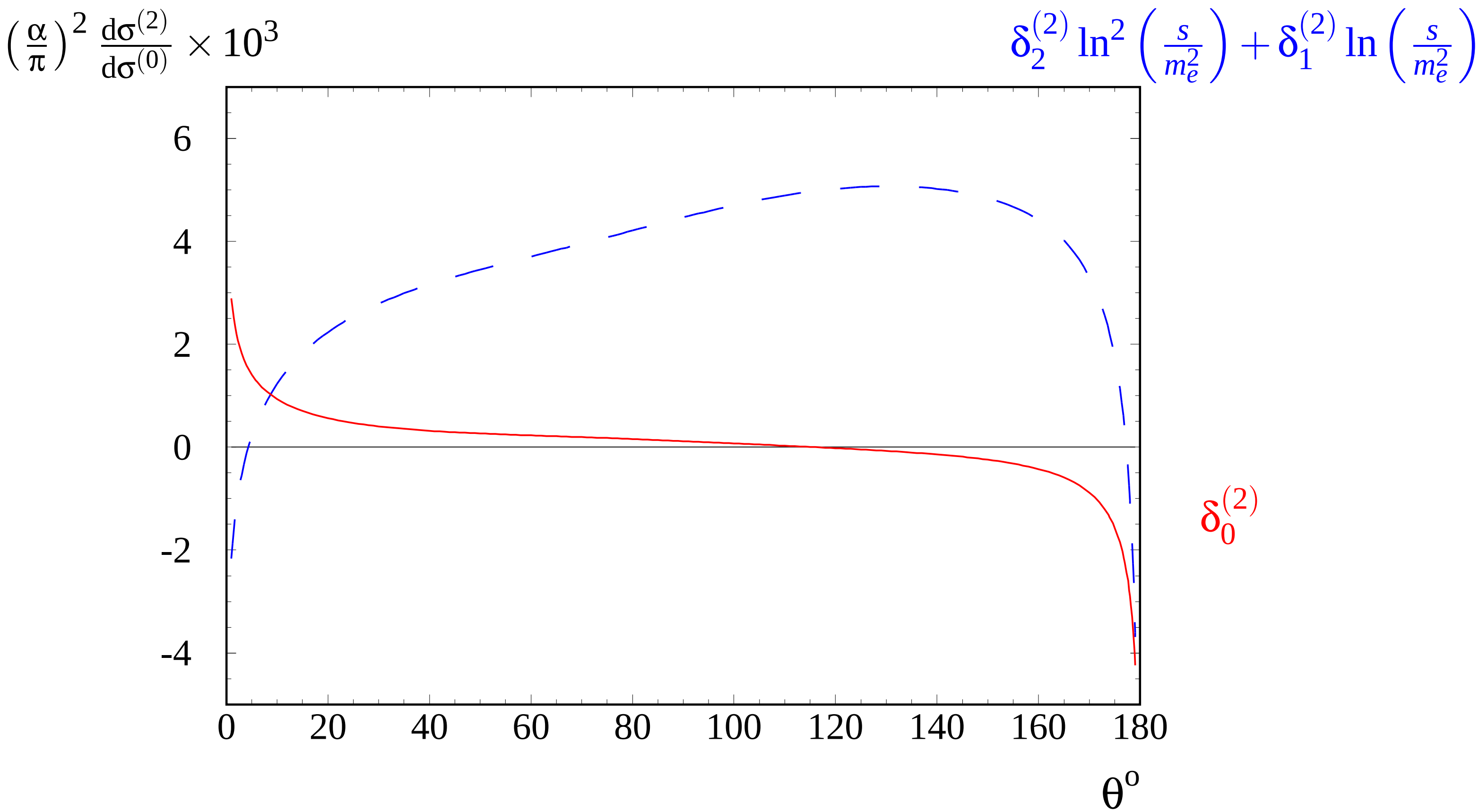
$\left(\frac{\alpha}{\pi}\right)^2 \frac{d\sigma^{(2)}}{d\sigma^{(0)}} \times 10^3$

$\delta_2^{(2)} \ln^2\left(\frac{s}{m_e^2}\right) + \delta_1^{(2)} \ln\left(\frac{s}{m_e^2}\right)$



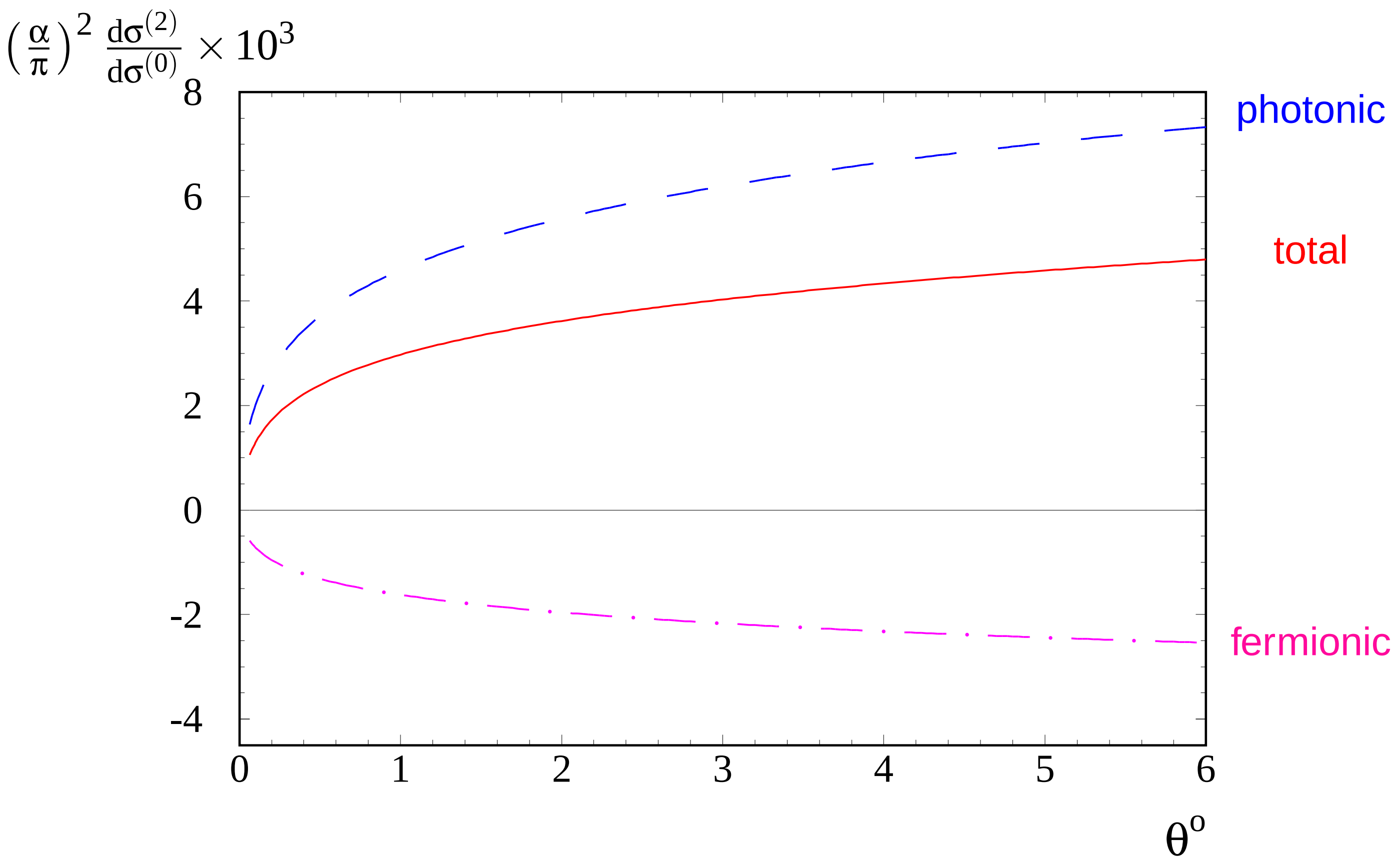
$\sqrt{s} = 1$ GeV, $\quad \ln\left(\frac{\mathcal{E}_{cut}}{\mathcal{E}}\right) = 0$

Two-loop photonic corrections to LA Bhabha scattering



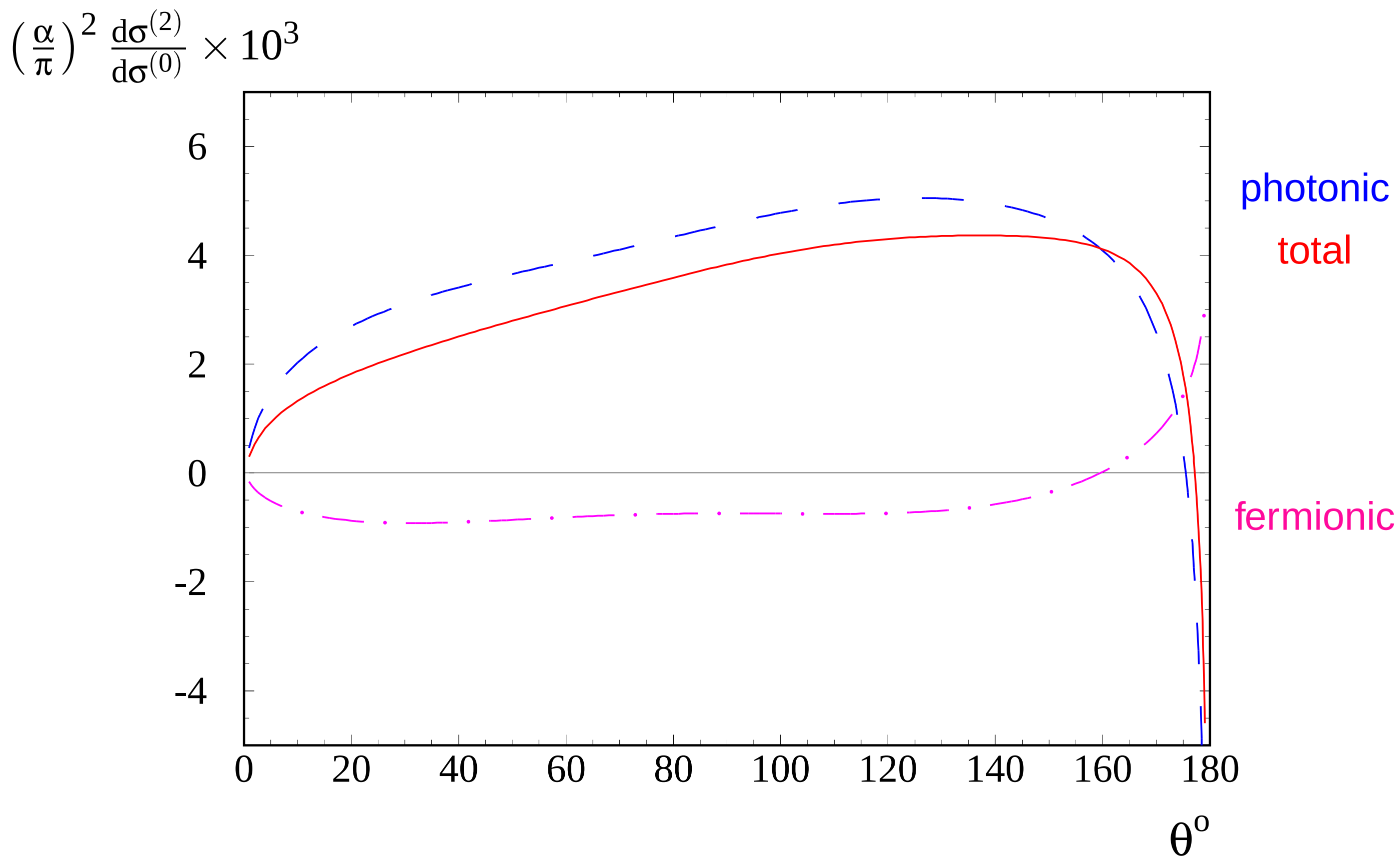
$$\sqrt{s} = 1 \text{ GeV}, \quad \ln\left(\frac{\mathcal{E}_{cut}}{\mathcal{E}}\right) = 0$$

Two-loop corrections to SA Bhabha scattering



$$\sqrt{s} = 100 \text{ GeV}, \quad \ln\left(\frac{\mathcal{E}_{cut}}{\mathcal{E}}\right) = \ln\left(\frac{\mathcal{E}_{cut}^{e^+e^-}}{\mathcal{E}}\right) = 0$$

Two-loop corrections to LA Bhabha scattering



$$\sqrt{s} = 1 \text{ GeV}, \quad \ln\left(\frac{\mathcal{E}_{cut}}{\mathcal{E}}\right) = \ln\left(\frac{\mathcal{E}_{cut}^{e^+e^-}}{\mathcal{E}}\right) = 0$$

Summary

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- After long way the two-loop QED corrections to Bhabha scatterng are here.

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- They should be incorporated into the Monte Carlo event generators.